PHYSICAL REVIEW D VOLUME 32, NUMBER 4 15 AUGUST 1985

Impossibility of supersymmetry restoration in quantum-mechanical systems

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(Received 4 January 1985)

We show that supersymmetry restoration is not possible for a large class of O'Raifeartaigh-type quantum-mechanical models. The vacuum energy is bound from below by its classical value. An exception is discussed.

I. INTRODUCTION

Supersymmetry, in particular, local supersymmetry, is a very beautiful mathematical construct; it contains in it some very suggestive solutions to phenomenological questions such as the hierarchy problem. The possibility that it may lead to a well-defined theory of quantum gravity is actively studied. Whatever awaits us at shorter distances, supersymmetry is definitely broken at an energy scale below a few GeV. The breaking of supersymmetry may be parametrized by considering softly broken theories of supergravity. A considerable amount of effort, however, was devoted to construct a realistic model in which global supersymmetry is spontaneously broken at a variety of scales. In some of these models the existence of symmetry breaking is identified at the tree level; the classical potential turns out to be positive definite. In this paper we wish to investigate the possibility that, although it may seem by inspection of the classical potential that supersymmetry is broken, the exact ground state has zero energy. This would maintain a zero Witten index. If this occurs, it would be the property of the exact solution and not of a small perturbation. In the absence of supersymmetry in nature our interest in restoring it requires some explanation. For compact bosonic symmetries it is the case that nonperturbative effects may restore a broken symmetry;[1] these phenomena are associated with nontrivial topological properties of the manifold. They result with the formation of a singlet ground state whose energy, in the semiclassical approximation, deviates only slightly from the perturbative energy.

In the case of supersymmetry the return to a singlet ground state requires a finite change in the ground-state energy. It is for supersymmetry breaking that it suffices that the vacuum energy change by a very small amount.[2] The question is whether or not the flat potentials employed to break supersymmetry spontaneously are unstable against an expansion in h. We will try to use the advantage that some exact statements can be made in quantum-mechanical supersymmetry.[2] For bosonic models, quantum-mechanical systems are the most aggressive in restoring symmetry. It was suggested in the past[3] that supersymmetry is restored in the Wess-Zumino model. A more careful theoretical analysis[4] found no evidence for this; a Monte Carlo study of two-dimensional supersymmetrical systems with one real scalar field also did not detect supersymmetry restoration.[5] It was also shown that to all orders in perturbation theory supersymmetry breaking by an F-term is not restored in four dimensions.[6] In Sec. II we find that for systems in which spontaneous supersymmetry breaking is achieved with the help of a flat tree-level potential, supersymmetry is not restored. In Sec. III we describe a class of models for which the symmetry is restored; these systems have, however, a compact flat direction and singular features.

The supersymmetry quantum-mechanical system that contains an equal number, N, of bosons and fermions is given by the Hamiltonian

$$H=\frac{1}{2}\left[\sum_{\alpha=1}^{N} p_\alpha{}^2+\left[\frac{\partial W}{\partial x_\alpha}\right]^2-2\sum_{\alpha\beta}B_{\alpha\beta}\frac{\partial^2 W}{\partial x_\alpha \partial x_B}\right]$$
$$=\tfrac{1}{2}\{Q,Q^\dagger\}\ , \tag{1.1}$$

where $W(\mathbf{x})$ is the superpotential and

$$B_{\alpha\beta}=\tfrac{1}{2}[\psi_\alpha^\dagger,\psi_\beta]\ ,$$
$$\{\psi_\alpha^\dagger,\psi_\beta\}=\delta_{\alpha\beta}\ ,\quad \{\psi_\alpha,\psi_\beta\}=\{\psi_\alpha^\dagger,\psi_\beta^\dagger\}=0\ . \tag{1.2}$$

The supersymmetry charge Q is given by

$$Q=\sum_{\alpha=1}^{N}\psi_\alpha^\dagger\left[-p_\alpha+\frac{\partial W}{\partial x_\alpha}\right]\ . \tag{1.3}$$

II. O'RAIFEARTAIGH-TYPE MODELS

A. The $N=1$ case

The simplest case to study is $N=1$ quantum mechanics. In that case Q_1 and H are given by

$$Q_1=\sigma_1 p+\sigma_2\frac{\partial W}{\partial x}\ ,$$
$$H=\frac{1}{2}\left[p^2+\left[\frac{\partial W}{\partial x}\right]^2\right]1+\tfrac{1}{2}\sigma_3\frac{\partial^2 W}{\partial x^2}\ . \tag{2.1}$$

The classical potential $V_{\rm cl}$ is defined to be

$$V_{\rm cl}=\left[\frac{\partial W}{\partial x}\right]^2\ . \tag{2.2}$$

We wish to investigate if it is possible that $V_{\rm cl}>0$ and yet supersymmetry is unbroken. For W of a polynomial form we know that the answer is negative. A necessary condition to have a positive definite $V_{\rm cl}$ is that the leading power x^n in W be odd, but in such a case Witten has shown that supersymmetry is broken. More generally, a positive $V_{\rm cl}$ implies, if W has a continuous derivative, that W is a monotonic function. The zero-energy wave function ϕ_0 always exists and is given by

$$\phi_0(x)=c\exp[\pm W(x)] .$$

The sign is chosen so that $\phi_0(x)$ is normalizable. This cannot be achieved for any monotonic $W(x)$. The closest one can get is to a plane-wave normalizable ground state. That is achieved if

$$\lim W(x) \underset{|x|\to\infty}{\longrightarrow} \text{const.} \tag{2.3}$$

In that case the classical potential tends to zero at infinity. A plane-wave normalizable wave functional would correspond to the absence of a mass scale; large field values are not suppressed. (For W with a discontinuous derivative, a formal zero-energy solution exists.) In higher dimensions, however, the minimal supersymmetrical realizations contain at least one complex scalar.

We thus turn to quantum-mechanical systems that contain more degrees of freedom. The spontaneous breaking of supersymmetry in field theories containing scalars and fermions is obtained through the O'Raifeartaigh mechanism.[7] For complex scalars the mechanism becomes only operative in the presence of three supermultiplets. In that case one can construct a positive-definite classical potential that has flat directions. We wish to abstract that feature and study multicomponent quantum-mechanical supersymmetrical systems whose classical potential has flat directions. This can be achieved already for $N=2$.

B. The $N=2$ case

$N=2$ supersymmetric quantum mechanics contains two bosonic and two fermionic degrees of freedom. The states of the system are given by four functions $\phi(n_1,n_2,x,y)$ ($n_1=0,1$; $n_2=0,1$) describing the amplitude to measure values x and y provided one is in a state containing n_1 fermions of type 1 and n_2 fermions of type 2. In the language of forms $\phi(0,0,x,y)$ is a zero-form, $\phi(1,1,x,y)$ is a two-form and the other two functions constitute a one-form. Q and $Q^\dagger$ can be represented in this basis by

$$Q=\hat{\psi}_i e^{-W}\partial_i e^{W} , \quad Q^\dagger=\hat{\psi}_i^\dagger e^{W}\partial_i e^{-W} . \tag{2.4}$$

$W(x,y)$ is the superpotential. The $\hat{\psi}_i$, $\hat{\psi}_i^\dagger$ obey

$$\{\hat{\psi}_i,\hat{\psi}_j^\dagger\}=\delta_{ij} , \quad \{\hat{\psi}_i,\hat{\psi}_j\}=0 . \tag{2.5}$$

They are chosen to be

$$\hat{\psi}_1=\begin{bmatrix}0&0&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&0\end{bmatrix} , \quad \hat{\psi}_2=\begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&0&0&0\\1&0&0&0\end{bmatrix} . \tag{2.6}$$

The Hamiltonian $H=\frac{1}{2}\{Q,Q^\dagger\}$ conserves the total number of fermions and thus factorizes into three blocks,

$$H=\begin{bmatrix}H_0&&\\&H_1&\\&&H_2\end{bmatrix} , \tag{2.7}$$

where H_0 is a one-dimensional matrix acting on zero-forms, H_1 is a two-dimensional matrix acting on one-forms, and H_2 is a one-dimensional matrix acting on two-forms. The explicit form of the H_i is

$$\begin{aligned} H_0&=\tfrac{1}{2}[-\Delta+(\nabla W)^2-\Delta W] ,\\ H_1&=\tfrac{1}{2}[-\Delta+(\nabla W)^2-\Delta W]1+2\begin{bmatrix}\partial_{11}W&\partial_{12}W\\\partial_{21}W&\partial_{22}W\end{bmatrix} ,\\ H_2&=\tfrac{1}{2}[-\Delta+(\nabla W)^2+\Delta W] . \end{aligned} \tag{2.8}$$

The number of zero-energy states in each p-form sector is also called the Betti number B_p. They are the solutions ψ to the equations

$$Q\psi=0 , \quad Q^\dagger\psi=0 , \tag{2.9}$$

which themselves are not given by $\psi=Q\tilde{\psi}$ or $\psi=Q^\dagger\tilde{\psi}$. In the zero-form and two-form sectors the equations are

$$\begin{aligned} &\nabla e^{W}\phi(0,0,x,y)=0 ,\\ &\nabla e^{-W}\phi(1,1,x,y)=0 . \end{aligned} \tag{2.10}$$

These equations are solved to give

$$\begin{aligned} &\phi(0,0,x,y)=ce^{-W} ,\\ &\phi(1,1,x,y)=ce^{+W} . \end{aligned} \tag{2.11}$$

The properties of W will determine if the functions are normalizable, if $|W|$ is increasing at infinity and W has a well-defined sign there then a normalizable solution exists, if W tends to a constant at infinity it is plane-wave normalizable. The O'Raifeartaigh potential we have in mind is

$$W_R(x,y)=x(y^2+c) . \tag{2.12}$$

It leads to the classical potential

$$V_{\rm cl}=\tfrac{1}{2}[(y^2+c)^2+4x^2y^2] . \tag{2.13}$$

For $c>0$ the minimum of $V_{\rm cl}$ is $c^2/2>0$. That value is obtained along the line $y=0$. For W_R there is no normalizable, zero-form or two-form, zero-energy state. In the one-form sector a zero-energy state satisfies the equations

$$\begin{aligned} &\partial_1 e^{W}\psi_2=\partial_2 e^{W}\psi_1 ,\\ &\partial_1 e^{-W}\psi_1=-\partial_2 e^{-W}\psi_2 . \end{aligned} \tag{2.14}$$

These equations are called pseudoanalytic equations by mathematicians; we do not know the general classification of their solutions, in the case when W is polynomial. However, assuming that in this case the Witten index can be calculated reliably in the classical limit, one can argue that no solutions exist to Eq. (2.14) for W_R given by Eq. (2.12). In the classical limit the index is zero. We have shown, by studying the zero-forms and two-forms, that no bosonic zero-energy solutions exist. Thus there are no zero-energy fermionic solutions as well. For the case $N=2$ there is no supersymmetry restoration if neither $e^{\pm W}$ are normalizable. To strengthen our proof and due to the fact that for $N>2$ we do not have such a general argument, we shall present also a variational argument that for a class of potentials that includes W_R there is no supersymmetry restoration.

C. A variational argument against supersymmetry restoration

In a supersymmetric quantum-mechanical system with N degrees of freedom the Hamiltonian is block diagonal in the form basis. As the total fermion number is conserved H acts only within a given p-form sector. The expectation value of the Hamiltonian in the p-form sector is given by

$$E_p[\phi_p]=\frac{\langle\phi_p\,|\,H\,|\,\phi_p\rangle}{\langle\phi_p\,|\,\phi_p\rangle}\ . \tag{2.15}$$

Assume that the superpotential W has a global direction $\mathbf{n}$ such that $(\mathbf{n}\cdot\nabla W)>0$ for all $\mathbf{x}$. Note that this is a sufficient condition for a positive $V_{\rm cl}$. Consider now the super potential $W^{(\lambda)}=W+\lambda n_i x_i$, where λ is a real number. The expectation value of $H(W^{(\lambda)})$ in ϕ_p is given by

$$E^{W^{(\lambda)}}[\phi_p]=E^{W^{(0)}}+\frac{\lambda^2}{2}+\lambda\frac{\int d^N x(\mathbf{n}\cdot\nabla W)\phi_p{}^2}{\int d^N x\,\phi_p{}^2}\ . \tag{2.16}$$

The main point to note is that the contribution from the "Yukawa" term is independent of λ.

If we now assume that $H(W^{(0)})$ has a zero-energy state we will run into a contradiction. On the one hand, since H is semipositive definite $E^{W^{(\lambda)}}\geq 0$; on the other hand, if one assumes $E^{W^{(0)}}=0$ for ϕ_p one has

$$E^{W^{(\lambda)}}=\frac{\lambda^2}{2}+\lambda\langle\mathbf{n}\cdot\nabla W\rangle\ . \tag{2.17}$$

By the assumption $\langle\mathbf{n}\cdot\nabla W\rangle$ is positive if it is also finite, one can always choose such a λ that $E^{W^{(\lambda)}}<0$.

For W_R choosing $\mathbf{n}=\hat{\mathbf{x}}$ one has

$$(\hat{\mathbf{x}}\cdot\nabla W)=y^2+c\ , \tag{2.18}$$

which is positive for all x,y as long as $c>0$. We have thus shown that for a flat potential of the O'Raifeartaigh type supersymmetry is not restored. This result holds also for $N>2$; for example, for $N=3$ a flat potential is derived from $W_R=ayz+bx(y^2+c)$, this potential has monotonic direction such that $\mathbf{n}\cdot\nabla W>0$, and therefore does not possess a zero-energy solution.

From the variational treatment one can actually set a lower bound to the ground-state energy. Substituting c of Eq. (2.12) instead of λ in Eq. (2.14), and calculating the expectation value of H in the ground state of $H(W^{(c)})$ one obtains

$$E^{W^{(c)}}=\langle\phi_p(W^{(c)})\,|\,H(W^{(0)})\,|\,\phi_p(W^{(c)})\rangle+\frac{c^2}{2}+c\langle\phi_p(W^{(c)})\,|\,\mathbf{n}\cdot\nabla W\,|\,\phi_p((c))\rangle\ . \tag{2.19}$$

For a positive c one has

$$E^{W^{(c)}}=\frac{c^2}{2}+b^2\ , \tag{2.20}$$

where b^2 is a positive number, that is, the exact ground-state energy of the flat potential W_R is larger than the classical value of the energy $c^2/2$. Recent Monte Carlo calculations[5] in a two-dimensional supersymmetric field theory are in agreement with this result.

III. AN EXAMPLE OF SUPERSYMMETRY RESTORATION

We conclude with an example for which supersymmetry is indeed restored. A flat noncompact potential was unable to sidestep the h expansion, that can be, however, achieved with a slightly singular potential W. Its crucial property will be that while the classical potential (the gradient squared of W) is finite and positive everywhere, the Yukawa term (essentially the Laplacian of W) is somewhat singular at the origin. This will be enough to maintain a well-defined Hamiltonian in which supersymmetry is restored. For a general N ($N>1$ however) we study as an example the radially symmetric potential

$$W(r)=\frac{r^2}{2}+g\ln(r+a^2)\ . \tag{3.1}$$

This superpotential leads to the classical potential

$$V_{\rm cl}(r)=\left[r+\frac{g}{r+a^2}\right]^2\ . \tag{3.2}$$

This potential has a minimum for $r=\sqrt{g}-a^2$ if $g>a^4$. The value of the potential at that distance is positive. For $0<g<a^4$ the minimum is at $r=0$ and is still positive. In a way this is a flat potential, but unlike higher-dimensional supersymmetric theories, this flat direction is compact. Affleck[8] has shown that for superpotentials $W(x_1,\ldots,x_N)$, which are O(N) symmetric, there are only four possible candidates for a zero-energy state. They are an O(N) singlet, zero-form, one-form, $(N-1)$-form, and N-form. For W as given in Eq. (3.1), there exists a zero-form, zero-energy, normalizable state for g positive. The form of this potential does not allow the application of the variational argument. For negative g supersymmetry is unbroken at the tree level. The classical value of the index is zero. All this is due to the singular nature of the potential. The zero-form wave function is given by

$$\phi_N(r)=\frac{\exp(-r^2/2)}{(r+a^2)^g}\ . \tag{3.3}$$

IV. DISCUSSION

We have found no evidence for supersymmetry restoration in quantum-mechanical models that have a flat polynomial potential. Actually we have found that in these models the tree-level value of the potential serves as a lower bound to the true ground-state energy. We have constructed systems in which such a restoration occurs; however, they have no obvious field-theoretical counterparts.

ACKNOWLEDGMENTS

We wish to thank S. Elitzur and V. Rittenberg for discussions. This work was supported in part by the Israeli Council for Basic Research.

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