Higher-derivative operators and DeWitt's WKB ansatz

Hae Won Lee
Department of Physics, Chungbuk National University, Chungbuk 310, Korea

Pong Youl Pac
Department of Physics, Seoul National University, Seoul 151, Korea
(Received 6 September 1985)

We investigate the short-distance behaviors of the higher-derivative operators appearing in some field theories which attracted much interest recently, for instance, higher-derivative quantum gravity. This study is important to find the short-distance structures of the related propagators and the one-loop divergences. We develop an algorithm which can be used to find asymptotic expansions of the heat kernels for higher-derivative operators. Our method is applicable both to flat— and to curved—space-time cases.

I. INTRODUCTION

In any quantum field theory, knowing one-loop divergences is important. In particular, whether a given theory is asymptotically free or not is usually determined from the one-loop counterterms. The proper-time method introduced by Schwinger[1] and developed further by DeWitt[2] has proven its power in calculating one-loop divergences of various models. According to this method, the one-loop effective Lagrangian is determined by the short-distance behaviors of the heat kernel,[3] $\langle x \mid \exp(-\tau M) \mid x' \rangle$, where the operator M is found from the kinetic term of the Lagrangian in the presence of suitable background fields. The one-loop divergences can be found by computing the first few terms in a power-series expansion of the heat kernel with respect to τ, which is usually called an "asymptotic expansion." So far the algorithm used to calculate this series is restricted to the cases of second-order operators.

Recently, the interest in higher-derivative field theories has increased. The difficulties in renormalizing Einstein gravity led many physicists to consider quantum gravity with a quadratic Lagrangian.[4] After some effort, this modified gravity turned out to possess several good reasons to be regarded as a promising candidate for a gravity theory. This model is renormalizable[5] and exhibits asymptotic freedom.[6] Moreover it seems to reproduce Einstein gravity at the low-energy limit. The main obstacle to construct consistent quantum gravity from this model is the lack of a manifest unitarity, which is common to all higher-derivative field theories. Some authors argue that the Lee-Wick mechanism[7] can be used to overcome this problem.[8] Conformal supergravity,[9] the dipole gluon model,[10] and a few other theories[11] are the other examples.

In this paper we shall generalize the algorithm of finding asymptotic series to the cases where the operator M has a higher order. For the theories mentioned in the preceding paragraph, M is a fourth-order elliptic differential operator. Our method can be used to compute the asymptotic series for an operator with any differential order and works both in flat and in curved space-time. If the operator can be written as a product of second-order operators, the original DeWitt algorithm may be used to find the one-loop divergences, as was done in Ref. 12. Recently, some authors developed a reduction scheme[13] with which the functional determinant of a higher-order operator can be expressed in terms of the Green's functions of a second-order operator and, combining this with the original Schwinger-DeWitt technique, the one-loop divergences can be obtained. In contrast with these works, with our method, one can compute the asymptotic expansion directly which contains much more information than the one-loop divergences. As will be seen in Sec. III, in some cases our results disagree with the previous one[12] in the boundary terms which have been neglected in Ref. 13.

Although we can calculate the one-loop divergences of the theories mentioned just before, using our elegant method, we do not because our aim is just to generalize the algorithm for the asymptotic expansions. In the next section, we derive a generalized algorithm in flat space-time. In Sec. III the generalization to curved space-time is given. The final section contains the conclusions.

II. FLAT SPACE-TIME

First we describe the proper-time method briefly.[1,2] Throughout the paper we work in Euclidean space-time with metric signature $(++++)$. The Minkowski—space-time cases can be obtained by an analytic continuation to imaginary time. For a positive operator M, the logarithm of the determinant is given by the formal expression

$$\ln(\det M) = -\int_{\xi}^{\infty} \frac{d\tau}{\tau} \operatorname{Tr}[\exp(-\tau M)] , \tag{1}$$

where ξ denotes the proper-time cutoff. Typical examples of second-order operator M are

$$\begin{gathered} -\partial^2, \quad -D_\mu D^\mu, \quad -D^2\delta_{\mu\nu}+2iF_{\mu\nu} , \\ -D^2-\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu} , \end{gathered} \tag{2}$$

where D_μ denotes the covariant derivative associated with the Yang-Mills field $A_\mu = A_\mu^a T^a$ with group generators T^a, $F_{\mu\nu}$ its field tensor, and $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$. Next we list some fourth-order operators:

$$(-\partial^2)^2,\ (-D^2)^2 ,$$
$$(-D^2\delta_{\mu\nu}+2iF_{\mu\nu})^2+2X\cdot D\delta_{\mu\nu}-4X_\nu D_\mu , \tag{3}$$

where $X_\lambda = -i[D_\mu,F_{\mu\nu}]$ and the third example appears in the dipole gluon model.[10]

In general, one may consider arbitrary $2d$th-order operators where d is a positive integer. Suppose that M is any $2d$th-order operator such that, when the background fields vanish, it reduces to

$$M_0=(-\partial^2)^d . \tag{4}$$

Here possible mass parameters are regarded as one kind of background field. The operators listed in Eqs. (2) and (3) satisfy this condition.

For the trivial operator M_0, the heat kernel can be written in the form

$$\langle x\tau | x'\rangle_0 \equiv \langle x \mid \exp(-\tau M_0) \mid x'\rangle$$
$$= \tau^{-2/d}\Phi(\tfrac{1}{2}z^2) , \tag{5}$$

where $z_\mu=(x-x')_\mu/\tau^{1/2d}$ and Φ is a C^∞ function at $z=0$ with an integral expression

$$\Phi(\tfrac{1}{2}z^2)=\frac{1}{4\pi^3}\int_0^\infty dp\, p^3 \int_{-1}^{1} ds(1-s^2)^{1/2}e^{ipzs}e^{-p^{2d}} . \tag{6}$$

When $d=1$, $\Phi(x)=(1/16\pi^2)\exp(-2^{-1}x)$. For $d\geq 2$, Φ is not integrable and is available only in series expansion. For example, when $d=2$,

$$\Phi(\tfrac{1}{2}z^2)=\frac{1}{32\pi^2}\left[1-\frac{\sqrt{\pi}}{8}z^2+\cdots\right] . \tag{7}$$

We will see that the detailed form of Φ is not relevant for our purpose.

Next, let us consider the heat kernel for general M. Without loss of generality, M is assumed to be expressed in terms of the covariant derivative $D_\mu \equiv \partial_\mu - iA_\mu$ and the other background fields ϕ, i.e., $M=M(D_\mu,\phi)$. Then it follows that

$$\langle x\tau | x'\rangle \equiv \langle x \mid \exp(-\tau M) \mid x'\rangle$$
$$=[\exp(-\tau M^x)\exp(\tau M_0^x)]\langle x\tau | x'\rangle_0 . \tag{8}$$

Every time D_μ^x and ∂_μ^x pass through $\langle x\tau|x'\rangle_0$, they are replaced, respectively, by

$$D_\mu^x+\tau^{-1/2d}(\tilde{\partial}_\mu+z_\mu\Psi) \text{ and } \partial_\mu^x+\tau^{-1/2d}(\tilde{\partial}_\mu+z_\mu\Psi) \tag{9}$$

with $\tilde{\partial}_\mu=\partial/\partial z^\mu$ and $\Psi=\Phi^{-1}\Phi'$. Here we have used Eq. (5) for $\langle x\tau|x'\rangle_0$. Ψ is also C^∞ at $z=0$. Introducing the operator

$$\Box_\mu=\tilde{\partial}_\mu+z_\mu\Psi(\tfrac{1}{2}z^2) , \tag{10}$$

we can rewrite the substitution rule (9) as

$$\tau^{1/2d}D_\mu^x \to \bar{D}_\mu+\Box_\mu \text{ and } \tau^{1/2d}\partial_\mu \to \bar{\partial}_\mu+\Box_\mu , \tag{11}$$

where $\bar{D}_\mu=\tau^{1/2d}D_\mu$ and $\bar{\partial}_\mu=\tau^{1/2d}\partial_\mu$.

The heat kernels satisfy the Schrödinger equation

$$M^x\langle x\tau|x'\rangle=-\frac{\partial}{\partial\tau}\langle x\tau|x'\rangle ,$$
$$M_0^x\langle x\tau|x'\rangle_0=-\frac{\partial}{\partial\tau}\langle x\tau|x'\rangle_0 . \tag{12}$$

Applying rule (11) to the second equation of Eq. (12), we obtain the following differential equation for Φ:

$$(1+\tfrac{1}{4}z\cdot\Box)1=\tfrac{1}{2}d(-\Box^2)^2 1 . \tag{13}$$

Denoting

$$\langle x\tau|x'\rangle=\langle x\tau|x'\rangle_0 H(x,x',\tau) , \tag{14}$$

we can prove, using rule (11), that

$$H(x,x',\tau)=\exp[-\bar{M}(\bar{D}+\Box)]\exp[-(\bar{\partial}+\Box)^2]^d 1$$
$$=\exp[-\bar{M}(\bar{D}+\Box)]\exp(-\Box^2)^d 1$$
$$=\exp(-\bar{m})1 , \tag{15}$$

with $\bar{m}=\bar{M}(\bar{D}+\Box)-(-\Box^2)^d$ and $\bar{M}=\tau M$. Here we have used

$$[\bar{D}_\mu,\Box_\nu]=0 \text{ and } [\Box_\mu,\Box_\nu]=0 . \tag{16}$$

Note also that

$$[\Box_\mu,z_\nu]=\delta_{\mu\nu} . \tag{17}$$

Now we return to Eq. (1). For the operators given in Eqs. (2) and (3), Eq. (1) gives the proper-time representation of the one-loop effective actions provided the overall factors are adjusted. From Eqs. (7) and (14), we can write the one-loop effective action as

$$-\int dx^4\int_\xi^\infty \frac{d\tau}{\tau}\frac{\Gamma(1+2/d)}{32\pi^2}\tau^{-2/d}\mathrm{tr}H(x,x,\tau) , \tag{18}$$

where tr means the trace over the internal indices. It is obvious from Eq. (18) that for the ultraviolet divergences we need only the terms in $H(x,x,\tau)$ with order up to $O(\tau^{2/d})$.

In calculating H using Eq. (15), H is expanded by $\tau^{1/2d}$ and z_μ rather than τ and $(x-x')_\mu$. For the $d=1$ case where $\Psi=-\frac{1}{2}$, H can be also expanded by τ and $(x-x')_\mu$, however, which are not appropriate as expansion parameters when $d\geq 2$. Now we illustrate our method when $M=-D^2$ and $d=1$. In this case,

$$\bar{m}=\bar{M}(\bar{D}+\Box)+\Box^2=-\bar{D}^2-2\bar{D}\cdot\Box \tag{19}$$

and

$$H=\sum_{n=0}\frac{1}{n!}(\bar{D}^2+2\bar{D}\cdot\Box)^n 1 . \tag{20}$$

Since $[\bar{D}_\mu,\Box_\nu]=0$, we may evaluate each term by computing strings of $\Box$'s and D's separately. Since

$$\bar{D}^2=\tau D^2,\ \bar{D}_\mu=\sqrt{\tau}D_\mu , \tag{21}$$

and

$$\Box_\mu=\frac{1}{\sqrt{\tau}}\frac{\partial}{\partial y^\mu}+\sqrt{\tau}y_\mu ,$$

with $y^\mu=(x-x')^\mu$, we conclude that

$$\overline{m}=\tau(D^2+yD)+D_\mu\frac{\partial}{\partial y^\mu} \tag{22}$$

and therefore H can be expanded by τ and y^μ as well.

Keeping the terms in the series (20) with order up to $O(\tau^2)$, we have

$$\begin{aligned}H(x,x,\tau)&=1+\overline{D}^2+\tfrac{1}{2}[(\overline{D}^2)^2+4(\Box\cdot\overline{D})^2]\\&\quad+\frac{4}{3!}[\{(\Box\cdot\overline{D})^2,\overline{D}^2\}+\Box\cdot\overline{D}\overline{D}^2\Box\cdot\overline{D}]\\&\quad+\frac{16}{4!}(\Box\cdot\overline{D})^4+O(\tau^{5/2})\\&=1-\frac{\tau^2}{12}F_{\mu\nu}F_{\mu\nu}+O(\tau^3)\,,\end{aligned} \tag{23}$$

where we have used the following identities, for $z=0$:

$$\begin{aligned}&\Box_\mu=0,\quad \Box_\mu\Box_\nu=-\tfrac{1}{2}\delta_{\mu\nu}\,,\\&\Box_\mu\Box_\nu\Box_\lambda\Box_\tau=\tfrac{1}{4}(\delta_{\mu\nu}\delta_{\lambda\tau}+\cdots)\,,\end{aligned} \tag{24}$$

and so on, which can be proved from Eq. (13). Using Eqs. (20) and (22), we can find $H(x,x',\tau=0)$

$$\begin{aligned}H(x,x',\tau=0)&=\sum_{n=0}\frac{(-1)^n}{n!}y_{\mu_1}\cdots y_{\mu_n}D^{\mu_1}\cdots D^{\mu_n}1\\&=P\exp\left[i\int_x^{x'}ds_\mu A^\mu(s)\right]\,.\end{aligned} \tag{25}$$

For general d, when τ, z_μ, and x_μ are regarded as independent variables, $\overline{m}$ is $O(\tau^{1/2d})$. Therefore Eq. (15) may be used to find H in power series of $\tau^{1/2d}$ and z_μ. Now we illustrate our method for the $d=2$ case. We consider here the simplest one $M=(D^2)^2$. Let us first find $\overline{m}=(\overline{D}+\Box)^4-(\Box^2)^2$,

$$\begin{aligned}\overline{m}&=(\overline{D}^2)^2+2\{\Box\cdot\overline{D},\overline{D}^2\}+4(\Box\cdot\overline{D})^2\\&\quad+2\Box^2\overline{D}^2+4\Box^2\Box\cdot\overline{D}\,.\end{aligned} \tag{26}$$

Inserting this into Eq. (15), we find

$$H(x,z=0,\tau)=I-\frac{\tau}{6}F_{\mu\nu}F_{\mu\nu}+O(\tau^{3/2})\,, \tag{27}$$

where we have used, for $d=2$ and $z=0$,

$$\Box^4=1,\quad \Box^8=2,\quad \Box^{12}=3!\,, \tag{28}$$

and so on. In conclusion, we have shown that, for any higher-order operator, our method gives a simple rule to calculate the asymptotic series of the heat kernel.

III. CURVED SPACE-TIME

The algorithm developed in the preceding section is easily generalized to curved space-time. Before doing this, let us be precise about the notation. We use a metric tensor $g_{\mu\nu}(x)$ with signature $(++++)$. A curvature tensor is defined by

$$R_{\mu\nu\sigma}{}^\tau=\Gamma_{\nu\sigma}{}^\tau{}_{,\mu}-\Gamma_{\mu\sigma}{}^\tau{}_{,\nu}+\Gamma_{\nu\sigma}{}^\rho\Gamma_{\mu\rho}{}^\tau-\Gamma_{\mu\sigma}{}^\rho\Gamma_{\nu\rho}{}^\tau\,, \tag{29}$$

where $\Gamma_{\mu\nu}{}^\lambda$ denotes the torsion-free affine connection preserving the metric $g_{\mu\nu}$ (Ref. 14). We use a comma for ordinary derivative and a period for covariant derivative. Ricci tensor and scalar curvature are defined by

$$R_{\mu\nu}=R_{\sigma\mu\nu}{}^\sigma\quad\text{and}\quad R=R_\mu{}^\mu\,. \tag{30}$$

In curved space-time, instead of $D_\mu=\partial_\mu-iA_\mu$, we must use

$$\nabla_\mu=\partial_\mu-iA_\mu+\Gamma_{\mu(\lambda)}{}^{(\tau)}\,, \tag{31}$$

where the indices within parentheses show the matrix nature of Γ_μ. Then operator M is expressed by ∇_μ, i.e., $M=M(\nabla_\mu,\phi)$. For examples, we may consider

$$\begin{aligned}&-\nabla_\mu\nabla^\mu,\quad -\nabla^2+\tfrac{1}{4}R,\quad (\nabla^2)^2\,,\\&\nabla^4+C_{\mu\nu}\nabla^\mu\nabla^\nu+D_\mu\nabla^\mu+E\,,\end{aligned} \tag{32}$$

where the second term operates on spinors and R denotes the curvature matrix with isospin and spinor indices. The last example in Eq. (32) appears in higher-derivative quantum gravity.[4]

The heat kernel $\langle x\tau\,|\,x'\rangle=\langle x\,|\,\exp(-\tau M)\,|\,x'\rangle$ satisfies the Schrödinger equation (12) and the boundary condition

$$\langle x\tau\,|\,x'\rangle\underset{\tau\to0+}{=}\frac{1}{\sqrt{g}}\delta(x-x')\,, \tag{33}$$

where $g=\det(g_{\mu\nu})$.

Analogous to the flat space-time cases, let us assume that $\langle x\tau\,|\,x'\rangle$ can be written in the form

$$\langle x\tau\,|\,x'\rangle=\tau^{-2/d}\Phi\left[\frac{\sigma}{\tau^{1/d}}\right]H(x,x',\tau)\,, \tag{34}$$

for some function H. Here σ is a biscalar $\sigma(x,x')$, half the square of geodesic length from x to x', which is a generalization of $(x-x')^2/2$ in flat space-time. Similarly, we introduce

$$\Box_\mu=g_{\mu\nu}(x)\tilde{\partial}^\nu+z_\mu\Psi\,, \tag{35}$$

where $z_\mu\equiv\tau^{-1/2d}\sigma_{.\mu}$ and $\tilde{\partial}^\nu=\partial/\partial z_\nu$ denotes the partial derivative by z_ν at fixed x. To be precise, we write

$$\tilde{\partial}_\mu=\left.\frac{\partial}{\partial z^\mu}\right|_x\,. \tag{36}$$

Note that biscalar σ is also defined by[2]

$$2\sigma=\sigma_{.\mu}\sigma^{.\mu}\quad\text{and}\quad\sigma(x,x')=\sigma(x',x)\,, \tag{37}$$

together with the boundary conditions

$$\sigma=\sigma_{.\mu}=0\quad\text{and}\quad\sigma_{.\mu\nu}=g_{\mu\nu}\quad\text{for } x=x'\,. \tag{38}$$

Hereafter we shall use the form $\sigma_\mu=\sigma_{.\mu}$ and $\sigma_{\mu\nu}=\sigma_{.\mu\nu}$.

The substitution rule (9) can be generalized to curved space-time by a slight modification. Operating ∇_μ on Eq. (34), we find that

$$\begin{aligned}\tau^{1/2d}\nabla_\mu\Phi&=\Phi(\tau^{1/2d}\nabla_\mu+z_\mu\Psi)\\&=\Phi(\overline{\nabla}_\mu+\sigma_{\mu\nu}\Box^\nu)\,,\end{aligned} \tag{39}$$

where $\overline{\nabla}_\mu=\tau^{1/2d}\nabla_\mu-\sigma_{\mu\nu}\tilde{\partial}^\nu$ and we have used

$$\sigma_{\mu\nu}z^\nu=z_\mu\,. \tag{40}$$

Then similar to Eqs. (16) and (17), we can find

$$[\overline{\nabla}_\mu,\Box_\nu]=0=[\overline{\nabla}_\mu,z_\nu] \tag{41}$$

and

$$[\Box_\mu,\Box_\nu]=0,\ [\Box_\mu,z_\nu]=g_{\mu\nu}(x)\ . \tag{42}$$

Derivation of Eq. (41) is not simple and left to readers as an exercise.

Analogous to Eq. (15) H defined in Eq. (34) is given by

$$H=\exp[-\overline{M}(\overline{\nabla}_\mu+\sigma_{\mu\nu}\Box^\nu)]\exp(-\Box^2)^d 1\ . \tag{43}$$

This can be proved by showing that H in Eq. (43) satisfies the Schrödinger equation (12) and the boundary condition (33). Here we omit the detailed proof. With Eq. (43), H is expanded by $\tau^{1/2d}$ and z_μ. Of course, this implies that H is C^∞ at $\tau=z_\mu=0$ with respect to the variables $\tau^{1/2d}$ and z_μ. When $d=1$, H can be also expanded by τ and σ_μ. Hereafter, we regard H as a function of τ, z_μ, and x^ν. For this, let us define

$$\hat{H}(x,z,\tau)=H(x,x',\tau)\ . \tag{44}$$

Similarly, we can view $\sigma_{\mu\nu}$ as a function of x, z, and τ. For later convenience, let us define

$$\begin{aligned}&\hat{\sigma}_\mu\equiv\sigma_{\mu\nu}\Box^\nu\ ,\\ &\hat{\sigma}_{\mu.\lambda}=[\overline{\nabla}_\lambda,\hat{\sigma}_\mu]\ ,\\ &\hat{\sigma}_{\mu.\bar{\lambda}}=[\Box_\lambda,\hat{\sigma}_\mu]\ ,\end{aligned} \tag{45}$$

and so on. Then it is clear that $\hat{\sigma}_{\mu.\lambda_1\lambda_2\cdots\lambda_k}=O(\tau^{k/2d})$ at least.

Now we prove that, in calculating $\hat{H}$ to a given power of τ, we need only finite terms in Eq. (43). Noting that $\Box_\mu$ is not a differential operator when operated on a function of x, we permute $\overline{\nabla}_\mu$'s and $\Box_\nu$'s in each term of Eq. (43) so that $\Box_\nu$'s are on the left-hand side of $\overline{\nabla}_\mu$'s. With this form, we can evaluate the strings of $\overline{\nabla}_\mu$'s and those of $\Box_\mu$'s separately, since strings of $\overline{\nabla}_\mu$'s evaluated on 1 depend only on x. In this sense, we may write

$$\overline{\nabla}_\mu=O(\tau^{1/2d})\ \text{and}\ \Box_\nu=O(\tau^0)\ . \tag{46}$$

Let $\overline{m}=\overline{M}-\overline{M}_0$ where $\overline{M}=\overline{M}(\overline{\nabla}_\mu+\hat{\sigma}_\mu)$ and $\overline{M}_0=(-\Box^2)^d$. Since $\overline{M}(\Box_\mu)=\overline{M}_0$ for vanishing background fields and $\overline{\nabla}_\mu+\hat{\sigma}_\mu=\Box_\mu+O(\tau^{1/2d})$, it is obvious that $\overline{m}=O(\tau^{1/2d})$. We calculate the series expansion in Eq. (43), using the formula

$$\exp(-\overline{M})\exp(\overline{M}_0)=\overline{T}\exp\left[\int_0^1(-\overline{m}_s)ds\right]\ , \tag{47}$$

where $\overline{m}_s=\exp(-\overline{M}_0 s)\overline{m}\exp(\overline{M}_0 s)$ and $\overline{T}$ denotes the anti-"time"-ordering operator defined by

$$\overline{T}A_{s_1}B_{s_2}=\theta(s_2-s_1)A_{s_1}B_{s_2}+\theta(s_2-s_1)B_{s_2}A_{s_1}\ . \tag{48}$$

Then Eq. (47) and $\overline{m}_s$ have the series expansions

$$\begin{aligned}\overline{T}\exp\left[\int_0^1(-\overline{m}_s)ds\right]=1&+\int_0^1 ds(-\overline{m}_s)\\ &+\int_0^1 ds_1\int_0^{s_1}ds_2\overline{m}_{s_2}\overline{m}_{s_1}\\ &+\cdots\end{aligned} \tag{49}$$

and

$$\begin{aligned}\overline{m}_s=\overline{m}&+(-s)[\overline{M}_0,\overline{m}]\\ &+\frac{1}{2!}(-s)^2[\overline{M}_0,[\overline{M}_0,\overline{m}]]+\cdots\ .\end{aligned} \tag{50}$$

Equations (43), (47), (49), and (50) form a main frame of our algorithm for the asymptotic expansion. We illustrate our method by calculating the coincidence limit of H, i.e., $H(x,x,\tau)=\hat{H}(x,z=0,\tau)$ up to $O(\tau^{2/d})$ for operators $-\nabla^2$ and $(-\nabla^2)^2$. To do this, let us first find $\overline{m}$. After a little bit of calculations, we have

$$\begin{aligned}\overline{m}&=-(\overline{\nabla}+\hat{\sigma})^2+\Box^2\\ &=-(\overline{\nabla}^2+2\hat{\sigma}_\mu\overline{\nabla}^\mu+\hat{\sigma}_{\mu.}{}^{\mu}+\hat{\sigma}^2-\Box^2)\ ,\end{aligned} \tag{51}$$

and

$$\begin{aligned}\overline{m}=(\overline{\nabla}+\hat{\sigma})^4-\Box^4=\overline{\nabla}^4&+2\Box_\mu\{\overline{\nabla}^\mu,\overline{\nabla}^2\}+2\hat{\sigma}^2\overline{\nabla}^2+4\hat{\sigma}^\mu\hat{\sigma}^\nu\overline{\nabla}_\mu\overline{\nabla}_\nu+2(\{\Box_\nu,\hat{\Delta}\}+\hat{\sigma}^2{}_{.\nu}+2\Box_\mu\hat{\sigma}_\nu{}^{.\mu}+\{\hat{\sigma}_\nu,\hat{\sigma}^2\})\overline{\nabla}^\nu\\ &+2\Box_\nu\hat{\Delta}^{.\nu}+\hat{\sigma}^{2.\alpha}{}_\alpha+\{\Box^2,\hat{\Delta}\}+2\Box_\mu\hat{\sigma}^{2.\mu}+\hat{\sigma}^4-\Box^4+O(\tau^{5/4})\ ,\end{aligned} \tag{52}$$

where $\hat{\Delta}\equiv\hat{\sigma}_\mu{}^{.\mu}$. Inserting these expansions into Eqs. (43), (47), (49), and (50), we can evaluate $\hat{H}$.

In order to calculate various quantities involving $\hat{\sigma}_\mu$'s, we need more techniques. We want to expand $\hat{\sigma}_\mu$ by z_μ's. This can be done by using the identity

$$[\hat{\sigma}_\mu,\hat{\sigma}_\nu]-\hat{\sigma}_{[\mu.\nu]}=\tau^{1/d}R_{\mu\nu\lambda\tau}z^\lambda\Box^\tau\ , \tag{53}$$

with $\hat{\sigma}_{[\mu.\nu]}=\hat{\sigma}_{\mu.\nu}-\hat{\sigma}_{\nu.\mu}$. From $\sigma_{.\mu}=\sigma^{.\nu}\sigma_{.\nu\mu}$, we also have

$$z\cdot\Box=z\cdot\hat{\sigma}\ . \tag{54}$$

Differentiating Eq. (54) with $\Box_\mu$s, we easily find that, when $z=0$,

$$\sum_{k=1}^{n}\hat{\sigma}_{\mu_k.\bar{\mu}_1\cdots\bar{\mu}_{k-1}\bar{\mu}_{k+1}\cdots\bar{\mu}_n}=0\ . \tag{55}$$

From Eq. (55) we have, for $z=0$,

$$\hat{\sigma}_{\mu_1.\bar{\mu}_2\cdots\bar{\mu}_n}=\frac{1}{n}\sum_{k=2}^{n}\hat{\sigma}_{[\mu_1.\bar{\mu}_k]\bar{\mu}_2\cdots\bar{\mu}_{k-1}\bar{\mu}_{k+1}\cdots\bar{\mu}_n}\ , \tag{56}$$

with

$$\hat{\sigma}_{[\mu.\bar{\nu}]}=\hat{\sigma}_{\mu.\bar{\nu}}-\hat{\sigma}_{\nu.\bar{\mu}}\ .$$

Denoting $\tilde{\sigma}_\mu=\hat{\sigma}_\mu-\Box_\mu$, Eq. (53) can be rewritten in the form

$$\hat{\sigma}_{[\mu.\bar{\nu}]}=[\tilde{\sigma}_\mu,\tilde{\sigma}_\nu]-\tilde{\sigma}_{[\mu.\nu]}-\tau^{1/d}R_{\mu\nu\lambda\tau}z^\lambda\Box^\tau\ . \tag{57}$$

From Eqs. (56) and (57), we can compute the series expansion of $\tilde{\sigma}_{\mu}$, recursively. First it is obvious that

$$\tilde{\sigma}_{\mu}=0 \text{ and } \tilde{\sigma}_{\mu.\bar{\nu}}=0 , \tag{58}$$

in the coincidence limit. When $n=3$, 4, and 5, we have

$$\begin{aligned}
\tilde{\sigma}_{\mu.\bar{\nu}\bar{\lambda}} &= \tfrac{1}{3}\tau^{1/d}(R_{\mu\nu\tau\lambda}+R_{\mu\lambda\tau\nu})\Box^{\tau} ,\\
\tilde{\sigma}_{\mu.\bar{\nu}\bar{\lambda}\bar{\tau}} &= \frac{\tau^{3/2d}}{12}(R_{\mu\lambda\tau\rho.\nu}+ \text{permuted terms})\Box^{\rho} ,\\
\tilde{\sigma}_{\mu.\bar{\nu}\bar{\lambda}\bar{\tau}\bar{\rho}} &= \frac{\tau^{2/d}}{5}(-\tfrac{1}{12}R_{\mu\tau\rho\alpha.\lambda\nu}+\tfrac{1}{9}R_{\lambda\mu\tau\sigma}R^{\sigma}{}_{\nu\rho\alpha}\\
&\qquad + \text{permuted terms})\Box^{\alpha} ,
\end{aligned} \tag{59}$$

for $z=0$. These results may be used to find

$$\begin{aligned}
\sigma_{\mu\nu} &= g_{\mu\nu}+\frac{\tau^{1/d}}{3}z^{\alpha}z^{\beta}R_{\mu\alpha\nu\beta}+\frac{\tau^{3/2d}}{12}z^{\alpha}z^{\beta}z^{\gamma}R_{\mu\alpha\beta\nu.\gamma}\\
&\quad +\frac{\tau^{2/d}}{15}z^{\alpha}z^{\beta}z^{\gamma}z^{\delta}(\tfrac{1}{4}R_{\mu\alpha\beta\nu.\gamma\delta}-\tfrac{1}{3}R_{\mu\alpha\beta\sigma}R^{\sigma}{}_{\gamma\delta\nu})\\
&\quad +\cdots .
\end{aligned} \tag{60}$$

Note also that, when $z=0$,

$$\begin{aligned}
&\sigma_{\alpha}{}^{\alpha}{}_{.\bar{\beta}}{}^{\bar{\beta}}{}_{\bar{\gamma}}{}^{\gamma} = -2\sigma_{\alpha\beta}{}^{.\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}}{}^{\gamma} = \tfrac{1}{3}R_{.\alpha}{}^{\alpha} ,\\
&\sigma_{\alpha\beta.\bar{\gamma}}{}^{\bar{\gamma}\bar{\alpha}\beta}=0 ,\\
&\sigma_{\mu}{}^{\mu}{}_{.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{\beta}} = -4\sigma_{\mu\nu.\bar{\alpha}}{}^{\bar{\alpha}\bar{\mu}\bar{\nu}}\\
&\qquad = -\tfrac{4}{15}(R_{.\alpha}{}^{\alpha}+\tfrac{2}{3}R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}) .
\end{aligned} \tag{61}$$

In Eq. (49), we frequently meet $(\hat{\sigma}_{\mu})_s$. For completeness, let us now write $(\hat{\sigma}_{\mu})_s$ in the coincidence limit for $d=1,2$. Using Eq. (50), we can find, for $d=1$

$$\begin{aligned}
(\hat{\sigma}_{\mu})_s &= \Box_{\mu}+R_{\mu\alpha}\Box^{\alpha}(-\tfrac{2}{3}s)+\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}}\Box^{\beta}2s^{2}\\
&\quad +\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{\beta}}\tfrac{1}{2}s^{2}+\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}\Box^{\beta}\Box^{\gamma}2s^{2}+\cdots ,
\end{aligned}$$

whereas for $d=2$

$$\begin{aligned}
(\hat{\sigma}_{\mu})_s &= \Box_{\mu}+R_{\mu\alpha}\Box^{\alpha}(\tfrac{4}{3}s)+\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}}\Box^{\beta}(-4s+8s^{2}\Box^{4})\\
&\quad +\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{\beta}}(-s+2s^{2}\Box^{4})\\
&\quad +\hat{\sigma}_{\mu.\bar{\alpha}}{}^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}}\Box^{\beta}\Box^{\gamma}\Box^{2}(24s^{2}-16s^{3}\Box^{4})+\cdots .
\end{aligned} \tag{62}$$

Now we can evaluate $\hat{H}(x,z=0,\tau)$ up to $O(\tau^{2/d})$ by direct calculations using Eqs. (43), (49), (50), (61), and (62). These calculations are relatively simple when $M=-\nabla^2$, whereas it requires some labor when $M=(\nabla^2)^2$. Here we omit the detailed calculations and report only the final results. For $M=-\nabla^2$, our result is in complete agreement with that obtained by DeWitt[2]

$$\hat{H}(z=0)=1+\tau a_1+\tau^2 a_2+\cdots ,$$

with

$$\begin{aligned}
a_1 &= \tfrac{1}{6}R ,\\
a_2 &= \tfrac{1}{30}R_{.\alpha}{}^{\alpha}+\tfrac{1}{72}R^{2}-\tfrac{1}{180}R_{\mu\nu}R^{\mu\nu}\\
&\quad -\tfrac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}+\tfrac{1}{12}F_{\mu\nu}F^{\mu\nu} .
\end{aligned} \tag{63}$$

For $M=(-\nabla^2)^2$, our result is

$$\hat{H}(z=0)=1+\tau^{1/2}b_1+\tau b_2+\cdots ,$$

with

$$b_1=-\tfrac{1}{3}\Box^2 R \text{ and } b_2=2a_2 . \tag{64}$$

Note that $\Box^2=-\sqrt{\pi}$ when $z=0$. The relation $b_2=2a_2$ was expected to hold, even before the calculations because $\ln[\det(-\nabla^2)^2]=2\ln[\det(-\nabla^2)]$.

Next, let us consider a more general $d=2$ case

$$M=\nabla^4+C_{\mu\nu}\nabla^{\mu}\nabla^{\nu}+D_{\mu}\nabla^{\mu}+E , \tag{65}$$

where $C_{\mu\nu}$, D_{μ}, and E denote arbitrary matrix fields with additional space-time and internal space indices. Then we should add to $\overline{m}$ given in Eq. (52) the term

$$\begin{aligned}
\Delta\overline{m} &= \sqrt{\tau}(C_{\mu\nu}\overline{\nabla}^{\mu}\overline{\nabla}^{\nu}+C_{(\mu\nu)}\Box^{\mu}\overline{\nabla}^{\nu}+C_{\mu\nu}\hat{\sigma}^{\mu}\hat{\sigma}^{\nu})\\
&\quad +\tau^{3/4}(D_{\mu}\overline{\nabla}^{\mu}+D_{\mu}\Box^{\mu})+\tau E ,
\end{aligned}$$

with

$$C_{(\mu\nu)}=C_{\mu\nu}+C_{\nu\mu} . \tag{66}$$

Now the calculations are not difficult because $\Delta\overline{m}=O(\sqrt{\tau})$. Following the similar steps to the previous cases, we have

$$\Delta b_1=-\tfrac{1}{4}C_{\alpha}{}^{\alpha}\Box^2$$

and

$$\begin{aligned}
\Delta b_2 &= \tfrac{1}{2}iC_{\alpha\beta}F^{\alpha\beta}+\tfrac{1}{2}D_{\alpha}{}^{.\alpha}+\tfrac{1}{9}C_{\alpha}{}^{\alpha}{}_{.\beta}{}^{\beta}\\
&\quad -\tfrac{5}{36}C_{(\mu\nu)}{}^{.\mu\nu}-E+\tfrac{1}{48}(C_{\alpha}{}^{\alpha})^2+\tfrac{1}{48}C_{\alpha\beta}C^{(\alpha\beta)}\\
&\quad +\tfrac{1}{12}(RC_{\alpha}{}^{\alpha}-R_{\alpha\beta}C^{(\alpha\beta)}) .
\end{aligned} \tag{67}$$

This result agrees with that given in Ref. 12 except for the coefficients of $C_{\alpha}{}^{\alpha}{}_{.\beta}{}^{\beta}$ and $C_{(\alpha\beta)}{}^{.\alpha\beta}$. These terms have no physical relevance when the boundary effects are ignored. Furthermore we can include the term $B_{\alpha\beta\gamma}\nabla^{\alpha}\nabla^{\beta}\nabla^{\gamma}$ in M. Calculation in this case is straightforward, but not pursued here.

IV. CONCLUSIONS

Schwinger's proper-time method combined with DeWitt's algorithm for asymptotic expansion of heat kernels is ideal for investigating the short-distance behavior of various Green's functions, especially to find the one-loop divergences of various field theories. We generalized DeWitt's algorithm, so far limited to second-order differential operators, to higher-derivative operators of any order. Straightforward generalization of DeWitt's algorithm is not possible. Indeed, the heat kernel $\langle x \mid \exp(-\tau M) \mid x' \rangle$ cannot be expanded by $(x-x')_{\mu}$ and τ when M is a higher-derivative operator. Instead, it should be expanded by $(x-x')_{\mu}\tau^{-1/2d}$ and $\tau^{1/2d}$, which has been shown in Sec. II. We invented a method which is elegant and produces the asymptotic series automatically. Our algorithm works both in flat and in curved space-time. Therefore our algorithm will be useful to higher-derivative quantum gravity. There we already discovered slight discrepancies between our result and the one obtained using a different method in Sec. III. Although we restricted ourselves to torsion-free affine connections, our method is easily generalized to general affine connections and then will be applicable to conformal supergravity.

ACKNOWLEDGMENTS

This work was supported in part by the Ministry of Education, Republic of Korea, and by the Korea Science and Engineering Foundation.

---

[1]J. Schwinger, Phys. Rev. **82**, 664 (1951).

[2]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); Phys. Rep. **19C**, 295 (1975).

[3]For convenience, we work in Euclidean space-time. In Minkowski space-time we must consider $\langle x \mid \exp(itM) \mid x' \rangle$.

[4]R. Utiyama and B. S. DeWitt, J. Math. Phys. **3**, 608 (1962); S. Deser, in *Proceedings of the Conference on Gauge Theories and Modern Field Theories,* edited by R. Arnowitt and P. Nath (MIT Press, Cambridge, 1975); S. Weinberg, in *Proceedings of the XVII International Conference on High Energy Physics,* edited by J. R. Smith (Chilton, Didcot, 1974).

[5]K. S. Stelle, Phys. Rev. D **16**, 953 (1977).

[6]E. Tomboulis, Phys. Lett. **97B**, 77 (1980); J. Julve and M. Tonin, Nuovo Cimento **B46**, 137 (1978); E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **104B**, 377 (1981).

[7]E. Tomboulis, Phys. Lett. **70B**, 361 (1977).

[8]T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969); **B10**, 1 (1969); Phys. Rev. D **2**, 1033 (1970).

[9]P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981), and references therein.

[10]S. Blaha, Phys. Rev. D **15**, 4268 (1977); H. W. Lee and P. Y. Pac, *ibid.* **29**, 2948 (1984).

[11]J. Kiskis, Phys. Rev. D **11**, 2178 (1975).

[12]E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **104B**, 377 (1981).

[13]A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **119**, 1 (1985).

[14]Most of the results in Sec. III hold for general metric-preserving affine connections as well. Generalization to these cases is simple, but not pursued here.