Feynman path integral and the interaction picture

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The role of interaction-picture fields in the construction of coherent states and in the derivation of the Feynman path integral for interacting scalar quantum fields is examined. Special attention is paid to the dependence of the integrand on the intermediate times and it is shown that the Feynman rules are valid prior to taking the limit wherein the number of intermediate times goes to infinity; thus, this number does not act as a cutoff in divergent amplitudes. Specific normalization factors are determined.

I. INTRODUCTION

Ever since Feynman put Dirac's conjecture[1] on the role of the action in quantum mechanics on a firm footing,[2]

$$\langle f \mid i \rangle \sim \sum_{\text{paths}} e^{iS/\hbar} , \tag{1}$$

surprisingly little work has been done towards deriving the Feynman path integral (FPI) for quantum field theory from elementary Hamiltonian quantum mechanics even though the FPI approach is now the basis of much of modern field-theory work.[3] Polkinghorne[4] showed for field operators possessing eigenkets[5]

$$\hat{\varphi}(\mathbf{x},t) \mid \phi',t \rangle = \phi'(\mathbf{x},t) \mid \phi',t \rangle \tag{2}$$

and satisfying the FPI postulate (or definition)

$$\langle \phi''t'' \mid \hat{\phi}(\mathbf{x},t) \mid \phi't' \rangle = N^{-1} \int_{\phi'}^{\phi''} \phi(\mathbf{x},t) e^{iS[\phi]/\hbar} \delta\varphi , \tag{3}$$

the equations of motion and equal-time commutators of conventional field theory[6] follow. For free canonical fields Eq. (2) has no solution; we show below that its derivation from (3) involves an unjustified interchange of limits. Mathews and Salam[7] postulated the FPI equations

$$\langle \phi''t'' \mid (\hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n))_+ \mid \phi't' \rangle = N^{-1} \int_{\varphi'}^{\varphi''} \varphi(x_1) \cdots \varphi(x_n) e^{iS[\varphi]/\hbar} \delta\varphi \tag{4}$$

and derived propagators as functional integrals over the c-number fields $\varphi(x)$. In the case of Fermi fields one had to assume the c-number fields anticommute.

The 1960, Klauder[8] invented the coherent states to serve as eigenstates for the positive-frequency part of the quantum field operator, i.e., the destruction-operator part, and derived the FPI for the Schrödinger one-particle field theory. In 1978, Hammer, Shrauner, and DeFacio[9] used coherent states with arbitrary time dependence and associated interpolating fields satisfying the Lehmann-Symanzik-Zimmermann weak asymptotic condition[10] to derive the FPI from the S matrix. Swanson,[11] in 1981, used similar time-dependent coherent states to discuss the scalar and electromagnetic fields.

In this paper we try to take the simplest route to the FPI principle using coherent states that are interaction-picture states at $t=0$. This allows us to discuss the dynamical evolution of coherent states in the Heisenberg picture and the simplicity of the equations allows us to examine in some detail the following.

(1) The derivation of the FPI postulate Eq. (4).

(2) The derivation of the Feynman rules before the limit $\epsilon \rightarrow 0$ is taken just in case the inverse of the time interval ϵ^{-1} acts as a cutoff in divergent amplitudes. (It does not.)

(3) The derivation of some intermediate forms of the FPI that are possibly more suitable for actual calculations.

(4) The modification of the wave functions of the initial and final states. The extension of these results to the complex scalar field is trivial; the special problems associated with the photon will form the subject of a subsequent paper.

II. COHERENT STATES

A coherent state may be defined by[8]

$$\mid \chi \rangle \equiv e^{\underline{a}^\dagger \cdot \underline{\chi}} \mid 0 \rangle , \tag{5}$$

where the "scalar product" is shorthand for

$$\underline{a}^\dagger \cdot \underline{\chi} \equiv \sum_\alpha a_\alpha^\dagger \chi_\alpha \tag{6}$$

with the sum extending over the denumerable normal modes of the system. The creation and destruction operators for these modes satisfy

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \quad [a_\alpha, a_\beta] = 0 . \tag{7}$$

The parameters χ_a are arbitrary complex numbers. The coherent state is constructed to be an eigenstate of the destruction operator,

$$a_\alpha \mid \chi \rangle = \chi_\alpha \mid \chi \rangle , \tag{8}$$

and the scalar product of two such states is

$$\langle \chi' \mid \chi \rangle = e^{\underline{\chi}'^* \cdot \underline{\chi}} . \tag{9}$$

They form an overcomplete set,

$$\int \mathscr{D}\chi \mid \chi \rangle \langle \chi \mid = 1 , \tag{10}$$

with the measure in

$$\mathscr{D}\chi \equiv \prod_{\alpha} \frac{d^2\chi_\alpha}{\pi} e^{-|\chi_\alpha|^2} \tag{11}$$

uniquely determined by (10). Here $d^2\chi_\alpha \equiv d\mathscr{R}\chi_\alpha \, d\mathscr{I}\chi_\alpha$.

The following useful integral can be established by elementary methods

$$\int \mathscr{D}\chi \, e^{\underline{\psi}^*\cdot\underline{\chi}} e^{\underline{\chi}^*\cdot\underline{\varphi}} = e^{\underline{\psi}^*\cdot\underline{\varphi}} , \tag{12}$$

and is equivalent to (10).

A free real scalar field is defined by

$$\hat{\varphi}(x) \equiv \underline{a}^\dagger \cdot \underline{\varphi}^{(-)}(x) + \underline{\varphi}^{(-)\dagger}(x) \cdot \underline{a} , \tag{13}$$

where $\varphi_\alpha^{(+)}(x) \equiv \varphi_\alpha^{(-)*}(x) \equiv \varphi_\alpha^{(-)\dagger}(x)$ and $\varphi_\alpha^{(-)}(x)$ are positive- and negative-energy solutions of the Klein-Gordon equation for the mode α which are normalized in the usual way. The coherent state is an eigenstate of the destruction-operator part of $\hat{\varphi}$:

$$\hat{\varphi}^{(+)}(x) | \chi \rangle = \chi(x) | \chi \rangle \tag{14}$$

with

$$\chi(x) \equiv \underline{\varphi}^{(-)\dagger}(x) \cdot \underline{\chi} \tag{15}$$

which is also a positive-frequency solution of the Klein-Gordon equation.

We identify the coherent state as an interaction-picture state at $t=0$:

$$| \chi, 0 \rangle_{\mathrm{IP}} \equiv | \chi \rangle . \tag{16}$$

The IP state at other times is determined by the usual time-development operator

$$| \chi, t \rangle_{\mathrm{IP}} = U(t) | \chi \rangle = e^{i\hat{H}_0^s t} e^{-i\hat{H}t} | \chi \rangle , \tag{17}$$

but it is important to note that it is not an eigenstate of $\hat{\varphi}^{(+)}(\mathbf{x},t)$. Time-dependent coherent interaction-picture states can be defined[9,11] but they are an unnecessary complication.

The Heisenberg picture

The Heisenberg-picture fields are defined by

$$\hat{\varphi}_H^{(\pm)}(\mathbf{x},t) \equiv e^{i\hat{H}t} \hat{\varphi}^{(\pm)}(\mathbf{x},0) e^{-i\hat{H}t} . \tag{18}$$

They are not necessarily positive- and negative-frequency components. One defines the Heisenberg state $| \chi t \rangle_H$ to be an eigenstate of $\hat{\varphi}_H^{(+)}(\mathbf{x},t)$

$$\hat{\varphi}_H^{(+)}(\mathbf{x},t) | \chi, t \rangle_H = \chi(\mathbf{x},0) | \chi, t \rangle_H \tag{19}$$

[note that the eigenvalue is $\chi(\mathbf{x},0)$ *not* $\chi(\mathbf{x},t)$] so it is related to the IP state by

$$| \chi, t \rangle_H = e^{i\hat{H}t} | \chi \rangle . \tag{20}$$

(Note that no eigenket of $\hat{\varphi}_H \equiv \hat{\varphi}_H^{(+)} + \hat{\varphi}_H^{(-)}$ exists.) Equation (20) gives us the dependence of the Heisenberg state on the parameter t; this should be contrasted with the Schrödinger-picture time development

$$| a, t \rangle_s = e^{-i\hat{H}t} | a, 0 \rangle_s \tag{21}$$

as the meaning of the t dependence is quite different in the two pictures.

A direct consequence of (20) is that the Heisenberg-picture states satisfy a completeness relationship of the form

$$\int \mathscr{D}\chi \, | \chi t \rangle_H \, {}_H\langle \chi t | = 1 . \tag{22}$$

In the following the only t-dependent states will be Heisenberg states and the t-independent states will be the IP states. Thus we can drop the subscripts H on the state vectors without ambiguity.

III. TRANSITION AMPLITUDES

The amplitude to go from the state $| a t_a \rangle$ to the state $| b t_b \rangle$ is $\langle b t_b | a t_a \rangle$. We can use the completeness of the Heisenberg states to write this as

$$\begin{aligned} \langle b t_b | a t_a \rangle &= \int \mathscr{D}\chi_a \mathscr{D}\chi_b \langle b t_b | \chi_b t_b \rangle \langle \chi_b t_b | \chi_a t_a \rangle \\ &\qquad \times \langle \chi_a t_a | a t_a \rangle \\ &= \int \mathscr{D}\chi_a \mathscr{D}\chi_b \Psi_b^*(\chi_b) \langle \chi_b t_b | \chi_a t_a \rangle (\chi_a) , \end{aligned} \tag{23}$$

where the wave function of the state a is defined as

$$\Psi_a(\chi) \equiv \langle \chi | a \rangle \equiv \langle \chi t_a | a t_a \rangle \tag{24}$$

and is normalized in the sense that

$$\int \mathscr{D}\chi \, | \Psi_a(\chi) |^2 = 1 . \tag{25}$$

These wave functions are closely related to the usual harmonic-oscillator wave functions. [See Eq. (76) below.]

Now we concentrate on the kernel $\langle \chi_b t_b | \chi_a t_a \rangle$. In the usual fashion[12] breakup the interval $T = t_b - t_a$ into $N+1$ equal parts of length ϵ and label the intermediate points by $t_k = t_a + k\epsilon$. Also let $\chi_a = \chi_0$ and $\chi_b = \chi_{N+1}$. Insert a complete set at each intermediate point:

$$\langle \chi_b t_b | \chi_a t_a \rangle = \int \prod_{j=1}^{N} \mathscr{D}\chi_j \prod_{j=0}^{N} \langle \chi_{j+1} t_{j+1} | \chi_j t_j \rangle . \tag{26}$$

A typical matrix element is then from (20)

$$\begin{aligned} \langle \chi_{j+1} t_{j+1} | \chi_j t_j \rangle &= \langle \chi_{j+1} | e^{-i\hat{H}(t_{j+1}-t_j)} | \chi_j \rangle \\ &= \langle \chi_{j+1} | e^{-i\epsilon\hat{H}} | \chi_j \rangle \\ &= \langle \chi_{j+1} | e^{-i\epsilon\hat{H}_{\mathrm{IP}}(0)} | \chi_j \rangle . \end{aligned} \tag{27}$$

In the last step we used $\hat{H} = \hat{H}_{\mathrm{IP}}(0)$ which follows from (18). The matrix elements of $\hat{H}_{\mathrm{IP}}(0)$ are easily calculated but the matrix elements of higher powers of $\hat{H}_{\mathrm{IP}}(0)$ are messy. Fortunately, these higher powers are not needed because of the following lemma:

$$\begin{aligned} &\prod_{j=0}^{N} \langle \chi_{j+1} | e^{-i\epsilon\hat{H}_{\mathrm{IP}}(0)} | \chi_j \rangle \\ &\qquad = \prod_{j=0}^{N} \langle \chi_{j+1} | \chi_j \rangle \exp\left[-i\epsilon \frac{\langle \chi_{j+1} | \hat{H}_{\mathrm{IP}}(0) | \chi_j \rangle}{\langle \chi_{j+1} | \chi_j \rangle} \right] \end{aligned} \tag{28}$$

which is *exact* in the limited $\epsilon\to 0$, $N\epsilon=T$. This is easily checked by direct calculation by noting that terms like $\epsilon\sum_j$ and $\epsilon^2\sum_{jk}$ are $O(\epsilon N)$ and are finite while terms like $\epsilon^2\sum_j$ are $O(\epsilon^2 N)$ and vanish in the limit. We discuss the possibility that the correction $O(\epsilon^2 N)$ might contribute in divergent amplitudes later on.

The interaction-picture Hamiltonian for real scalar fields is

$$\hat{H}_{\rm IP}=\int d^3x\,\{\tfrac{1}{2}:[\nabla\hat{\varphi}(\mathbf{x})]:+\tfrac{1}{2}m^2:[\hat{\varphi}(\mathbf{x})]^2:$$
$$+\tfrac{1}{2}:[\dot{\hat{\varphi}}(\mathbf{x})]^2:+:V(\hat{\varphi}(\mathbf{x}):\}\ . \qquad (29)$$

In evaluating the matrix elements of $\hat{H}_{\rm IP}$, special care must be given in the $\dot{\hat{\varphi}}$ term; this is trivial when the fields are expressed in their normal-mode form.

Collecting together these results one gets

$$\langle\chi_b t_b\,|\,\chi_a t_a\rangle$$
$$=\int\prod_{j=1}^{N}\mathscr{D}\chi_j\prod_{j=0}^{N}\langle\chi_{j+1}\,|\,\chi_j\rangle$$
$$\times\exp\left[-i\epsilon\frac{\langle\chi_{j+1}\,|\,\hat{H}_{\rm IP}(0)\,|\,\chi_j\rangle}{\langle\chi_{j+1}\,|\,\chi_j\rangle}\right]. \qquad (30)$$

Note there is no explicit dependence of the integrand on the intermediate times t_j. These times will reappear in the FPI when one defines a differentiable function $\chi_\alpha(t)$ which has values $\chi_{j\alpha}$ at times t_j but which is otherwise arbitrary.

IV. GREEN'S FUNCTIONS

Now consider the expectation value of the Heisenberg field $\langle\chi_b t_b\,|\,\hat{\varphi}_H(\mathbf{x},t)\,|\,\chi_a t_a\rangle$ for $t_b>t>t_a$. We write

$$\hat{\varphi}_H=\hat{\varphi}_H^{(+)}+\hat{\varphi}_H^{(-)} \qquad (31)$$

via (13) and (18). We break up the time interval as above and put in the N complete sets of intermediate states; identify t with one of the intermediate times, t_k, say. Then sandwich $\hat{\varphi}_H^{(+)}(\mathbf{x},t_k)$ between $\langle\chi_{k+1}t_{k+1}\,|$ and $|\,\chi_k t_k\rangle$ and sandwich $\hat{\varphi}_H^{(-)}(\mathbf{x},t_k)$ between $\langle\chi_k t_k\,|$ and $|\,\chi_{k-1}t_{k-1}\rangle$. In the first case the eigenvalue $\chi_k(\mathbf{x},0)$ is produced and in the second case $\chi_k^*(\mathbf{x},0)$. Defining

$$\varphi_k(x)=\chi_k(x)+\chi_k^*(x) \qquad (32)$$

one has the result

$$\langle\chi_b t_b\,|\,\hat{\varphi}_H(\mathbf{x},t)\,|\,\chi_a t_a\rangle$$
$$=\int\varphi_k(\mathbf{x},0)\prod_1^N\mathscr{D}\chi_j\prod_0^N\langle\chi_{j+1}t_{j+1}\,|\,\chi_j t_j\rangle \qquad (33)$$

which is just what one would have obtained if $\hat{\varphi}_H(\mathbf{x},t)$ had the eigenvalue $\varphi_k(\mathbf{x},0)$. (Note that $t_b>t>t_a$ is necessary for the separate values of $\hat{\varphi}^{(+)}$ and $\hat{\varphi}^{(-)}$; if $t=t_a$ or t_b the above proof does not go through. Therefore one must take the limit $t\to t_a$ or t_b *after* the limits implied in the functional integrations.) In a similar fashion one derives

$$\langle\chi_b t_b\,|\,[\hat{\varphi}_H(x_1)\cdots\hat{\varphi}_H(x_n)]_+\,|\,\chi_a t_a\rangle$$
$$=\int\varphi(x_1)\cdots\varphi(x_n)\prod_1^N\mathscr{D}\chi_j\prod_0^N\langle\chi_{j+1}t_{j+1}\,|\,\chi_j t_j\rangle\ , \qquad (34)$$

where we have put $\varphi_k(\mathbf{x},0)\equiv\varphi(\mathbf{x},t_k)$ for simplicity. This result obviously will lead to the FPI equations [Eq. (4)].

V. NORMAL MODES

Since $\chi_j(\mathbf{x},t)$ satisfies the KG equation

$$(\nabla^2-m^2)\chi_j(\mathbf{x},t)=\frac{\partial^2}{\partial t^2}\chi_j(\mathbf{x},t) \qquad (35)$$

and is positive energy, we can put

$$\chi_j(\mathbf{x},t)=\sum_n\frac{\psi_n(\mathbf{x})\chi_{jn}e^{-i\omega_n t}}{\sqrt{2\omega_n}} \qquad (36)$$

with $\psi_n(\mathbf{x})$ a real function satisfying

$$(\nabla^2-m^2)\psi_n(\mathbf{x})=-\omega_n{}^2\psi_n(\mathbf{x})\ , \qquad (37)$$

$$\int d^3x\,\psi_n(\mathbf{x})\psi_m(\mathbf{x})=\delta_{nm}\ , \qquad (38)$$

$$\sum_n\psi_n(\mathbf{x})\psi_n(\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})\ . \qquad (39)$$

For example, normalizing in a cube of side L gives

$$\psi_n(\mathbf{x})=\left[\frac{2}{L}\right]^{3/2}\sin\frac{n_x\pi x}{L}\sin\frac{n_y\pi y}{L}\sin\frac{n_z\pi z}{L}\ . \qquad (40)$$

The free field expansion is then

$$\hat{\varphi}(\mathbf{x},t)=\sum_n\frac{\psi_n(\mathbf{x})}{\sqrt{2\omega_n}}(a_n e^{-i\omega_n t}+a_n^\dagger e^{i\omega_n t}) \qquad (41)$$

and the free-particle Hamiltonian takes the usual form

$$\hat{H}_0=\sum_n\omega_n a_n^\dagger a_n\ . \qquad (42)$$

The total Hamiltonian in the interaction picture, $\hat{H}_{\rm IP}(0)$, has matrix elements

$$\frac{\langle\chi_{j+1}\,|\,\hat{H}_{\rm IP}(0)\,|\,\chi_j\rangle}{\langle\chi_{j+1}\,|\,\chi_j\rangle}$$
$$=\sum_n\omega_n\chi_{j+1n}^*\chi_{jn}+\int d^3x\,V[\chi_{j+1}^*(\mathbf{x})+\chi_j(\mathbf{x})]\ . \qquad (43)$$

The dependence of the transition amplitude on the intermediate times comes from its dependence on the index j not from the t dependence of $\hat{\varphi}(\mathbf{x},t)$.

VI. FREE-PARTICLE KERNEL

When $V=0$ one has immediately from (43), (30), and (9)

$$\langle\chi_b t_b\,|\,\chi_a t_a\rangle_0=\int\prod_{j=1}^N\mathscr{D}\chi_j\prod_{j=0}^N\prod_n e^{\gamma_n\chi_{j+1n}^*\chi_{jn}}$$
$$=\prod_n e\gamma_n^{N+1}\chi_{bn}^*\chi_{an}\ , \qquad (44)$$

where

$$\gamma_n \equiv 1 - i\epsilon\omega_n \ . \tag{45}$$

In the last step in (44) we used (12) repeatedly. Now take the limit $N \rightarrow \infty$ and get the result

$$\langle \chi_b t_b \,|\, \chi_a t_a \rangle_0 = \exp\left[\sum_n \chi_{bn}^* \chi_{an} e^{-i\omega_n T}\right] . \tag{46}$$

VII. FREE-PARTICLE PROPAGATOR

One can write the free kernel in (44) in the form

$$\langle \chi_b t_b \,|\, \chi_a t_a \rangle_0 = e^{|\chi_a|^2 + |\chi_b|^2} \int \prod_{j=1}^{N} \frac{d^2\chi_j}{\pi} e^{-\chi^\dagger M \chi} . \tag{47}$$

Here $\chi^\dagger M \chi$ is a matrix product in the space of the subscript j; the normal-mode dependence is suppressed as it plays no role here:

$$\chi^\dagger \equiv (\chi_0^*, \chi_1^*, \ldots, \chi_{N+1}^*) , \tag{48}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\gamma & 1 & 0 & 0 & \cdots \\ 0 & -\gamma & 1 & 0 & \cdots \\ 0 & 0 & -\gamma & 1 & \cdots \\ \vdots & & & & \end{bmatrix} . \tag{49}$$

Define

$$\langle \chi_b t_b \,|\, \chi_a t_a \rangle_\xi = e^{|\chi_a|^2 + |\chi_b|^2} \int \prod_{1}^{N} \frac{d^2\chi_j}{\pi} e^{-\chi^\dagger M \chi + \chi^\dagger \zeta + \bar{\zeta}\chi} . \tag{50}$$

Set

$$\chi = \tilde{\chi} + M^{-1}\zeta \tag{51}$$

and identify

$$\bar{\zeta} = \zeta^\dagger (M^{-1})^\dagger M \tag{52}$$

and one gets

$$\langle \chi_b t_b \,|\, \chi_a t_a \rangle_\zeta = \langle \chi_b t_b \,|\, \chi_a t_a \rangle_0 e^{\bar{\zeta} M^{-1} \zeta} . \tag{53}$$

Now examine $\langle \chi_b t_b \,|\, [\hat{\varphi}_H(\mathbf{x},t_j)\hat{\varphi}_H(\mathbf{y},t_k)]_+ \,|\, \chi_a t_a \rangle_0$. From (34) and (53) one sees that the operation of $\varphi_H(\mathbf{x},t_j)$ is equivalent to

$$\hat{\varphi}_H(\mathbf{x},t) \rightarrow \sum_n \frac{\psi_n(\mathbf{x})}{\sqrt{2\omega_n}} \left[\frac{\partial}{\partial \zeta_{jn}} + \frac{\partial}{\partial \bar{\zeta}_{jn}}\right] \tag{54}$$

on (50) and hence on (53). Specifically one has

$$\frac{\langle \chi_b t_b \,|\, [\hat{\varphi}_H(\mathbf{x},t_j)\hat{\varphi}_H(\mathbf{y},t_k)]_+ \,|\, \chi_a t_a \rangle_0}{\langle \chi_b t_b \,|\, \chi_a t_a \rangle_0} = \sum_n (M^{-1}{}_{kj} + M^{-1}{}_{jk}) \frac{\psi_n(\mathbf{x})\psi_n(\mathbf{y})}{2\omega_n} . \tag{55}$$

Because

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \gamma & 1 & 0 & 0 & \cdots \\ \gamma^2 & \gamma & 1 & 0 & \cdots \\ \gamma^3 & \gamma^2 & \gamma & 1 & \cdots \\ \vdots & & & & \cdots \end{bmatrix} \tag{56}$$

for $j > k$ one has

$$M^{-1}{}_{jk} = \gamma^{j-k}, \quad M^{-1}{}_{kj} = 0 . \tag{57}$$

The index j is determined by

$$j = \frac{t_j - t_a}{\epsilon} = \frac{t_j - t_a}{T}(N+1) \tag{58}$$

so

$$\gamma^{j-k} = \gamma^{(t_j - t_k)(N+1)/T} \rightarrow e^{-i\omega(t_j - t_k)} \quad \text{as } \epsilon \rightarrow 0 . \tag{59}$$

Thus, we have

$$\begin{aligned} \langle \chi_b t_b \,|\, [\hat{\varphi}_H(\mathbf{x},t_j)\hat{\varphi}_H(\mathbf{y},t_k)]_+ \,|\, \chi_a t_a \rangle &= \sum_n \frac{\psi_n(\mathbf{x})\psi_n(\mathbf{y})}{2\omega} e^{-i\omega_n |t_j - t_k|} \\ &\equiv -i\Delta_c(\mathbf{x}-\mathbf{y}, t_j - t_k) . \end{aligned} \tag{60}$$

Note that the only place where terms of order ϵ were dropped was in taking the limit in (59).

In the calculation of transition amplitudes via perturbation methods, the matrix element of V that enters is $V[\chi_{j+1}^* + \chi_j]$ rather than $V[\chi_j^* + \chi_j]$ so the propagator that replaces (55) is proportional to $M^{-1}{}_{kj} + M^{-1}{}_{j+1,k+1}$ instead of $M^{-1}{}_{kj} + M^{-1}{}_{jk}$. Because of the form (56) of M^{-1} these are identical so *the Feynman rules are exact prior to taking the limit to* $\epsilon \rightarrow 0$. Thus ϵ^{-1} cannot act as a cutoff for the integrations over intermediate energies. At this point, we can also appreciate that corrections to (28) or order $\epsilon^2 N$ cannot contribute even in divergent amplitudes because they essentially involve $\epsilon^2 V^2[\chi_{j+1}^* + \chi_j]$ and will be connected in the amplitude to other vertices via the exact propagator discussed above.

VIII. FORMAL DERIVATION OF THE FPI

Once it is appreciated that ϵ^{-1} is not a cutoff in divergent amplitudes the limit $\epsilon \rightarrow 0$ can be taken without apprehension. Because we are dealing with the *real* scalar field the interaction in (29) is a function only of $\mathscr{R}\chi_{jn}$ in the limit and the intergration over $\mathscr{I}\chi_{jn}$ can be carried out explicitly. (For a complex scalar field one has two sets of complex parameters like χ_{jn} and two sets of intergrations can be carried out at this stage.) The complete expression for the kernel is

$$\begin{aligned} \langle \chi_b t_b \,|\, \chi_a t_a \rangle = \int \prod_{jn} \frac{d^2\chi_{jn}}{\pi} \exp\Bigg[& \sum_{j=0}^{N} \chi_{j+1n}^* \chi_{jn} - \sum_{1}^{N} |\chi_{jn}|^2 \\ & - i\epsilon \sum_{0}^{N} \omega_n \chi_{j+1n}^* \chi_{jn} \\ & - i\epsilon \sum_{0}^{N} \int d^3x \, V_j(\mathbf{x}) \Bigg] , \end{aligned} \tag{61}$$

where

$$V_j(\mathbf{x})\equiv V[\chi^*_{j+1}(\mathbf{x})+\chi_j(\mathbf{x})] \ . \tag{62}$$

First one finds

$$\sum_0^N \chi^*_{j+1n}\chi_{jn}-\sum_1^N |\chi_{jn}|^2 = \tfrac{1}{2}|\chi_{bn}|^2+\tfrac{1}{2}|\chi_{an}|^2-\tfrac{1}{2}\sum_0^N |\chi_{j+1n}-\chi_{jn}|^2 + \tfrac{1}{2}\sum_0^N(\chi^*_{j+1n}\chi_{jn}-\chi^*_{jn}\chi_{j+1n}) \ . \tag{63}$$

When time derivations are defined by

$$\chi_{j+1n}\equiv\chi_n(t_{j+1}) = \chi_n(t_j+\epsilon) = \chi_n(t_j)+\epsilon\dot{\chi}_n(t_j)+\cdots\equiv\chi_{jn}+\epsilon\dot{\chi}_{jn} \tag{64}$$

and third term in (63) is $O(\epsilon^2)$ and the last $O(\epsilon)$ while the first two terms relate to the norms of the initial and final states.

To lowest order in ϵ, V is a function of $\mathscr{R}\chi$ so we perform the $\mathscr{I}\chi$ integrations: Let $\chi_{jn}\equiv x_{jn}+iy_{jn}$ and define for each mode

$$I_0\equiv\exp\sum_{j=0}^N[\tfrac{1}{2}(\chi^*_{j+1}\chi_j-\chi^*_j\chi_{j+1})-i\epsilon\omega|\chi_j|^2] \ ,$$
$$I_1\equiv\int I_0 dy_1 \ , \tag{65}$$
$$I_2\equiv\int I_1 dy_2 \ ,$$

etc. One easily finds

$$I_N=\left[\frac{\pi}{i\epsilon\omega}\right]^{N/2}\exp\left[\frac{i}{4\epsilon\omega}(x_{j+1}-x_{j-1})^2-i\epsilon\omega x_j^{\,2}\right] \times\exp(-i\epsilon\omega|\chi_a|^2+iy_ax_1-iy_bx_N) \ . \tag{66}$$

The first exponent corresponds to a simple harmonic oscillator with coordinate $(2/\omega_n)^{1/2}x_{jn}$. The second is

$$i(y_ax_a-y_bx_b)+O(\epsilon) \tag{67}$$

and will modify the initial- and final-state wave functions as will be shown in Eq. (76).

By reversing the normal-mode decomposition procedure one finds directly with $\varphi(\mathbf{x},t_j)\equiv\chi^*_j(\mathbf{x})+\chi_j(\mathbf{x})$

$$\sum_{j,n}\frac{(x_{j+1n}-x_{j-1n})^2}{4\epsilon\omega_n}=\tfrac{1}{2}\int d^4x\,\dot{\varphi}^2(x) \ , \tag{68}$$

$$\sum_{j,n}\epsilon\omega_n x_{jn}{}^2=\tfrac{1}{2}\int d^4x[m^2\varphi^2(x)+(\nabla\varphi)^2] \ , \tag{69}$$

so that

$$S[\varphi]=\int d^4x[\tfrac{1}{2}\dot{\varphi}^2-\tfrac{1}{2}(\nabla\varphi)^2-\tfrac{1}{2}m^2\varphi^2-V(\varphi)] \tag{70}$$

is the exponent in (66):

$$\prod_n e^{-|\chi_{bn}|^2/2}e^{-|\chi_{an}|^2/2}e^{-i(y_{an}x_{an}-y_{bn}x_{bn})} \times\langle\chi_bt_b|\chi_at_a\rangle=\int\mathscr{D}\varphi^{iS[\varphi]} \ , \tag{71}$$

where

$$\mathscr{D}\varphi\equiv\prod_n\prod_{j=1}^N\frac{d\varphi_{jn}}{(4\pi i\epsilon\omega_n)^{1/2}}=\prod_n\prod_{j=1}^N\frac{d(\varphi_{jn}/\sqrt{2\omega_n})}{\sqrt{2\pi i\epsilon}} \tag{72}$$

and $\varphi_{jn}=2x_{jn}$.

Now consider the modification of, for example, the initial wave function by the factors $e^{+|\chi_a|^2/2}$ and $e^{ix_{an}y_{an}}$ and by the integration over $\mathscr{I}\chi_{an}\equiv y_{an}$. Specifically consider

$$I\equiv\int\frac{dy}{\pi}e^{-(x^2+y^2)/2}e^{ixy}\Psi(\chi) \ . \tag{73}$$

Let Ψ_a be the wave function of some general state such as

$$|a\rangle=\prod_n\frac{(a_n^\dagger)^{k_n}}{\sqrt{k_n!}}|0\rangle \ , \tag{74}$$

where $k_n=0,1,2,\ldots$ is the occupation number of the nth mode:

$$\Psi_a(\chi)\equiv\langle\chi|a\rangle=\prod_n\frac{(\chi_n^*)^{kn}}{\sqrt{k_n!}} \ , \tag{75}$$

then define $\Psi(\chi)$ as the wave function of a specific mode, i.e., $\Psi(\chi)\equiv(\chi^*)^k/\sqrt{k}!$. We need the formula

$$\int\frac{dy}{\pi}e^{ixy}\frac{(x-iy)^k}{\sqrt{k!}}e^{-(x^2+y^2)/2}=\frac{H_k(\sqrt{2}x)e^{-x^2}}{(2^{k-1}\pi k!)^{1/2}} \ , \tag{76}$$

where H_k is the Hermite polynomial. With the normalization

$$\int d\varphi U_k{}^2\left[\frac{\varphi}{\sqrt{2\omega}}\right]=1$$

we have

$$U_k\left[\frac{\varphi}{\sqrt{2\omega}}\right]=(\sqrt{2\pi}2^kk!)^{-1/2}H_k\left[\frac{\varphi}{\sqrt{2}}\right]e^{-\varphi^2/4} \tag{77}$$

so that

$$I=2^{3/4}\pi^{-1/4}U_k\left[\frac{\varphi}{\sqrt{2\omega}}\right] \ . \tag{78}$$

Thus we finally get

$$\langle bt_b|at_a\rangle=\int\prod_n\frac{d\varphi_{an}}{(2\pi)^{1/4}}\frac{d\varphi_{bn}}{(2\pi)^{1/4}}U_{k_n}\left[\frac{\varphi_{an}}{\sqrt{2\omega_n}}\right] \times U_{l_n}\left[\frac{\varphi_{bn}}{\sqrt{2\omega_n}}\right] \times\prod_{j=1}^N\frac{d\varphi_{jn}}{(4\pi i\epsilon\omega_n)^{1/2}}e^{iS[\varphi]} \ . \tag{79}$$

At this point we have derived the FPI in its usual form (79) with all the normalization constants explicitly shown. For practical purposes the transition amplitude is some-times more conveniently expressed in terms of the complex parameters χ_{jn} than the real parameters φ_{jn}. Compare, for example, the harmonic-oscillator wave functions in complex form $(\chi^*)^k$ with the real form $H_k(\varphi/\sqrt{2\omega})e^{-\varphi^2/4}$. Thus the kernel in the form of Eq. (61) may be more useful than the form (79).

IX. DISCUSSION

The purpose of this work has been to relate the FPI formulation of relativistic quantum field theory to elementary canonical field theory. No attempt is made at this point to treat either theory with more than the conventional minimum level of rigor although some discussion of divergent amplitudes seemed appropriate. Also, no discussion of renormalization was attempted since the problems seem to be the same in the two theories and are not relevant to the relationship between them.

The formulation of the FPI for fermions has not been addressed here. It is clear that a relationship of fermion transition amplitudes to some form of path-integral principle with commuting c-number fields must exist and its form for free fields is easily derived by following the procedure of this paper with the appropriate modification of the basis states $|\chi\rangle$ to take into account Fermi statistics. The form of the path-integral action principle for interacting fermions is not self-evident.

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