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Internal symmetries of non-Abelian gauge field configurations

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In a topologically nontrivial gauge theory not all gauge transformations are symmetries (as defined by Forgács and Manton and by Schwarz) of a given field configuration: first, there may be an obstruction to implement gauge transformations on the fields; next, even those transformations which can be implemented may fail to be symmetries. For a test particle in such a background field, those gauge transformations which are symmetries generate ordinarily conserved Noether currents, one of which is the usual electric current. These results shed a new light on the problem of "global color" in monopole theory and explain why the electric charge of a nucleon is not in general conserved in the non-Abelian Aharonov-Bohm experiment of Wu and Yang.

I. INTRODUCTION

In non-Abelian gauge theories, the word "symmetry" has two meanings: on the one hand, it means a transformation which changes the Lagrangian to an equivalent one. This is what we call a symmetry of the theory. On the other hand, this same word is used to refer to a transformation which leaves a specific field configuration invariant. Typical examples are provided by space-time symmetries.[1-7] Here we extend this analysis to internal transformations. More exactly, we are concerned with the following question: Which gauge transformation are symmetries for a given non-Abelian field configuration? This problem is closely related to that of "global color" in monopole theory:[8-13] the first step in defining a symmetry of a given field configuration is, in fact, the implementation of this transformation. However, a topological obstruction may prevent us from doing so. Next, an implementable transformation may fail to be a symmetry. Those which are symmetries form a (generally proper) subgroup H of the gauge group, which we shall call an internal-symmetry group, since its action on space-time is trivial.

The physical importance of internal symmetries is understood by noting that, for a test particle moving in a non-Abelian background field, the internal-symmetry group of a given configuration becomes a symmetry for the particle Lagrangian. So, by the Noether theorem, a conserved current is associated to each symmetry generator. In particular, we can get conserved electric charge. So internal symmetries generate electric charge in the same way rotations generate angular momentum.

The main application of our theory is to the "color problem"[8,9] in monopole theory. We show first that a subgroup K of G is implementable if and only if the standard transition function[14-17] $h(t)=\exp(4\pi Qt)$, $0\leq t\leq 1$, where Q is the "non-Abelian charge" of Goddard, Nuyts, and Olive,[17] is homotopic to a loop in

$$Z_G(K)=\{g\in G\mid gk=kg,\forall k\in K\}\ , \tag{1.1}$$

the centralizer of K in G. In particular, G itself is implementable if and only if $h(t)$ is homotopic to a loop in $Z(G)$, the center of G.[13] Translated to terms of Q, this latter condition is expressed as

$$\exp[4\pi z(Q)]=1\ , \tag{1.2}$$

where $z:\mathcal{G}\to Z(\mathcal{G})$ is the projection onto the center $Z(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. Equation (1.2) is, despite its form, a constraint on the Higgs charge (see Sec. VI). Next, K is a symmetry if and only if it is a subgroup of

$$Z_G(Q)=\{g\in G\mid g^{-1}Qg=Q\}\ , \tag{1.3}$$

the centralizer of the non-Abelian charge Q. The whole G is a symmetry if and only if Q belongs to the center.

In each topological sector, there is only one stable monopole.[16,18,19] We show that, for the unique stable monopole of a given topological sector, G is implementable exactly when G is an internal symmetry (for a subgroup K of G the situation is more complicated).

A second illustration is provided by the non-Abelian Aharonov-Bohm experiment proposed by Wu and Yang.[20,21,34] No topological obstruction arises in this case for implementing SU(2)-gauge transformations. There is however an ambiguity: SU(2) admits two inequivalent implementations, and, for a given field configuration, none of the implementations is a symmetry in general. This explains why the electric charge of a nucleon is not in general conserved.[20,21,34]

II. IMPLEMENTABILITY OF GAUGE TRANSFORMATIONS

Let G denote a compact and connected Lie group and let us consider a gauge theory with gauge group G over (possibly a portion of) space-time M. Let us choose a covering of M by contractible open sets V_α. In each V_α the Yang-Mills field is given by a gauge potential A_μ^α, which satisfies, with the transition functions $h_{\alpha\beta}$: $V_\alpha\cap V_\beta\to G$, the consistency relation

$$A^{\alpha}_{\mu}(x)=(h_{\alpha\beta})^{-1}(x)A^{\beta}_{\mu}(x)h_{\alpha\beta}(x)+(h_{\alpha\beta})^{-1}(x)\partial_{\mu}h_{\alpha\beta}(x) \quad (2.1)$$

for all $x\in V_{\alpha}\cap V_{\beta}$. Similarly, a matter field Φ is specified by giving, in each V_{α}, a local representative Φ^{α} which transforms according to a unitary representation $\Phi\to g\cdot\Phi$ of G. The Φ^{α}'s satisfy the consistency relation

$$\Phi^{\alpha}(x)=h_{\alpha\beta}(x)\cdot\Phi^{\beta}(x) . \quad (2.2)$$

Let K be a group and consider a fixed field configuration (A_{μ},Φ). Let us assume that (i) K acts on M, $x\to k\cdot x$; (ii) in each V_{α} a G-valued function τ^{α}_{k} is associated to each $k\in K$ such that

$$\tau^{\alpha}_{k_1k_2}(x)=\tau^{\alpha}_{k_1}(k_2\cdot x)\tau^{\alpha}_{k_2}(x) , \quad (2.3)$$

subject to the consistency condition

$$\tau^{\alpha}_{k}(x)=[h_{\alpha\beta}(k\cdot x)]^{-1}\tau^{\beta}_{k}(x)h_{\alpha\beta}(x) . \quad (2.4)$$

Equations (2.3) and (2.4) imply that K is implementable: for each $x\in V_{\alpha}$

$$(k\cdot A_{\mu})^{\alpha}(x)=\tau^{\alpha}_{k}(x)k^{\nu}_{\mu}(x)A^{\alpha}_{\nu}(x)[\tau^{\alpha}_{k}(x)]^{-1}-\partial_{\mu}\tau^{\alpha}_{k}(x)[\tau^{\alpha}_{k}(x)]^{-1} \quad (2.5)$$

and

$$(k\cdot\Phi)^{\alpha}(x)=\tau^{\alpha}_{k}(x)\Phi^{\alpha}(x)$$

[where $k^{\nu}_{\mu}(x)$ is the matrix of the linear map on $T_xM\to T_{k\cdot x}M$ induced by $x\to k\cdot x$] have the correct transformation rules (2.1) and (2.2). Furthermore, $k\to k\cdot A_{\mu}$ (and $k\to k\cdot\Phi$) is a group action.

(iii) Following Schwarz[1] K is called a symmetry group for the configuration (A_{μ},Φ) with respect to this implementation if, in addition,

$$A^{\alpha}_{\mu}(k\cdot x)=(k\cdot A)^{\alpha}_{\mu}(x) ,$$
$$\Phi^{\alpha}(k\cdot x)=(k\cdot\Phi)^{\alpha}(x) . \quad (2.6)$$

We want to apply this general definition to a subgroup K of G, acting trivially on M: $x\to k\cdot x=x,\forall k\in K$. Notice that the conditions above are trivially satisfied by $\tau^{\alpha}_{k}(x)=1\,\forall\alpha,k,x$. This is, however, a trivial action. To have a sensible theory, some regularity condition has to be imposed. In this paper we make the usual requirement:[8,9,13] for each x, $\tau_{(.)}{}^{\alpha}(x)$ is the restriction to K of an automorphism of G. (This is automatic for symmetries, see Sec. III.) So $K\subseteq G$ implementable means now the existence of a family of G automorphisms $\tau^{\alpha}(x)$ such that

$$\tau^{\alpha}_{k_1k_2}(x)=\tau^{\alpha}_{k_1}(x)\tau^{\alpha}_{k_2}(x) , \quad (2.3')$$

satisfying the consistency condition

$$\tau^{\alpha}_{k}(x)=[h_{\alpha\beta}(x)]^{-1}\tau^{\beta}_{k}(x)h_{\alpha\beta}(x) . \quad (2.4')$$

This action is an internal symmetry if, in addition,

$$A^{\alpha}_{\mu}(x)=\tau^{\alpha}_{k}(x)A^{\alpha}_{\mu}(x)[\tau^{\alpha}_{k}(x)]^{-1}-\partial_{\mu}\tau^{\alpha}_{k}(x)[\tau^{\alpha}_{k}(x)]^{-1} , \quad (2.6')$$
$$\Phi^{\alpha}(x)=\tau^{\alpha}_{k}(x)\cdot\Phi^{\alpha}(x) .$$

We study first implementability. Following the pattern in monopole theory, we show that a topological obstruction may prevent us from implementing K.[8,9,13] Indeed, let us assume that K is implementable, and let $x_0\in M$ be an arbitrary reference point. There is no loss of generality in assuming $\tau^{\alpha}_{k}(x_0)=k$ since this can always be achieved by replacing $\tau_{(.)}{}^{\alpha}(x)$ by $\tau_{(.)}{}^{\alpha}(x)[\tau_{(.)}{}^{\alpha}(x_0)]^{-1}$. $\tau^{\alpha}(x)$ belongs then, for each x, to $(\mathrm{Aut}\ G)_0$, the connected component of the group of automorphisms of G. $(\mathrm{Aut}\ G)_0$ is known, however, to consist of inner automorphisms for any compact and connected G.[22,23] It follows that, for each $x\in V_{\alpha}$, there exists an $h_{\alpha}(x)\in G$ such that

$$\tau^{\alpha}_{k}(x)=h_{\alpha}(x)kh^{-1}_{\alpha}(x) .$$

The h_{α}'s can be chosen to be smooth since the V_{α}'s are contractible by assumption. The h_{α} define hence a gauge transformation in each V_{α}. In the new gauge (we still denote it by α) the action (2.5) of k becomes rigid, i.e., position independent:

$$(k\cdot A_{\mu})^{\alpha}(x)=kA^{\alpha}_{\mu}(x)k^{-1}$$

and (2.7)

$$(k\cdot\Phi)^{\alpha}(x)=k\cdot\Phi^{\alpha}(x) .$$

The consistency condition (2.4) requires now

$$k^{-1}h_{\alpha\beta}(x)k=h_{\alpha\beta}(x),\quad \forall k\in K,\quad x\in V_{\alpha}\cap V_{\beta} , \quad (2.8)$$

where the $h_{\alpha\beta}$ are the new transition functions between the rigid gauges in V_{α} and V_{β}. By reversing the argument we see that, by (2.8), K is implementable if and only if there exist gauges such that all transition functions $h_{\alpha\beta}(x)$ take their values in

$$Z_G(K)=\{g\in G\mid g^{-1}kg=k,\forall k\in K\} , \quad (2.9)$$

the centralizer of K in G. In particular, G itself is implementable if and only if there is a gauge where all transition functions belong to the center of G.

Geometrically, this condition means that the principal G-bundle P which carries the gauge structure reduces to a $Z_G(K)$ bundle.[23] This reduction is defined by a section τ—whose local representatives are the τ^{α}'s above—of the associated bundle

$$P\times_G\{(\mathrm{Aut}G)_0\mid K\}\simeq P\times_G\{\mathrm{Int}G\mid K\}\approx P\times_G\{G/Z_G(K)\}\simeq P/Z_G(K) . \quad (2.10)$$

III. INTERNAL SYMMETRIES

Let K be a connected Lie group with Lie algebra $\mathscr{K}$, and assume that K is implementable. Let its internal action be given by a family τ^{α}. Set

$$\omega^{\alpha}_{\kappa}(x)=\frac{d}{dt}\Big|_{t=0}\tau^{\alpha}_{[\exp(-t\kappa)]}(x),\quad \kappa\in\mathscr{K},\ x\in V_{\alpha} . \quad (3.1)$$

The infinitesimal action of $\kappa\in\mathscr{K}$ corresponding to the considered internal action of $K\subseteq G$ is given by

$$(\kappa\cdot A_{\mu})=D_{\mu}\omega_{\kappa}\ \text{ and }\ (\kappa\cdot\Phi)=\omega_{\kappa}\cdot\Phi \quad (3.2)$$

(to keep the notation simple, we dropped the index α). Equation (2.4′) implies that $\omega_\kappa(x)$ is a "Higgs" field of the adjoint type. The property (2.3) requires now

$$\omega_{[\kappa_1,\kappa_2]}(x)=[\omega_{\kappa_1}(x),\omega_{\kappa_2}(x)],\quad \kappa_1,\kappa_2\in\mathscr{K},\quad x\in M\ . \tag{3.3}$$

So, taking into account our regularity and normalization conditions, $\kappa\to\omega_\kappa(x)$ is, for each x, the restriction to $\mathscr{K}$ of a Lie algebra automorphism of $\mathscr{G}$ satisfying $\omega_\kappa(x_0)=\kappa$. If K acts by internal symmetries, then, by (3.2),

$$D_\mu\omega_\kappa=0 \quad\text{and}\quad \omega_\kappa\cdot\Phi=0\ . \tag{3.4}$$

Conversely, any normalized solution of (3.4) provides us with an internal action of $k=\exp(-2\pi\kappa)$. Indeed, (3.4) is solved by parallel transport,

$$\omega_\kappa(x)=g(x)\kappa g^{-1}(x)\ , \tag{3.5}$$

where $g(x)$ is the nonintegrable phase factor

$$g(x)=P\exp\left[-\int_{x_0}^{x}A_\mu dx^\mu\right]. \tag{3.6}$$

Equation (3.6) is in general path dependent. Let $\kappa\in\mathscr{K}$ be such that (3.5) is nevertheless path independent, and assume $\omega_\kappa\cdot\Phi=0$ is also satisfied [this is automatic if $D_\mu\Phi=0$, since in this case $\Phi(x)=g(x)\Phi_0 g(x)^{-1}$ and so

$$\begin{aligned}\omega_\kappa(x)\cdot\Phi(x)&=g(x)\kappa g(x)^{-1}g(x)\Phi_0 g(x)^{-1}\\ &=\kappa\cdot\Phi_0=0\ ,\end{aligned}$$

since $\kappa\in\mathscr{K}\subset\mathscr{G}$]. $k=\exp(-2\pi\kappa)$ is now implementable: $\tau_k(x)=\exp[-\omega_\kappa(x)]$ admits, as one proves easily, the properties (2.3′) and (2.4′). The corresponding action of k on the fields is plainly a symmetry: it satisfies (2.7′) and (2.8′). Those $\eta\in\mathscr{G}$ for which (3.5) is path independent and $\omega_\kappa\cdot\Phi=0$ is also satisfied generate a connected subgroup H of G. [By (3.5) the group property (3.3) is now automatic.] Some differential geometry shows that H is the centralizer of the holonomy algebra of the Yang-Mills potential.[7,23]

H is the maximal internal-symmetry group of the Yang-Mills-Higgs (YMH) configuration we consider: any subgroup K of G which is a symmetry group is plainly a subgroup of H.

Observe that $\kappa\to\omega_\kappa(x)$ given by (3.5) is always an automorphism, justifying a posteriori our regularity assumption in Sec. II.

IV. CONSERVED CHARGES

Consider now a test particle ψ moving in a background YMH field (A_μ,Φ). For the sake of simplicity we consider only a spin-$\frac{1}{2}$ Dirac particle, with Lagrangian

$$-\mathscr{L}=\bar\psi(\gamma^\mu D_\mu+c\Phi+m)\psi\ , \tag{4.1}$$

where c is a group-independent constant, and ψ is assumed to transform according to a unitary representation U of G. ψ is just another matter field, so, in each V_α, it is described by a local representative ψ^α. The consistency condition (2.2) becomes now $\psi^\alpha(x)=U(h_{\alpha\beta})\psi^\beta(x)$, $x\in V_\alpha\cap V_\beta$.

Let us assume that K is a connected group of internal symmetries for the given background YMH configuration (A_μ,Φ), implemented by $\tau(x)\in \mathrm{Aut}G\,|\,K$ (restriction of an automorphism of G to K; the superscript α is dropped again). K acts on ψ as $(k\cdot\psi)(x)=U(\tau_k(x))\psi(x)$; the infinitesimal action reads $(\kappa\cdot\psi)(x)=(\omega_\kappa(x))\psi(x)$. It is straightforward to show that this implementation leaves invariant the Lagrangian (4.1). Hence the internal symmetry of the background field becomes a Noether symmetry for the test particle. Consequently, for each $\kappa\in\mathscr{K}$, the current

$$j_\kappa^\mu=\frac{\delta\mathscr{L}}{\delta(\partial_\mu\psi)}(\kappa\cdot\psi)=\bar\psi\gamma^\mu\omega_\kappa\cdot\psi \tag{4.2}$$

is conserved, $\partial_\mu j_\kappa^\mu=0$ (Ref. 24).

Let us consider the non-Abelian current

$$J_a^\mu=\psi\gamma^\mu\tau_a\cdot\psi,\quad a=1,\ldots,\dim\ ,$$

where the τ_a's are a basis of the Lie algebra. The gauge invariance of the Lagrangian (4.1) implies that J_a^μ is covariantly conserved, $D_\mu J_a^\mu=0$. However, the ω_κ component

$$j_\kappa^\mu=\mathrm{Tr}(\omega_\kappa J^\mu) \tag{4.3}$$

is already ordinarily conserved since $D_\mu\omega_\kappa=0$ by assumption. It is straightforward to verify that (4.3) is just (4.2), as anticipated by the notation. Interestingly, this expression has been proposed previously to define conserved electric charge.[25] Now we understand its origin: it is the current associated to an internal-symmetry generator. This sheds a new light on the role of internal symmetries. It is instructive to pursue this direction. Let us assume in fact that $D_\mu\Phi=0$ and so the YM field satisfies the vacuum field equations

$$D_\mu F^{\mu\nu}=0,\quad D_\mu(\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})/2=0 \tag{4.4}$$

and identify the electromagnetic field as the ω_κ component of $F_{\mu\nu}$:

$$\mathscr{F}_{\mu\nu}(x)=(1/e)\,\mathrm{Tr}(F_{\mu\nu}\omega_\kappa/|\omega_\kappa|)\ , \tag{4.5}$$

where e is a coupling constant. Equation (4.4) implies that $\mathscr{F}_{\mu\nu}$ satisfies the vacuum Maxwell equations

$$\partial_\mu\mathscr{F}^{\mu\nu}=0,\quad \partial_\mu(\epsilon^{\mu\nu\rho\sigma}\mathscr{F}_{\rho\sigma})/2=0\ . \tag{4.6}$$

Let us define the electric charge operator by

$$Q_{\mathrm{em}}(x)=e\omega_\kappa(x)/|\omega_\kappa(x)|\ . \tag{4.7}$$

As demonstrated in Refs. 26 and 27, the eigenvalues of (4.7) are quantized if and only if κ generates a U(1) (rather then merely a torus) subgroup of G. If so, all electric charges are integer multiples of

$$q_{\min}=e/|\kappa_0|\ , \tag{4.8}$$

where κ_0 is a "minimal" U(1) generator [i.e., such that $\exp(2\pi\kappa t)=1$ the first time for $t=1$] parallel to κ. Indeed, let us assume that ψ is an eigenstate of Q_{em} with eigenvalue $nq_{\min}$. The particle's electric charge is hence

$$q=\int_{R^3} j_\kappa^0(x)d^3x=\int_{R^3}\bar{\psi}\gamma^0\omega_\kappa\cdot\psi$$
$$=nq_{\min}\int_{R^3}\psi^\dagger\psi=nq_{\min} . \quad (4.9)$$

V. ASYMPTOTIC PROPERTIES OF MONOPOLE CONFIGURATIONS

In this section we review briefly those properties of non-Abelian monopoles we need in the sequel. (For reviews see, e.g., Refs. 14—16.)

Let us consider a YMH theory with a compact, connected and simply connected (and hence semisimple) "unifying" gauge group $\widetilde{G}$. At some energy scale ($\sim 10^{14}$ GeV) the $\widetilde{G}$ symmetry is spontaneously broken to a subgroup G of $\widetilde{G}$ by the vacuum expectation value (VEV) of the Higgs field Φ. Consequently, the asymptotic values of the Higgs field provide us with a map $\Phi:S^2\to\widetilde{G}\cdot\Phi_0\simeq\widetilde{G}/G$. Magnetic monopoles are everywhere-regular, static, finite-energy, purely magnetic solutions to the YMH equations, satisfying also the "finite-energy" condition $D_\mu\Phi=0$ on S^2. Φ provides us with the fundamental topological invariant $[\Phi]\in\pi_2(\widetilde{G}/G)$ we call the Higgs charge. The injective homomorphism $\delta:\pi_2(\widetilde{G}/G)\to\pi_1(G)$ is now an isomorphism since $\widetilde{G}$ is assumed to be simply connected.

We have shown in a previous paper,[28] that, for any compact and connected Lie group G, $\pi_1(G)=\pi_1(G)_{\text{free}}+\pi_1(G_{\text{ss}})$ (direct sum). $\pi_1(G)_{\text{free}}\simeq Z^p$, where p is the dimension of the center $Z(\mathscr{G})$ of the Lie algebra of G, and G_{ss} is the subgroup of G generated by $[\mathscr{G},\mathscr{G}]$. G_{ss} is semisimple, so $\pi_1(G_{\text{ss}})$ is a finite Abelian group. The isomorphism $\pi_1(G)_{\text{free}}\simeq Z^p$ is established explicitly as follows: let $\Gamma=\{\xi\in\mathscr{G}\mid\exp(2\pi\xi)=1\}$ denote the unit lattice of G, and consider the image $z(\Gamma)$ of Γ under the projection map $z:\mathscr{G}\to Z(\mathscr{G})$. $z(\Gamma)$ is a p-dimensional lattice in $z(\Gamma)$, and, as we have shown in Ref. 28,

$$\rho([\gamma])=\frac{1}{2\pi}\int_\gamma z(g^{-1}dg)\in Z(\mathscr{G}) , \quad (5.1)$$

where γ is a loop in G, is an isomorphism between $\pi_1(G)_{\text{free}}$ and $z(\Gamma)$. Any loop in G is known to be homotopic to one of the form $\gamma(t)=\exp(2\pi\xi t),\xi\in\mathscr{G}$, for which $\rho(\gamma)=z(\xi)$. If $\zeta_1,\ldots,\zeta_p$ is a Z-basis for the lattice $z(\Gamma)$, then $\rho([\gamma])=z(\xi)=\sum m_j\zeta_j$; $[\gamma]\to(m_1,\ldots,m_p)$ is the aforementioned isomorphism.

On S^2 the YMH equations decouple and we are left with a pure G-valued Yang-Mills theory with field equation $D_jF^{jk}=0$. The general solution of this equation has been found in Ref. 17: let us cover S^2 with the contractible open sets $V_1=S^2\setminus\{\text{south pole}\}$ and $V_2=S^2\setminus\{\text{north pole}\}$. There exist gauges over $V_{1,2}$—the so-called U gauge—such that $\Phi=\Phi_0$ and the solution is

$$A_\theta^{1,2}=0,\ \ A_\phi^{1,2}=\pm Q(1\mp\cos\theta) . \quad (5.2)$$

Q—the non-Abelian charge—is a constant vector in the Lie algebra. Q can be chosen, with no loss of generality, in any given Cartan subalgebra. To have a well-defined theory, Q must be quantized, $\exp(4\pi Q)=1$. A loop in $\delta[\Phi]$ representing the Higgs charge is then expressed as

$$h(t)=\exp(4\pi Qt),\ \ 0\le t\le 1 . \quad (5.3)$$

Equation (5.3) is the transition function between the U gauges over V_1 and V_2. So

$$\rho(\Phi)=\rho(\delta[\Phi])=2z(Q) . \quad (5.4)$$

Let us decompose Q as $Q=z(Q)+Q'$, where Q' belongs to the derived algebra. The result of Brandt and Neri[18] tells us that the monopole is stable if and only if, for any root α of the semisimple Lie algebra $[\mathscr{G},\mathscr{G}]$, $2\alpha(Q')=0$ or 1 for any root α of $[\mathscr{G},\mathscr{G}]$, cf. Ref. 19. In each topological sector there exists hence exactly one stable monopole.[16]

VI. THE PROBLEM OF GLOBAL COLOR FOR MONOPOLES

Let us now consider a non-Abelian monopole (A_j,Φ), and let G denote the little group of the Higgs field at infinity. Let K be a subgroup of G. According to the general theory of Sec. II, K is implementable if and only if, in V_α ($\alpha=1,2$), there exist G automorphisms $\tau^\alpha(x)$ which satisfy the consistency condition (2.4′) with the transition function (5.3). Both V_1 and V_2 are contractible, so we can go to rigid gauges so that the consistency condition reads $kh(x)=h(x)k$, $\forall k\in K$, $x\in V_1\cap v_2$, where h is the transition function for the new (rigid) gauges. The homotopy class of the transition function is however independent of the choice of a gauge, so this condition holds if and only if any transition function—in particular (5.3)—is homotopic to one in

$$Z_G(K)=\{g\in G\mid gkg^{-1}=k,\forall k\in K\} , \quad (6.1)$$

the centralizer in G of K. The full "residual" group G is implementable if and only if (5.3) is homotopic to a loop in the center of G (Ref. 13).

Requiring the implementability of a subgroup K is a topological constraint on the Higgs charge.[8,9] Indeed, (5.3) homotopic to a curve in $Z_G(K)$ means exactly that

$$\delta[\Phi]=[h(t)]\in\operatorname{Im}i_* , \quad (6.2)$$

where i_* is the homomorphism i_*: $\pi_1(Z_G(K))\to\pi_1(G)$ induced by the inclusion map $i:Z_G(K)\to G$. In particular, G itself is implementable if and only if $\operatorname{Ad}h$ is contractible in $\operatorname{Aut}G_0\simeq\operatorname{Int}G$.[8] Indeed, the exact sequence $Z(G)\to G\to\operatorname{Int}G$ of groups yields, using $\pi_2(G)=0$,

$$0\to\pi_1(Z(G))\xrightarrow{i_*}\pi_1(G)\xrightarrow{\operatorname{Ad}_*}\pi_1(\operatorname{Int}G)\to\cdots \quad (6.3)$$

so (6.2) implies that $[h]=i_*[\gamma]$ for a γ in $Z(G)$, and hence $[\operatorname{Ad}h]=\operatorname{Ad}_*[h]=(\operatorname{Ad}\circ i)_*[\gamma]=0$. Conversely, if $\operatorname{Ad}_*[h]=0$, then $[h]=i_*[\gamma]$ for a γ in the center, yielding (6.3). Alternatively, this statement follows from the triviality of (2.10) which is now a principal $\operatorname{Int}G$ bundle, having transition function $\operatorname{Ad}h$.

To translate (6.2) to more down-to-earth terms, let us study first the case $K=G$. $i_*\pi_1(Z(G))$ lies in the free part, so

(i) if G is implementable, $\delta[\Phi]\in\pi_1(G)_{\text{free}}$.

This implies at once that if $\pi_1(G)$ is finite (as it happens in some grand unified theories (GUT's), see Sec. VIII) then G is never implementable for topologically nontrivial Higgs fields.

Let $\zeta_1, \ldots, \zeta_p$ be a Z basis of $z(\Gamma)$. For each $j=1,\ldots,p$ there exists a least positive integer M_j such that $\exp(2\pi\zeta_j M_j)=1$.[28] The loops $\gamma_j(t)=\exp(2\pi\zeta_j M_j t)$, $j=1,\ldots,p$ generate $\pi_1(Z(G))$, and thus also its image under i_*. $[\gamma_j]\in\pi_1(G)_{\text{free}}\simeq Z^p$ has "quantum" numbers $(0,\ldots,M_j,\ldots,0)$. The parameter space of $\text{Im}\, i_*$ consists hence of integer combinations of these p-tuples. Equation (6.2) means thus that

(ii) there exist integers $n_1,\ldots,n_p$

such that $[\Phi]\simeq(m_1,\ldots,m_p)$ satisfies

$$m_j=n_jM_j,\quad j=1,\ldots,p\ . \tag{6.4}$$

The physically most interesting situation is when $Z(\mathscr{G})$ is one dimensional. In this case (6.4) is simply

$$m=nM\ , \tag{6.5}$$

where M labels the homotopy class of the central U(1). This generalizes the results in Refs. 8 and 9.

The condition of implementability has a nice expression in terms of the non-Abelian charge Q. Indeed, if $Z(\mathscr{G})\neq 0$, (6.2) is equivalent to

$$\exp(4\pi Q't),\quad 0\le t\le 1\ , \tag{6.6}$$

is contractible in G_{ss} and

$$\exp[4\pi z(Q)]=1\ . \tag{6.7}$$

Indeed, (6.6) is exactly (i) above. On the other hand, (5.3) homotopic to a curve $\gamma(t)$ in $Z(G)$ means that (5.3) and $\gamma(t)$ have the same image under ρ. But a $\gamma(t)$ in $Z(G)$ is homotopic to a loop of the form $\gamma(t)=\exp(2\pi\zeta t)$, with $\zeta\in Z(\mathscr{G})$, whose image under ρ is ζ itself. Hence $\rho(\gamma(t))=2z(Q)=\zeta$. However, $\exp(2\pi\zeta)=1$, proving (6.7). Conversely, if (6.6) and (6.7) are satisfied, then (5.3) is homotopic to $\gamma(t)=\exp[4\pi z(Q)t]\subseteq Z(G)$ since they have the same image under ρ. If (5.3) has no center, $Z(G)$ is a discrete subgroup of G and thus $\pi_1(G)$ is finite, so that the constraint (6.2) is violated.

Similar, although slightly more complicated results hold for a general K. Let us assume, for simplicity, that $\pi_1(G)$ is free, Z^p. (This happens, for example, if Φ is in the adjoint representation.) $\rho[i_*(\pi_1(Z_G(K))]$ is a sublattice in $z(\Gamma)$, so it is generated by elements $\xi_j\in Z(\mathscr{G}), j=1,\ldots,r<p$. There is no loss of generality in assuming that each ξ_j is parallel to a suitable ζ_j, $\xi_j=c_j\zeta_j$. The coefficient c_j here is an integer, since the ζ_k's form a Z basis in $z(\Gamma)$. Denote L_j the least common multiple $L_j=[c_j,M_j], j=1,\ldots,r$ with M_j as above, and let M be the least common multiple of the L_j/c_j's. K implementable means now the quantization condition

$$\exp[4\pi Mz(Q)]=1\ . \tag{6.8}$$

Alternatively, the implementability condition (6.2) is also expressed as

$$m_j=\begin{cases} c_jn_j & \text{for some integer } n_j, j=1,\ldots,r\\ 0 & \text{for } j=r+1,\ldots,p\ .\end{cases} \tag{6.9}$$

For $K=G, c_j=M_j$ so $M=1$ and (6.8) reduces to (6.7).

Having settled the problem of implementability, let us ask if K is a symmetry group. Using the infinitesimal approach of Sec. III we see that this happens if and only if (3.5) is path independent for each generator κ of K (since $D_\mu\Phi=0$ and thus $\omega_\kappa\cdot\Phi=0$ is automatically satisfied). This is a gauge-invariant condition so we can work in the U gauge (5.2), where $\omega_\kappa=\kappa$ in V_1 *and* in V_2, so path independence means simply

$$\kappa\in\mathscr{K}=Z_{\mathscr{G}}(Q)=\{\eta\in\mathscr{G}\mid[\eta,Q]=0\}\ . \tag{6.10}$$

We conclude that any symmetry group K must belong to the centralizer of Q in G. Notice that $Z(\mathscr{G})$ is always in (6.10). In particular, the whole of G is a symmetry with respect to the internal action defined by (3.2) if and only if Q is in the center of the Lie algebra.

What is the difference between implementability and symmetry? Observe that G is simultaneously implementable or not implementable for an entire topological sector. Let us assume $[\Phi]$ satisfies (6.6) and (6.7) and thus G is implementable for all monopoles in this homotopy class. In particular, $[\Phi]$ belongs to the free part of $\pi_2(\widetilde{G}/G)$. There is exactly one stable monopole in this homotopy sector—the one with $Q'=0$. So, for the unique stable monopole, symmetry and implementability are the same. For the other (unstable) monopoles the two statements are different. The general situation $K\subset_p G$ is more complicated and the conclusion is different. (Here $\subset_p$ is the proper subset.) Again, the full topological sector is simultaneously implementable or not. The non-Abelian charge of the unique stable monopole of our homotopy class may however not belong to $Z_{\mathscr{G}}(\mathscr{K})$, and thus K may fail to be a symmetry for the stable monopole. If, on the other hand, we choose Q in $Z_{\mathscr{G}}(\mathscr{K})$, K is a symmetry, but the corresponding monopole is generally unstable (see Sec. VIII).

If we work with the unifying group $\widetilde{G}$, there are no transition functions. How does our obstruction manifest itself in this picture? The problem is now to associate an element g_x in G_x (the stability subgroup of the Higgs field at x) in a smooth way. As explained in Refs. 8 and 9, $G_x=u(x)G_{x_0}u(x)^{-1}$. The map $x\to g_x=u(x)gu(x)^{-1}$ becomes, however, singular somewhere, except when our condition is satisfied. Alternatively, $u(x)$ can be used to gauge $G_x(g_x)$ to G (g) in a contractible subset. But, to cover the two-sphere, we need at least two such subsets, and we reintroduce thus transition functions just like before.

VII. CONSERVED CHARGES AND ELECTROMAGNETIC PROPERTIES IN THE FIELD OF A MONOPOLE

Let us consider now a spin-$\frac{1}{2}$ Dirac field ψ coupled to a background monopole field (A_j,Φ). As explained in Sec. IV, to any symmetry generator η, i.e., to any η which commutes with the non-Abelian charge vector Q, is associated a conserved current. In the U gauge this current is simply

$$j^\mu_\eta=e\bar{\psi}\gamma^\mu\eta\psi\ . \tag{7.1}$$

In particular, a generator ζ of the center is an internal symmetry direction for all monopoles created when the symmetry is spontaneously broken to G. In other words, ζ is an admissible electromagnetic direction for all monopoles in the theory. [This is the choice made in Ref. 28—the generalization of the standard approach[26] valid when Φ is in the adjoint representation and $Z(\mathcal{G})$ is one-dimensional.] For a fixed monopole configuration we have slightly more freedom: any vector which commutes with the non-Abelian charge is admissible.

Monopoles carry also a magnetic charge. This is defined by the flux integral

$$g=\frac{1}{4\pi e}\int_{S^2}\mathcal{F}_{\mu\nu}\,, \tag{7.2}$$

where the electromagnetic field $\mathcal{F}_{\mu\nu}$ is defined by (4.5). In the U gauge (7.2) is calculated at once:

$$g=\frac{1}{e\,|\eta|}\,\mathrm{Tr}(Q\eta)\,. \tag{7.3}$$

Observe, that the magnetic charge is quantized: indeed, $Q=(n/2)Q_0$ for some integer n, where Q_0 is a minimal U(1) generator parallel to Q. So g is an integer multiple of

$$g_{\min}=\frac{1}{2e\,|\eta_0|}\,\mathrm{Tr}(Q_0\eta_0)\,, \tag{7.4}$$

where η_0 is a minimal U(1) generator parallel to η. The comparison of (7.4) with (4.8) shows that the electric and magnetic charges, respectively, satisfy the generalized Dirac condition

$$2g_{\min}g=\frac{\mathrm{Tr}(2Q\eta_0)}{|\eta_0|^2}\,. \tag{7.5}$$

Notice that the value of (7.5) depends in general on Q and not only on the Higgs charge. In other words, it is not a topological invariant. If, however, η is in the center, $\eta\in Z(\mathcal{G})$, then the right-hand side of (7.5) satisfies

$$\mathrm{Tr}(2Q\zeta)=\mathrm{Tr}(2z(Q)\zeta)=\mathrm{Tr}(\rho(\Phi)\zeta)\,, \tag{7.6}$$

so (7.5) becomes rather

$$2q_{\min}g=\frac{\mathrm{Tr}(\rho(\Phi)\eta_0)}{|\eta_0|^2}\,, \tag{7.7}$$

which is already a topological invariant: it depends only on $\rho(\Phi)$, the free part of the Higgs charge (cf. Ref. 28). In the particular case when $\pi_1(G)=\pi_1(G)_{\mathrm{free}}\simeq Z$, let $[\Phi]\simeq m$; (7.7) is simply

$$2q_{\min}g=m/M\,, \tag{7.8}$$

where the integer M labels the homotopy class of the central U(1). On the other hand, G implementable means now that $m=nM$ [cf. (6.5)]. We conclude that, in this special case, G is implementable exactly when the generalized Dirac condition (7.7) reduces to the original (integer) Dirac condition. [If $Z(\mathcal{G})$ is not one-dimensional, this conclusion is, however, false; see the SO(10) example below.]

VIII. EXAMPLE: GRAND UNIFIED MONOPOLES

As a first illustration, we consider monopoles in the $\tilde{G}=\mathrm{SU}(5)$ GUT (Refs. 29 and 30). Following the general pattern, let us assume Φ is in the **24** (adjoint) representation; the choice $\Phi_0=vi\,\mathrm{diag}(2,2,2,-3,-3)$ yields the little group

$$\begin{aligned}G&=S[\mathrm{U}(3)\times\mathrm{U}(2)]\\&=[\mathrm{SU}(3)_c\times\mathrm{SU}(2)_W\times\mathrm{U}(1)_Y]/Z_6\,.\end{aligned} \tag{8.1}$$

$Z(\mathcal{G})$ is generated by Φ_0 itself, and $\pi_1(G)\simeq Z$. The "quantum number" $[\Phi]\simeq m$ is calculated by $m=\mathrm{Tr}_3(\rho(\Phi))/i=2\,\mathrm{Tr}Q/i$ (trace on the upper 3×3 block, cf. Refs. 27 and 28. The generating loop $\exp(2\pi\zeta_0 t)=\exp(2\pi\Phi_0/vt)$ of the center of G has quantum number $M=6$, so, according to (6.5), G is implementable if and only if m is an integer multiple of $6, m=6n$.[8,9,13] This is seen alternatively from (6.7), observing that $z(Q)=(m/6)M$. G contains the color subgroup

$$\mathrm{SU}(3)_c=\begin{bmatrix}A & \\ & 1_2\end{bmatrix},\quad A\in\mathrm{SU}(3) \tag{8.2}$$

whose centralizer is

$$Z_G(\mathrm{SU}(3)_c)=\mathrm{U}(2)_{\mathrm{WS}}=\begin{bmatrix}(\det B)^{-1}1_3 & \\ & B\end{bmatrix},\quad B\in\mathrm{U}(2)\,. \tag{8.3}$$

$\pi_1(\mathrm{U}(2)_{\mathrm{WS}})\simeq Z$ is generated, e.g., by

$$\gamma(t)=\exp\left[2\pi i\begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&-3&\\&&&&0\end{bmatrix}t\right],\quad 0\le t\le 1\,, \tag{8.4}$$

whose homotopy class is labeled by $c=3$. Hence, by (6.9), $\mathrm{SU}(3)_c$ is implementable if and only if $m=3n$. [Alternatively, this follows from (6.8) noting that $M=2$ now.] Similarly, consider

$$\mathrm{SU}(2)_W=\begin{bmatrix}1_3 & \\ & B\end{bmatrix}\quad (B\in\mathrm{SU}(2)) \tag{8.5}$$

the subgroup of weak interactions. $Z_G(\mathrm{SU}(2)_W)$ is just

$$\mathrm{U}(3)=\begin{bmatrix}A & \\ & (\det A)^{-1/2}1_2\end{bmatrix},\quad A\in\mathrm{U}(3)\,. \tag{8.6}$$

$\pi_1(\mathrm{U}(3))\simeq Z$ is generated, e.g., by

$$\gamma(t)=\exp[2\pi it\,\mathrm{diag}(-2,0,0,1,1)]\,,$$

whose class in $\pi_1(G)$ is $c=2$. Thus $\mathrm{SU}(2)_W$ is implementable if and only if $m=2n$. G is an internal-symmetry group only for the stable charge-6 monopole[31] given by

$$Q=\zeta/2=i\,\mathrm{diag}(1,1,1,-\tfrac{3}{2},-\tfrac{3}{2})\,. \tag{8.7}$$

$\mathrm{SU}(3)_c$ is a symmetry if and only if $Q\in\mathrm{U}(2)_{\mathrm{WS}}$. This is realized by two different charge-3 monopoles:

$$Q_1=(i/2)\,\mathrm{diag}(1,1,1,0,-3) \qquad (8.8)$$

and

$$Q_2=(i/2)\,\mathrm{diag}(1,1,1,-1,-2)\ . \qquad (8.9)$$

Only (8.9) satisfies the Brandt-Neri (BN) condition and is thus stable.[31] $SU(3)_c$ is hence a symmetry group simultaneously for a stable and an unstable monopole. Similarly, $SU(2)_W$ is a symmetry if and only if $Q\in U(3)$ in (8.6). This condition is met by two charge-2 monopoles:

$$Q_1=(i/2)\,\mathrm{diag}(2,0,0,-1,-1) \qquad (8.10)$$

and

$$Q_2=(i/2)\,\mathrm{diag}(1,1,0,-1,-1)\ . \qquad (8.11)$$

Both monopoles are $SU(2)_W$ symmetric, but only (8.11) is stable. For the "elementary" monopole we have $2Q=i\,\mathrm{diag}(1,0,0,0,-1)$, so the maximal symmetry group is

$$H=\begin{bmatrix} U(1) & & & \\ & U(2)_c & & \\ & & U(1) & \\ & & & \cdot \end{bmatrix}. \qquad (8.12)$$

At much lower energies (~ 100 GeV) the symmetry is further broken to $G=U(3)=[SU(3)_c\times U(1)_{em}]/Z_3$ by a Higgs **5**. $\pi_1(U(3))\simeq Z$, and the quantum number m is calculated by the same formula.[28] $Z(U(3))$ is generated by

$$Q_{em}=i\,\mathrm{diag}(1,1,1,-3,0) \qquad (8.13)$$

(a minimal generator). Equation (8.13) is the standard choice for the electromagnetic direction. The central U(1) has quantum number 3, so $G=U(3)$ is implementable for the U(3) monopole if and only if $m=3n$.[13] This is seen alternatively from (6.6) since $2z(Q)=mQ_{em}/3$ in this case. The color subgroup $SU(3)_c$ belongs to U(3); its centralizer in U(3) is

$$Z_{U(3)}(SU(3)_c)=U(1)_{em}=U(1)_{center}\ , \qquad (8.14)$$

so $SU(3)_c$ is implementable if and only if $m=3n$ [alternatively, $M=1$ in (6.8)]. $G=U(3)$ is an internal symmetry if and only if $Q\in Z(U(3))$, i.e., if $Q=(m/2)Q_{em}$. But this is simultaneously the centralizer for $SU(3)_c$, so they are simultaneously symmetries or not. The charge-3 monopole given by $2Q=Q_{em}$ is stable by the BN condition, and is thus U(3) symmetric. The elementary $S(U(3\times U(2))$ monopole survives the "phase transition" $S(U(3)\times U(2))\to U(3)$. The maximal symmetry group (8.12) is reduced however to

$$H'=U(2)_c\times U(1)_{em} \qquad (8.15)$$

since $U(2)_{WS}$ is broken to $U(1)_{em}$ in this process.

Let us consider the SU(5) 5-plet $\psi=(d_R,d_B,d_G,e^-,\nu_e)_L$, where R, B, G refer to the quark colors. Equation (8.15) is generated by

$$\sigma_0=\begin{bmatrix}0&&&&\\&1&&&\\&&1&&\\&&&-2&\\&&&&0\end{bmatrix},\quad \sigma_1=\begin{bmatrix}0&&&&\\&&i&&\\&i&&&\\&&&0&\\&&&&0\end{bmatrix},$$

$$\sigma_2=\begin{bmatrix}0&&&&\\&&-1&&\\&1&&&\\&&&0&\\&&&&0\end{bmatrix}, \qquad (8.16)$$

$$\sigma_3=\begin{bmatrix}0&&&&\\&i&&&\\&&-i&&\\&&&0&\\&&&&0\end{bmatrix},\quad Q_{em}=\begin{bmatrix}i&&&&\\&i&&&\\&&i&&\\&&&-3i&\\&&&&0\end{bmatrix}.$$

All these generators are internal symmetries for ψ—considered as a test particle in the field of an SU(5) GUT monopole. Q_{em} is the standard choice for the electromagnetic direction; the corresponding electric charge is quantized in units of

$$q_{min}=e/2\sqrt{3}=q/3\ . \qquad (8.17)$$

The electromagnetic current is thus expressed as

$$j^\mu_{em}=qi\left[\tfrac{1}{3}(\bar d_R\gamma^\mu d_R+\bar d_B\gamma^\mu d_B+\bar d_G\gamma^\mu d_G)-\bar e\gamma^\mu e\right]\ . \qquad (8.18)$$

Equation (8.18) is conserved in all background monopole fields, but only for the elementary one. The four other currents

$$j^\mu_0=ci[(\bar d_B\gamma^\mu d_B+\bar d_G\gamma^\mu d_G)-2\bar e\gamma^\mu e]\ , \qquad (8.19a)$$

$$j^\mu_1=ci(\bar d_B\gamma^\mu d_G+\bar d_G\gamma^\mu d_B)\ , \qquad (8.19b)$$

$$j^\mu_2=c(-\bar d_B\gamma^\mu d_G+\bar d_G\gamma^\mu d_B)\ , \qquad (8.19c)$$

$$j^\mu_3=ci(\bar d_B\gamma^\mu d_B-\bar d_G\gamma^\mu d_G) \qquad (8.19d)$$

(where $c=e/\sqrt{2}$) are however conserved only for the elementary monopole. The corresponding "magnetic" charges defined as the flux integral of the corresponding "electromagnetic" fields are

$$g^{em}=1/2q,\ \ g_0=1/6c,\ \ g^1=g^2=g^3=0\ , \qquad (8.20)$$

so the generalized Dirac conditions read

$$2q^{em}_{min}g^{em}=\tfrac{1}{3},\ \ 2q^0_{min}g^0=\tfrac{1}{3},\ \ 2q^j_{min}g^j=0,\ \ j=1,2,3\ . \qquad (8.21)$$

As a second example, consider $\tilde G=$ Spin10 [the double covering of SO(10)] broken to $G=[\mathrm{Spin}6\times\mathrm{Spin}4]/Z_2$ by a Higgs **54** (10×10 symmetric matrices) with base point[32,33]

$$\Phi_0=\mathrm{diag}(2,2,2,2,2,2,-3,-3,-3,-3)\ .$$

$\pi_1(G)\simeq Z_2$. The Lie algebra $\mathscr{G}=so(6)\times so(4)$ has trivial center so G is never implementable. Let us consider the (stable) elementary monopole given by $Q=(J_{56}-J_{78})/2$, where the J_{ab} are the usual rotation generators [antisymmetric, imaginary, 10×10 matrices $(J_{ab})_{ij}=-i(\delta_{ai}\delta_{bj}-\delta_{aj}\delta_{bi})$]. The only vectors in $\mathcal{G}$ which commute with Q are the multiples of Q, so the maximal symmetry algebra is the one generated by Q. This is the only choice of electromagnetic direction. Electric and magnetic charges are quantized in units

$$q_{\min}=e/|Q|=e/2,\quad g=|Q|/2e=1/e\ , \tag{8.22}$$

so the original (integer) Dirac condition is satisfied. Implementability and integer Dirac condition are hence different in this case.

IX. THE NON-ABELIAN AHARONOV-BOHM EXPERIMENT

Another tricky example is provided by the non-Abelian Aharonov-Bohm experiment proposed by Wu and Yang in their celebrated paper on the nonintegrable phase factor.[20] They suggest in fact to set up an SU(2)-gauge field confined to a cylinder. If a nucleon beam is scattered around this flux line, a nontrivial interference would prove the existence of Yang-Mills fields.

It is not difficult to show[21,34] that there exists a gauge—analogous to the U gauge (5.2) for monopoles—where the gauge field with $F_{ij}=0$ in $M=R^3\setminus\{\text{cylinder}\}$ is simply

$$A_r=0,\quad A_\theta=0,\quad A_\phi=\alpha\sigma_3/i\ . \tag{9.1}$$

α here is a real parameter, defined modulo integers.

Let us try to implement $G=\mathrm{SU}(2)$ by an $\mathrm{Aut}G_0$-valued "Higgs" field $\tau_{(\,)}(x)$ on M. As explained in Sec. III, we can gauge any such $\tau_g(x)$ to identically g simultaneously in

$$V_1=\{(r,\theta,\phi)\,|\,0\le\phi<\pi+\epsilon\}$$

and

$$V_1=\{(r,\theta,\phi)\,|\,\pi-\epsilon<\phi\le2\pi\}$$

since both V_1 and V_2 are contractible. The price to pay for this is that we introduce a transition function h—which is now just a constant element of SU(2). Consistency requires now $gh=hg$, h must be in the center of SU(2). So we have two solutions: $h=1$ or $h=-1$. We conclude that, although there is no obstruction to implement $G=\mathrm{SU}(2)$, there is an ambiguity. In the U gauge (9.1) the two implementations are found explicitly as either

$$\tau_g{}^1(x)=g \tag{9.2}$$

or

$$\tau_g{}^2(x)=\begin{bmatrix}\exp(i\phi/2) & 0\\ 0 & \exp(-i\phi/2)\end{bmatrix}\times g\begin{bmatrix}\exp(-i\phi/2) & 0\\ 0 & \exp(i\phi/2)\end{bmatrix}=\begin{bmatrix}g_{11} & \exp(i\phi)g_{12}\\ \exp(-i\phi)g_{21} & g_{22}\end{bmatrix}, \tag{9.3}$$

where $x=(r,\theta,\phi)$ and $g=(g_{ij})$ a matrix.

Are these implementations symmetries? The corresponding local expressions read

$$\omega_\eta^1=\eta,\quad \eta\in\mathrm{su}(2) \tag{9.4}$$

and

$$\omega_\eta^2=\begin{bmatrix}\eta_{11} & \exp(i\phi)\eta_{12}\\ \exp(-i\phi)\eta_{21} & \eta_{22}\end{bmatrix},\quad \eta\in\mathrm{su}(2)\ . \tag{9.5}$$

To be an internal symmetry, ω_η must be covariantly constant. However,

$$D\omega_\eta^1(x)=\frac{\alpha}{i}[\sigma_3,\eta]=\frac{1}{i}\begin{bmatrix}0 & 2\alpha\\ 2\alpha\eta_{21} & 0\end{bmatrix} \tag{9.6}$$

and

$$D\omega_\eta^2(x)=\frac{1}{i}\begin{bmatrix}0 & (2\alpha-1)\eta_{12}\\ -(2\alpha-1)\eta_{21} & 0\end{bmatrix}, \tag{9.7}$$

respectively. We conclude that either (i) 2α is neither 0 nor 1, and then $\eta_{12}=\eta_{21}=0$ so the only symmetry direction is the one given by the field (9.1) itself; or (ii) $2\alpha=0$ or $2\alpha=1$. The whole SU(2) is then a symmetry. However, for $\alpha=0$ only the implementation (9.6), for $\alpha=\frac{1}{2}$ only the implementation (9.7) is a symmetry.

These, at first sight rather abstract, statements have a fundamental importance. Indeed, the nucleons in the generalized Aharonov-Bohm experiment can be viewed as test particles moving in a background YM vacuum.[21,34] How can we tell which of the nucleons is a proton, which is a neutron? As emphasized by Yang and Mills in the very first paper on gauge theory,[35] this can be done only by measuring the electric charge. The charge of the particle alone can be conserved however, only if the background field has symmetries. In case (ii) the choice of the electromagnetic direction is ambiguous. To choose (9.5) is however unphysical, since it would imply charge nonconservation also for a free particle. So the correct choice is (9.5). Interestingly, for $\alpha=\frac{1}{2}$, the electric charge of the particle is actually not conserved: protons can be turned to neutrons.[20,21,34]

REMARKS

Using the same technique as for monopoles, one can show[36] that, for an SU(2) instanton, $G=\mathrm{SU}(2)$ is never implementable. The physical consequences of this fact are not entirely clear however.

The theory outlined in this paper admits a nice fiber-bundle interpretation. This is explained in a companion paper.[23]

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