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Classical spin and its quantization

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We develop a new formulation of the quantum mechanics of a relativistic fermion based on the existence of an extended set of underlying classical coordinates. We enlarge the classical theory to a superspace built out of ordinary space-time coordinates and a set of Grassmann coordinates which transform like Majorana spinors and describe a classical spin. We introduce a natural supersymmetry between the Grassmann and the space-time coordinates. We quantize the theory both canonically and via a relativistic particle-mechanics path-integral prescription. The quantization of the motion of the coordinates in the superspace gives rise to the standard Dirac structure of a relativistic quantum-mechanical particle with spin one-half.

I. INTRODUCTION

Half-integral spin occupies a somewhat anomalous position in quantum mechanics since, unlike position and momentum and even unlike integral angular momentum, it appears to possess no classical analog. For this reason half-integral spin is generally regarded as being intrinsically quantum mechanical in origin. Since integral spin can be introduced into quantum theory by a canonical quantization of a purely classical spin it would therefore be of some interest to be able to introduce half-integral spin into quantum theory by an analogous procedure.

In order to try to find a way to do this we recall how the standard spin-one-half theory was developed. The postulates of quantum mechanics replace the real classical coordinates and momenta q_i and p_j (which are ordinary commuting numbers which obey

$$q_i p_j - p_j q_i = 0 \tag{1.1}$$

for all i,j) by operators $\hat{q}_i$ and $\hat{p}_j$ which obey

$$\hat{q}_i \hat{p}_j - \hat{p}_j \hat{q}_i = i\hbar\delta_{ij} \ . \tag{1.2}$$

The observable real classical c numbers q_i and p_j are then the eigenvalues of the Hermitian operators $\hat{q}_i$ and $\hat{p}_j$, viz.,

$$\hat{q}_i \,|\, q_i\rangle = q_i \,|\, q_i\rangle \ , \quad \hat{p}_j \,|\, p_j\rangle = p_j \,|\, p_j\rangle \ . \tag{1.3}$$

This quantization of the motion of the classical trajectory in the p,q phase space in turn entails the quantization of the components of the angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ according to

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \tag{1.4}$$

and leads to the quantization of integral spin, with the associated $|\, l,m\rangle$ eigenvectors being given in the coordinate representation by the spherical harmonics, viz.,

$$\langle \theta,\phi \,|\, l,m\rangle = Y_l^m(\theta,\phi) \ . \tag{1.5}$$

Thus the quantization of position, momentum, and integral angular momentum is completely canonical, and has a direct connection to the motion of a classical particle in phase space.

It was noted by Pauli that the angular momentum commutation relations of Eq. (1.4) also admit of complex half-integer spin representations in which the operators additionally also satisfy anticommutation relations (restricting to spin one-half)

$$S_i S_j + S_j S_i = \frac{\hbar^2}{2}\delta_{ij} \tag{1.6}$$

and have Hilbert-space eigenvectors $|\, S,S_z\rangle$. Unlike the integral-spin case it is not possible to attach a meaning to $\langle \theta,\phi \,|\, S,S_z\rangle$.

Attempts to produce a classical picture of half-integral spin centered on a mechanical picture of a particle possessing an intrinsic spin as it rotated about its own axis (thus unfortunately giving spin its misleading name). For a point classical electron moving in a circular orbit of radius r with a period T the angular momentum is given by $L = mvr = 2\pi m r^2/T$, while the magnetic moment $\mu = IA = \pi r^2 e/T$, so that

$$\mu = \frac{eL}{2m} \ . \tag{1.7}$$

Thus the mechanical picture predicts a gyromagnetic ratio for spin equal to one. Experimentally the g factor is close to two, and so the mechanical picture was ruled out. Moreover, it was laid to rest completely when Dirac calculated the correct g factor from his wave equation, thus establishing both the relativistic and the quantum-mechanical nature of spin.

Since Dirac's work it has generally been thought that half-integral spin is intrinsically quantum mechanical and that it possesses no classical analog. However, the above discussion of the g factor and the lack of any meaning to $\langle \theta,\phi \,|\, S,S_z\rangle$ only in fact indicate that there is no appropriate classical limit which can be associated with a motion in coordinate space. The analysis does not preclude motion in some other type of classical space. In particular, noting the connection between Eqs. (1.2) and (1.1), the anticommutation relations of Eqs. (1.6) suggest looking at spaces whose coordinates obey

$$\alpha_i\beta_j+\beta_j\alpha_i=0\ . \tag{1.8}$$

Such spaces exist and are nonempty. They were first studied by Grassmann and are known as Grassmann spaces with the α_i and β_j being the Grassmann coordinates. In this paper we shall construct and study such a space and show that the quantization of the motion of a classical Grassmann particle in Grassmann space leads to the standard Dirac theory of a spin-one-half particle. Thus just as the existence of quantization of energy obliged us to change the space appropriate for physics from coordinate space to Hilbert space, we see that the existence of half-integral spin requires us to extend the space of classical theory to include Grassmann space as well.

In setting up an appropriate classical Grassmann space we need to specify how the Grassmann coordinates transform under Lorentz transformations. Since classical physics deals with real quantities only, we must require that the Grassmann coordinates only acquire a real phase under an arbitrary Lorentz transform, and that they be Hermitian in the Grassmann space itself. Thus it is immediately suggested[1] to introduce a set of four Grassmann coordinates ξ_μ ($\mu=0,1,2,3$), which transform the same way under the Lorentz group as the space-time coordinates x_μ [i.e., according to the real $D(\frac{1}{2},\frac{1}{2})$ vector representation] and which obey

$$\xi_\mu\xi_\nu+\xi_\nu\xi_\mu=0 \tag{1.9}$$

for all μ,ν. A canonical quantization of Eq. (1.9) would replace the coordinates by Hilbert-space operators $\hat{\xi}_\mu$ which obey

$$\hat{\xi}_\mu\hat{\xi}_\nu+\hat{\xi}_\nu\hat{\xi}_\mu=2\hbar g_{\mu\nu} \tag{1.10}$$

in an appropriate normalization. This is initially very encouraging since the $\hat{\xi}_\mu$ satisfy the same algebra as the Dirac gamma matrices γ_μ, viz.,

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}\ . \tag{1.11}$$

However, the difficulty with Eq. (1.10) is that it admits of no analog to Eq. (1.3). Specifically, according to Eq. (1.10) each $\hat{\xi}_\mu$ obeys $\hat{\xi}_\mu{}^2=\pm\hbar$. Hence the eigenvalues of $\hat{\xi}_\mu$ could not be the ξ_μ Grassmann numbers of Eq. (1.9) since they obey $\xi_\mu{}^2=0$. We must therefore seek another set of coordinates.

Now it was noted by Majorana that the real $SO(3,1;R)$ Lorentz group actually admits of two inequivalent real irreducible four-dimensional representations, namely the above $D(\frac{1}{2},\frac{1}{2})$ representation and also one other representation, the Majorana representation, which transforms like $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$ and acts on Hermitian Majorana spinors ψ_α ($\alpha=1,2,3,4$). For such Majorana spinors $\psi_\alpha^T C_{\alpha\beta}\psi_\beta$ is Lorentz invariant, where $C_{\alpha\beta}$ is the transposition matrix for the Dirac gamma matrices. Since $C_{\alpha\beta}$ is antisymmetric, $\psi_\alpha^T C_{\alpha\beta}\psi_\beta$ is only nonzero if the components of ψ_α anticommute with each other, and so the Majorana spinors are just the natural candidates that we are seeking for our desired Grassmann coordinates. Thus our classical Grassmann space will be built out of anticommuting Majorana spinors. For such Majorana spinors we will find that it is possible to construct an analog of Eq. (1.3) and effect a complete canonical quantization.

It is important to note here the role played by the relativistic Minkowski metric. Specifically, the four-dimensional rotation group $SO(4;R)$ only admits one real irreducible four-dimensional representation, namely, the one which acts on the four space coordinates. (The Majorana representation would become complex under the change in the metric.) Thus the existence of the real Majorana representation is intimately connected with the Minkowski nature of the metric. Thus, unlike the Dirac (or Pauli) theories, spinors already enter our theory prior to quantization.

The viewpoint that we are developing here is based on the primacy of particles and coordinates over waves and fields. Specifically, we first specify a classical particle mechanics described by the motion of classical coordinates x_μ and ψ_α under an $\hbar$-independent Lagrangian and then quantize it. We never refer to fields $\psi_\alpha(x_\mu)$ at all with ψ_α being independent of and on an equal footing with x_μ.[2] To effect the actual quantization of the classical theory it is convenient to use Feynman's path-integral formulation of quantum mechanics. To this end we note that there are two main cases in which path-integral quantization is usually considered in the literature. Specifically, Feynman path-integral quantization is usually only applied either to nonrelativistic classical particles or to relativistic first-quantized fields. In the nonrelativistic case we start with purely classical particles whose Lagrangian ($m\dot{x}^2/2$) is purely classical, and then interfere the various paths via the path integration to obtain the nonrelativistic quantum propagator. In this first quantization (particles into waves) the only reference to Planck's constant is in the explicit $\hbar$ factor in the $iS_{\rm CL}/\hbar$ phase associated with each path. In the relativistic case we start with c-number fields (i.e., fields whose commutators or anticommutators are independent of $\hbar$) but use a Lagrangian (Klein-Gordon or Dirac) which depends on $\hbar$ explicitly, i.e., we start with a theory which is already first quantized. The path integration over the c-number fields then yields the relativistic quantum propagator to give a second quantization (fields into particles) of the theory. Thus, in the main the literature appears to ignore a possible relativistic first quantization of purely classical particle coordinates which obey relativistic kinematics and possess a purely classical (i.e., $\hbar$ independent) Lagrangian. For our Grassmann Majorana spinor coordinates such a formulation is crucial and the bulk of the work in this paper will be in setting up and then actually integrating the appropriate Feynman path integral. To this end in turn we will need first to develop the path-integration formalism for the relativistic space-time coordinates x_μ.

The present paper is therefore organized as follows. First, in Sec. II, we study the Feynman path-integration prescription for the coordinates x_μ by developing a five-dimensional formalism. The actual path integration then yields the Klein-Gordon propagator. In Sec. III we study the path-integral quantization of the pure Grassmann sector of the theory, and then in Sec. IV present its canonical quantization. In Sec. V we study the Feynman path-integral quantization in the complete superspace of the combined Majorana spinor and space-time sectors of the theory to obtain the Dirac propagator. Finally, in Sec. VI we present some general comments on our work. We have already presented some of our results briefly[3] and in this paper we give the details.

II. FEYNMAN PATH INTEGRATION FOR SPACE-TIME COORDINATES

In this section we quantize the relativistic classical mechanics of the four space-time coordinates by path integration. Our philosophy here and throughout will be to first identify the correct underlying classical particle theory by constructing the space of paths needed to apply Hamilton's variational principle in the classical theory, and to then integrate over the self-same set of paths in order to obtain the quantum propagator. Thus all the requisite information needed to specify the measure for the Feynman path integral must be contained in the classical theory and Planck's constant must only enter via the $iS_{\rm CL}/\hbar$ phase factor associated with each classical path.

The usual classical action for a free spinless relativistic particle is

$$S=-m\int ds=-m\int dt(1-\mathbf{v}^2)^{1/2} . \tag{2.1}$$

Its noncovariant variation with respect to $\mathbf{x}(t)$ gives the equation of motion $\mathbf{v}_{\rm ST}=\text{const}$ (ST denotes stationary). For the purposes of path integration we will need a covariant measure which treats $\mathbf{x}$ and t equivalently and so we must rewrite the action in a more convenient form. To this end we express the Hamiltonian form of the action, viz.,

$$S=\int dt[\mathbf{p}\cdot\mathbf{v}-(\mathbf{p}^2+m^2)^{1/2}] , \tag{2.2}$$

where $\mathbf{p}=\partial\mathscr{L}/\partial\mathbf{v}$ in the manifestly covariant form [here and throughout we use the metric $g_{\mu\nu}=(1,-1,-1,-1)$]

$$S=-\int ds\, p_\mu \frac{dx^\mu}{ds} . \tag{2.3}$$

Though now covariant the above action is a constrained action since $ds=dt(1-\mathbf{v}^2)^{1/2}$ is the proper time interval and $p_0=(\mathbf{p}^2+m^2)^{1/2}$. Since we would like to be able to perform a free covariant variation to obtain the equations of motion in the classical theory we must remove these constraints. Thus we shall reinterpret the action as an action in the five-dimensional space introduced by Feynman[4] and Nambu[5] in which ds is replaced by a Lorentz scalar parameter $d\tau$ which can then be varied independently of dx_μ ($d\tau$ is thus not the proper time), in which p_0 is allowed to vary independently of $\mathbf{p}$, and in which an extra $(p^2-m^2)/2m$ kinetic energy term is introduced. With these requirements our new 5-space action takes the form

$$S=\int d\tau \left[-p_\mu(\tau)\frac{dx^\mu(\tau)}{d\tau}+\frac{p_\mu(\tau)p^\mu(\tau)-m^2}{2m} \right] . \tag{2.4}$$

Trajectories in the 5-space are given by the sets of functions $x_\mu(\tau)$ and $p_\mu(\tau)$. While there is a connection between $\mathbf{x}$ and t for the stationary path associated with the 4-space action of Eq. (2.1), we note that for the arbitrary 4-space path $\mathbf{x}$ and t are independent, and we thus need some additional parameter to specify the arbitrary path. The parameter τ then serves this purpose. The parameter τ serves to parametrize covariant trajectories $x_\mu(\tau)$ just as the ordinary time t serves to parametrize trajectories $\mathbf{x}(t)$ in nonrelativistic classical mechanics. Indeed, while the 5-space action is Lorentz covariant in the four-dimensional space associated with the x_μ and p_μ variables, it is not five-dimensionally covariant in the full 5-space x_μ and τ variables, but rather it is "nonrelativistic" in τ.

Having now specified the 5-space action we vary it freely in a Lorentz-covariant manner between end points $(x_\mu^i, \tau=0)$ and $(x_\mu^f, \tau=T)$. The independent phase-space variations of $x_\mu(\tau)$ and $p_\mu(\tau)$ yield the equations of motion

$$x_\mu^{\rm ST}(\tau)=\frac{(x_\mu^f-x_\mu^i)\tau}{T}+x_\mu^i ,$$
$$p_\mu^{\rm ST}(\tau)=\frac{m(x_\mu^f-x_\mu^i)}{T} . \tag{2.5}$$

In the stationary path the ordinary velocity 3-vector

$$\mathbf{v}_{\rm ST}=\frac{(\mathbf{x}_f-\mathbf{x}_i)}{(t_f-t_i)} \tag{2.6}$$

is constant as it should be. Additionally we note that $\mathbf{v}_{\rm ST}$ is independent of T. Thus all values of T lead to the usual classical minimum associated with the action of Eq. (2.1) with the magnitude of T thus being unobservable. The parameter T is thus only a convenient intermediate parameter and must be integrated out in the 5-space path-integral measure. Finally, we note that in the stationary path the stationary action takes the value

$$S_{\rm ST}=-\frac{m(x_\mu^f-x_\mu^i)^2}{2T}-\frac{mT}{2} . \tag{2.7}$$

Now while τ is not the proper time for the arbitrary path we can still identify it with the proper time for the stationary path. Thus if we were to normalize T to the stationary proper time by setting $(x_\mu^f-x_\mu^i)^2=T^2$, we would then recover the constraint $(p_\mu^{\rm ST})^2=m^2$. Additionally the stationary action would take the value $S_{\rm ST}=-mT$, so that we would then also recover the standard value for $S_{\rm ST}$ that would be obtained by varying the action of Eq. (2.1) noncovariantly with respect to $\mathbf{x}(t)$. This then normalizes the value of the coefficient of the (p^2-m^2) term in Eq. (2.4) to $1/2m$.

To obtain the Feynman propagator we now simply integrate back over all the paths needed for the 5-space variation, viz.,

$$G_{if}(x,T)=G(x_\mu^f,T;x_\mu^i,0)$$
$$=\int[dx][dp]\exp(iS/\hbar) . \tag{2.8}$$

To perform the explicit integration it is convenient to parametrize the arbitrary coordinate path in terms of the stationary path and a complete basis which vanishes at the end points, viz.,

$$x_\mu(\tau)=x_\mu^{\rm ST}(\tau)+\sum_{n=1}^{\infty}a_\mu^n\sin\frac{n\pi\tau}{T} . \tag{2.9}$$

Here the variation coefficients a_μ^n can take on all values between $-\infty$ and $+\infty$. In order to parametrize the arbitrary momentum path we recall that while p_μ is defined as $m\dot{x}_\mu$ for each path $x_\mu(\tau)$ in a Lagrangian formulation, in a Hamiltonian formulation of the action principle $p_\mu(\tau)$ is varied independently of $x_\mu(\tau)$. Since the momentum is not constrained at the end points of the motion the arbitrary momentum path can be parametrized as

$$p_\mu(\tau)=p_\mu^{\rm ST}(\tau)+\sum_{n=0}^{\infty} b_\mu^n \cos\frac{n\pi\tau}{T} . \tag{2.10}$$

Here the coefficients b_μ^n can take on all values between $-\infty$ and $+\infty$. Because of the different boundary conditions we obtain one extra set of coefficients, the $n=0$ terms, for $p_\mu(\tau)$. For the arbitrary path the action is readily evaluated as

$$S=S_{\rm ST}+\frac{Tb_\mu^{02}}{2m}+\sum_{n=1}^{\infty}\left[\frac{Tb_\mu^{n2}}{4m}-\frac{n\pi}{2}b_\mu^n a_n^\mu\right] . \tag{2.11}$$

With this form for the action we confirm that under arbitrary variations of the a_μ^n and b_μ^n, $S_{\rm ST}$ is indeed the stationary action.

In terms of the coefficients a_μ^n and b_μ^n the path-integration measure is given by

$$\int [dx][dp]=A\lim_{N\to\infty}\left[\frac{1}{2\pi\hbar}\right]^{4(N+1)} \times\prod_\mu\prod_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} db_\mu^0 da_\mu^n db_\mu^n , \tag{2.12}$$

where the irrelevant normalization constant A is a pure number. Because of the one extra set of db_μ^0 integrations the measure is not dimensionless. Since the arbitrary action of Eq. (2.11) is quadratic the path integration can be performed analytically to yield (for a suitable choice of A)

$$G_{if}(x,T)=-i\left[\frac{m}{2\pi i\hbar T}\right]^2 \exp(iS_{\rm ST}/\hbar) , \tag{2.13}$$

where $S_{\rm ST}$ is given by Eq. (2.7). The $-i(m/2\pi i\hbar T)^2$ factor is due to the db_μ^0 integrations. Performing a four-dimensional Fourier transform to ordinary momentum space yields the dimensionless propagator

$$\tilde{G}(p,T)=\int d^4(x^f-x^i)G_{if}(x,T)\exp[ip\cdot(x^f-x^i)/\hbar] =\exp\left[\frac{i(p^2-m^2)T}{2m\hbar}\right] , \tag{2.14}$$

which we recognize as being of the form

$$\tilde{G}(p,T)=\exp\left[-\frac{iH_{\rm CL}(p)T}{\hbar}\right] . \tag{2.15}$$

Here $H_{\rm CL}(p)$ is the classical Hamiltonian associated with the action of Eq. (2.4).

As such $\tilde{G}(p,T)$ would be the propagator associated with nonrelativistic (in T) quantum mechanics in the 5-space with T serving as the "time" interval. However, our interest is in recovering the usual Minkowski theory and so, as discussed earlier, we integrate out the unobservable T. Since the proper time ds is intrinsically positive and since the 5-space theory is nonrelativistic in T, we must only consider positive values for $d\tau$ and T. Thus we obtain, finally [using a convenient dimensionless integration measure which takes into account the normalization of Eq. (2.20) below],

$$\frac{-im}{2\hbar}\int_0^\infty dT\,\tilde{G}(p,T)=\frac{m^2}{(p^2-m^2+i\epsilon)} , \tag{2.16}$$

which we recognize as the Klein-Gordon propagator. As well as obtaining the required Klein-Gordon propagator of a free spinless particles of mass m we note that we have also obtained the correct $i\epsilon$ prescription. Indeed, this prescription follows naturally in our formulation since forward propagation in T involves both forward and backward propagation in ordinary time t (see Refs. 4 and 5); just as in ordinary nonrelativistic quantum mechanics forward propagation in t involves forward and backward propagation in $\mathbf{x}$. Hence our theory is a one-body theory in the fifth coordinate T but is many body in the Lorentz coordinates x_μ. Quantum fields are thus not necessary to describe particle creation and annihilation.

It is of interest to compare our analysis with that given by Feynman.[4] Feynman used a first-quantized approach based on an $\hbar$-dependent Klein-Gordon equation in which the important factor $\exp(-imT/2\hbar)$ in the path-integral action was obtained by imposing a quantization eigenvalue condition in T space. Specifically, Feynman intuited the existence of a quantum-mechanical 5-space with a wave function $\phi(x_\mu,T)$ which satisfied

$$i\hbar\frac{\partial\phi(x_\mu,T)}{\partial T}=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right]\phi(x_\mu,T). \tag{2.17}$$

By analogy with the path-integral formulation of ordinary nonrelativistic quantum mechanics he then obtained a 5-space propagator

$$D(x_\mu^f,T;x_\mu^i,0)=-i\theta(T)\left[\frac{m}{2\pi i\hbar T}\right]^2 \times\exp\left[-\frac{im(x_\mu^f-x_\mu^i)^2}{2\hbar T}\right] , \tag{2.18}$$

which obeys the differential equation (f refers to the final x_μ^f)

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{\hbar^2}{2m}\partial_\mu^f\partial_f^\mu\right]D(x_\mu^f,T;x_\mu^i,0)=i\hbar\delta(T)\delta^4(x^f-x^i) . \tag{2.19}$$

[This differential equation thus differs from the one associated with the $G_{if}(x,T)$ propagator of Eq. (2.13), viz.,

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{\hbar^2}{2m}\partial_\mu^f\partial_f^\mu-\frac{m}{2}\right]G_{if}(x,T)=i\hbar\delta(T)\delta^4(x^f-x^i) \tag{2.20}$$

because of the extra $mG_{if}(x,T)/2$ factor.] Feynman then projected out the dependence on T in Eq. (2.17) by defining

$$\phi(x_\mu)=\int_{-\infty}^{\infty} dT\,\phi(x_\mu,T)\exp(-imT/2\hbar) \qquad (2.21)$$

and

$$D(x_\mu^f;x_\mu^i)=-\frac{im}{2\hbar}\int_{-\infty}^{\infty} dT\,D(x_\mu^f,T;x_\mu^i,0)\exp\left[-\frac{imT}{2\hbar}\right]. \qquad (2.22)$$

The four-dimensional wave function $\phi(x_\mu)$ satisfies the Klein-Gordon equation, while the ordinary four-dimensional Fourier transform of $D(x_\mu^f;x_\mu^i)$ of Eq. (2.22) gives the Klein-Gordon propagator of Eq. (2.16). In Feynman's approach the eigenvalue condition of Eq. (2.21) was introduced *a priori* by hand because of the requirement that $\phi(x_\mu)$ obey the first-quantized Klein-Gordon equation. In our approach, on the other hand, no such knowledge of the structure of the first-quantized theory is required, with all the relevant information being contained in the underlying classical theory. Our analysis thus serves to complement Feynman's work.[6]

In order to get some further insight into the role of this fifth coordinate T we recall Nambu's analysis.[5] In a covariant relativistic quantum mechanics the energy and the time become quantum operators since they transform as Lorentz 4-vectors together, respectively, with the three-dimensional momentum and position operators. Consequently, the canonical commutation relations of Eq. (1.2) must be extended to the covariant

$$\hat{x}_\mu\hat{p}_\nu-\hat{p}_\nu\hat{x}_\mu=-i\hbar g_{\mu\nu}\,. \qquad (2.23)$$

Since the time is an operator some other quantity has to be introduced to parametrize the dynamics, namely T, with T being the nonrelativistic variable associated with a five-dimensional Schrödinger equation.[5] In terms of the associated five-dimensional wave function $\psi(x_\mu,T)$ a four-dimensional wave function is then defined as[5]

$$\psi(x_\mu)=\int_{-\infty}^{\infty} dT\,\psi(x_\mu,T) \qquad (2.24)$$

so that the four-dimensional propagator is given by

$$G(x_\mu^f;x_\mu^i)=-\frac{im}{2\hbar}\int_{-\infty}^{\infty} dT\,\theta(T)\langle x_\mu^f\,|\exp(-i\hat{H}T/\hbar)\,|\,x_\mu^i\rangle. \qquad (2.25)$$

Thus for the particular case where the quantum-mechanical Hamiltonian $\hat{H}$ is given by

$$\hat{H}=-\frac{(\hat{p}_\mu\hat{p}^\mu-m^2)}{2m}\,, \qquad (2.26)$$

inserting complete sets of eigenstates of the momentum operator $\hat{p}_\mu$ into Eq. (2.25) yields

$$G(x_\mu^f;x_\mu^i)=-\frac{im}{2\hbar}\int_{-\infty}^{\infty} dT\,\theta(T)\int\frac{d^4p}{(2\pi\hbar)^4}\exp\left[-\frac{iH_{\rm CL}(p)T}{\hbar}-\frac{ip\cdot(x^f-x^i)}{\hbar}\right]. \qquad (2.27)$$

Since Eq. (2.27) gives the same propagator as that of Eq. (2.16) we thus establish the complete equivalence between integrating out T in the wave function using quantum-mechanical information [i.e., Eq. (2.24)] and integrating out T in the path-integral measure using classical information [the redundancy in T in fixing the classical minimum in Eq. (2.6)].

We thus see the need for a fifth coordinate in both the classical theory (where it serves to parametrize the variational paths required for Hamilton's principle) and the quantum theory (where it serves to parametrize the dynamics of operators and states). Since Feynman's path-integral prescription forms a bridge between classical and quantum mechanics it is thus natural that the parameter T will play a role in both theories. As we noted earlier it is only for the stationary classical path that we may identify T with the proper time of classical special relativity. In the path-integral language the nonstationary classical paths represent fluctuations in τ space around the proper time. It is thus the interference of the fluctuations around the proper time which produces the quantum mechanics, with its Schrödinger equation then being parametrized by the same parameter T.

III. GRASSMANN MAJORANA SPINORS

In this section we introduce the Grassmann coordinates appropriate to fermions and quantize their dynamics. We recall that the usual discussion of spin one-half deals with Dirac spinor fields $\psi_\alpha(x)$ which are tensor representations of the Lorentz group which depend on both spin and space. Additionally, a first-quantized $\hbar$-dependent Dirac Lagrangian (with inverse length scale $mc/\hbar$) is used to describe the dynamics. It is our wish here to describe spin one-half by classical coordinates rather than by quantum fields. As we noted in the Introduction the 4-vector ξ_μ is not suitable for this purpose, and so we turn instead to the Majorana spinors ψ_α.

In his study of the Dirac equation Majorana constructed a real representation of the Lorentz-group commutation algebra

$$[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}]=-g_{\mu\rho}\Sigma_{\nu\sigma}+g_{\nu\rho}\Sigma_{\mu\sigma}-g_{\mu\sigma}\Sigma_{\rho\nu}+g_{\nu\sigma}\Sigma_{\rho\mu} \qquad (3.1)$$

by finding a basis for the gamma matrix algebra of Eq. (1.11) in which the gamma matrices are all pure imaginary, viz.,

$$\gamma_0=\begin{bmatrix}0&\sigma_2\\ \sigma_2&0\end{bmatrix},\quad \gamma_1=-i\begin{bmatrix}\sigma_3&0\\0&\sigma_3\end{bmatrix},$$
$$\gamma_2=\begin{bmatrix}0&\sigma_2\\-\sigma_2&0\end{bmatrix},\quad \gamma_3=i\begin{bmatrix}\sigma_1&0\\0&\sigma_1\end{bmatrix}, \qquad (3.2)$$
$$\gamma_5=\begin{bmatrix}-\sigma_2&0\\0&\sigma_2\end{bmatrix},\quad C=\begin{bmatrix}0&\sigma_2\\ \sigma_2&0\end{bmatrix}.$$

[The matrix C in Eq. (3.2) transposes the γ_μ according to $C^{-1}\gamma_\mu C=-\gamma_\mu^T$.] Consequently, in the Majorana basis the Lorentz generators $\Sigma_{\mu\nu}(=\gamma_\mu\gamma_\nu/2)$ are purely real with

$$\Sigma_{01}=\frac{1}{2}\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix},\quad \Sigma_{02}=\frac{1}{2}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$
$$\Sigma_{03}=\frac{1}{2}\begin{bmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{bmatrix},\quad \Sigma_{12}=\frac{1}{2}\begin{bmatrix}0 & -\sigma_1\\ \sigma_1 & 0\end{bmatrix}, \tag{3.3}$$
$$\Sigma_{23}=\frac{1}{2}\begin{bmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{bmatrix},\quad \Sigma_{31}=-\frac{i}{2}\begin{bmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{bmatrix}$$

and furnish a four-dimensional irreducible representation of the real Lorentz SO(3,1) group. Under a Lorentz transformation an arbitrary Lorentz-group spinor transforms as

$$\psi\to e^{\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi\ , \tag{3.4}$$

where the rotation angles $\omega_{\mu\nu}$ are real. The Hermitian and anti-Hermitian parts of a general Lorentz spinor thus transform independently, with a Hermitian Majorana spinor then transforming irreducibly under the real Lorentz group.[7] Majorana spinors are thus suitable candidates for classical coordinates.

Altogether the Lorentz group admits of two real four-dimensional irreducible representations, namely the Majorana one and also the standard one which acts on the coordinates x_μ. The coordinate representation leaves invariant $x^\mu g_{\mu\nu}x^\nu$ while the Majorana one leaves invariant

$$\psi_\alpha^T C_{\alpha\beta}\psi_\beta=-i\psi_1\psi_4+i\psi_2\psi_3-i\psi_3\psi_2+i\psi_4\psi_1\ . \tag{3.5}$$

Since $C_{\alpha\beta}$ is antisymmetric, the coordinates ψ_α must therefore anticommute according to

$$\psi_\alpha\psi_\beta+\psi_\beta\psi_\alpha=0\ . \tag{3.6}$$

The Majorana spinors must thus be Grassmann numbers, and they are thus the natural coordinates needed for Eq. (1.8). Further, if the anticommuting components of ψ_α are all Hermitian in the Grassmann space the quantity $\psi_\alpha^T C_{\alpha\beta}\psi_\beta$ will be Hermitian also. The square root of -1, while foreign to the classical coordinate space associated with real x_μ, is thus naturally present in the classical Grassmann space associated with Hermitian ψ_α. Finally, with the anticommutator of the Grassmann number ψ_α with the Grassmann derivative operator $\partial/\partial\psi_\beta$ being $\delta_{\alpha\beta}$, we thus see that the generators

$$\Sigma_{\mu\nu}=\frac{1}{4}\psi^T C[\gamma_\mu,\gamma_\nu]C^{-1}\partial/\partial\psi$$

furnish a differential operator representation of the Lorentz algebra of Eq. (3.1). This is then the Grassmann spinor analog of the ordinary coordinate differential $x_\mu\partial_\nu-x_\nu\partial_\mu$ representation of the Lorentz group.

In the usual discussion of the Lorentz group we look for the most general set of transformations in coordinate space which leave the length $x^\mu g_{\mu\nu}x^\nu$ invariant and obtain the algebra of Eq. (3.1). We then look for other representations of the algebra. We now see that we could instead just as well have started with the invariance structure of $\psi_\alpha^T C_{\alpha\beta}\psi_\beta$ in a Grassmann space and still have gotten to the algebra of Eq. (3.1). From the point of view of coordinate space we call x_μ the vector representation and $g_{\mu\nu}$ the metric. Hence, from the point of view of Grassmann space we can think of ψ_α as the "vector" representation in that space and $C_{\alpha\beta}$ as the "metric." The Lorentz group may be realized in two distinct spaces, namely coordinate space and Grassmann space, and there is no *a priori* reason why one is any more fundamental than the other. Spinless particles only exist in coordinate space while spin one-half particles exist in both spaces simultaneously.

While on an equal footing with the coordinates x_μ the Majorana spinors are distinct from and independent of the x_μ, so that ψ_α is not a function of x_μ. In order to discuss the dynamics of ψ_α the spinors will have to depend on some parameter, and so we are led again to our fifth coordinate, the Lorentz scalar parameter τ, with ψ_α then being a function of τ. Thus τ will provide the bridge between x_μ and ψ_α to link coordinate space and Grassmann space dynamics.

To complete our description of the underlying classical theory we must still specify the actual set of coordinates which are to be associated with physical particles such as the electron. For a single Hermitian Grassmann Majorana spinor the Lorentz vector current $\psi^T C\gamma_\mu\psi$ is zero identically because of the antisymmetry properties of Eq. (3.6), so that a single ψ_α would have no coupling to electromagnetism. An axial-vector current is allowed and yields the only available Hermitian coupling to the momentum, viz.,

$$\mathcal{L}=\psi^T C\gamma_5\gamma_\mu p^\mu\psi+m\psi^T C\psi \tag{3.7}$$

(up to an irrelevant chiral rotation on ψ_α). However, the mass shell associated with Eq. (3.7) is the tachyonic spacelike mass spectrum $p^2+m^2=0$.[8] With the use of two Hermitian spinors we can construct both a nontrivial electromagnetic current and timelike solutions. Thus we must associate the physical electron with not one but two Hermitian Majorana Grassmann spinors, $\psi^R(\tau)$ and $\psi^I(\tau)$, which we shall combine into the conventional Dirac form of $\psi(\tau)$ and $\bar{\psi}(\tau)$ by identifying

$$\psi(\tau)=\psi^R(\tau)+i\psi^I(\tau)\ .$$

Our theory thus has the same number of degrees of freedom as the standard Dirac theory. In our 5-space formalism then the complete description of a classical electron is the set of 12 coordinates $x_\mu(\tau)$, $\psi_\alpha^R(\tau)$, and $\psi_\alpha^I(\tau)$. Before discussing the full dynamics based on all these coordinates we turn first to a discussion of the dynamics of the pure Grassmann sector.

Hamilton's principle for the spin kinetic energy of the pure Grassmann sector requires us to vary the action between fixed end points $(\eta_\alpha^i,\bar{\eta}_\alpha^i,0)$ and $(\eta_\alpha^f,\bar{\eta}_\alpha^f,T)$. Since the initial and final values of the Grassmann coordinates are to be independent, the equation of motion for $\psi_\alpha(\tau)$ must be second order in τ, so that the Lagrangian cannot be of the familiar Dirac $\bar{\psi}\psi$ form. By analogy with $m\dot{x}^2/2$ we instead take as our action

$$S=f\int d\tau\,\dot{\bar{\psi}}(\tau)\dot{\psi}(\tau)\ . \tag{3.8}$$

In this action f is an ordinary commuting number and carries a dimension. It is the analog for Grassmann coordinates of the mass parameter m and serves the same purpose. Nothing appears to be known about the value or the dimension of the parameter f. In terms of the dimension of f the dimension of $\psi_\alpha(\tau)$ is given by

$$D[\psi]=\left[\frac{[\text{action}]\times[\tau]}{[f]}\right]^{1/2} . \tag{3.9}$$

Similarly, canonical conjugates for $\psi_\alpha(\tau)$ and $\bar{\psi}_\alpha(\tau)$, viz.,

$$\pi_\alpha(\tau)=\frac{\partial\mathscr{L}}{\partial\dot{\bar{\psi}}_\alpha(\tau)}=f\dot{\psi}_\alpha(\tau) ,$$
$$\bar{\pi}_\alpha(\tau)=\frac{\partial\mathscr{L}}{\partial\dot{\psi}_\alpha(\tau)}=f\dot{\bar{\psi}}_\alpha(\tau) , \tag{3.10}$$

have dimension

$$D[\pi]=\left[\frac{[\text{action}]\times[f]}{[\tau]}\right]^{1/2} . \tag{3.11}$$

Thus the dimension of $\bar{\psi}\pi$ type products is that of the action and is independent of the dimension of f.

It is possible to avoid the introduction of a new parameter f if we instead take for the action

$$S=T\int d\tau\,\dot{\bar{\psi}}(\tau)\dot{\psi}(\tau) . \tag{3.12}$$

Then $\psi_\alpha(\tau)$ and $\pi_\alpha(\tau)$ have the same spinor dimension of $[\text{action}]^{1/2}$ as the spinors in the standard Dirac theory. This possibility is very intriguing since we can then introduce $\hbar$ into classical physics as the dimension of a coordinate. The quantization of the theory is then in units of this same $\hbar$. We shall consider both actions in this paper treating the quantization of the f-dependent action of Eq. (3.8) first so as to not lose generality.

In terms of the canonical conjugates of Eq. (3.10) the Hamiltonian form of the action of Eq. (3.8) is

$$S=\int d\tau\left[\dot{\bar{\psi}}(\tau)\pi(\tau)+\bar{\pi}(\tau)\dot{\psi}(\tau)-\frac{\bar{\pi}(\tau)\pi(\tau)}{f}\right] . \tag{3.13}$$

Varying the action between its end points gives the equations of motion

$$\psi_\alpha^{\text{ST}}(\tau)=\frac{(\eta_\alpha^f-\eta_\alpha^i)\tau}{T}+\eta_\alpha^i ,$$
$$\bar{\psi}_\alpha^{\text{ST}}(\tau)=\frac{(\bar{\eta}_\alpha^f-\bar{\eta}_\alpha^i)\tau}{T}+\bar{\eta}_\alpha^i ,$$
$$\pi_\alpha^{\text{ST}}(\tau)=\frac{f(\eta_\alpha^f-\eta_\alpha^i)}{T} , \tag{3.14}$$
$$\bar{\pi}_\alpha^{\text{ST}}(\tau)=\frac{f(\bar{\eta}_\alpha^f-\bar{\eta}_\alpha^i)}{T} ,$$

so that in the stationary path

$$S_{\text{ST}}=\frac{f(\bar{\eta}^f-\bar{\eta}^i)(\eta^f-\eta^i)}{T} . \tag{3.15}$$

By complete analogy to Eqs. (2.9) and (2.10) we parametrize the arbitrary τ-space path by

$$\psi_\alpha(\tau)=\psi_\alpha^{\text{ST}}(\tau)+\sum_{n=1}^{\infty}\beta_\alpha^n\sin\frac{n\pi\tau}{T} ,$$
$$\bar{\psi}_\alpha(\tau)=\bar{\psi}_\alpha^{\text{ST}}(\tau)+\sum_{n=1}^{\infty}\bar{\beta}_\alpha^n\sin\frac{n\pi\tau}{T} ,$$
$$\pi_\alpha(\tau)=\pi_\alpha^{\text{ST}}(\tau)+\sum_{n=0}^{\infty}\gamma_\alpha^n\cos\frac{n\pi\tau}{T} , \tag{3.16}$$
$$\bar{\pi}_\alpha(\tau)=\bar{\pi}_\alpha^{\text{ST}}(\tau)+\sum_{n=0}^{\infty}\bar{\gamma}_\alpha^n\cos\frac{n\pi\tau}{T} ,$$

where β_α^n, $\bar{\beta}_\alpha^n$, γ_α^n, and $\bar{\gamma}_\alpha^n$ are mutually anticommuting Grassmann numbers which obey

$$\int d\beta_\alpha^m\beta_\beta^n=D^2\delta_{mn}\delta_{\alpha\beta} ,$$
$$\int d\gamma_\alpha^m\gamma_\beta^n=\tilde{D}^2\delta_{mn}\delta_{\alpha\beta} . \tag{3.17}$$

[D and $\tilde{D}$ are dimensionful parameters which are introduced to carry the dimensions of $\psi_\alpha(\tau)$ and $\pi_\alpha(\tau)$, respectively.] In the arbitrary path the action takes the form

$$S=S_{\text{ST}}-\frac{T\bar{\gamma}^0\gamma^0}{f}+\sum_{n=1}^{\infty}\left[-\frac{T\bar{\gamma}^n\gamma^n}{2f}+\frac{n\pi}{2}(\bar{\beta}^n\gamma^n+\bar{\gamma}^n\beta^n)\right] . \tag{3.18}$$

Variation of S then confirms Eq. (3.15).

The Feynman path-integral prescription enables us to introduce a quantum propagator via

$$G_{if}(\eta,\bar{\eta},T)=G(\eta_\alpha^f,\bar{\eta}_\alpha^f,T;\eta_\alpha^i,\bar{\eta}_\alpha^i,0)$$
$$=\theta(T)\int[d\psi][d\bar{\psi}][d\pi][d\bar{\pi}]\exp(iS/\hbar) . \tag{3.19}$$

In terms of the Grassmann variation parameters the path-integral measure is given by

$$\int[d\psi][d\bar{\psi}][d\pi][d\bar{\pi}]=B\lim_{N\to\infty}\left[\frac{1}{\hbar}\right]^{8(N+1)}\prod_\alpha\prod_{n=1}^{N}\int\cdots\int d\gamma_\alpha^0 d\bar{\gamma}_\alpha^0 d\beta_\alpha^n d\bar{\beta}_\alpha^n d\gamma_\alpha^n d\bar{\gamma}_\alpha^n , \tag{3.20}$$

where the irrelevant normalization constant B is a pure number. To check the Lorentz invariance of the measure we note that

$$(\psi_\alpha^T C_{\alpha\beta}\psi_\beta)^2 = 8\psi_1\psi_2\psi_3\psi_4 \tag{3.21}$$

according to Eqs. (3.5) and (3.6). Thus $d\gamma_1^n d\gamma_2^n d\gamma_3^n d\gamma_4^n$, etc., are Lorentz scalars as required. Because of the simple form of the arbitrary action in Eq. (3.18) the path integration can be performed analytically and yields [for a suitable choice of B which absorbs a dimensionless $(D\tilde{D}/\hbar)^{16(N+1)}$ factor]

$$G_{if}(\eta,\bar{\eta},T) = \theta(T)\frac{\hbar^4 T^4}{D^{16}f^4}\exp(iS_{ST}/\hbar) , \tag{3.22}$$

where S_{ST} is given in Eq. (3.15). Finally, a Grassmann Fourier-space propagator may also be defined via

$$\tilde{G}(\pi,\bar{\pi},T) = \int d^4(\eta_f-\eta_i)d^4(\bar{\eta}_f-\bar{\eta}_i)G_{if}(\eta,\bar{\eta},T) \times\exp\left[-\frac{i(\bar{\eta}_f-\bar{\eta}_i)\pi}{\hbar}-\frac{i\bar{\pi}(\eta_f-\eta_i)}{\hbar}\right] , \tag{3.23}$$

where we normalize according to

$$\int d\eta_\alpha\eta_\beta = D^2\delta_{\alpha\beta} , \quad \int d\pi_\alpha\pi_\beta = \tilde{D}^2\delta_{\alpha\beta} , \tag{3.24}$$

and yields

$$\tilde{G}(\pi,\bar{\pi},T) = \theta(T)\exp\left[-\frac{iT\bar{\pi}\pi}{\hbar f}\right] . \tag{3.25}$$

We recognize $\tilde{G}(\pi,\bar{\pi},T)$ as being of the form

$$\tilde{G}(\pi,\bar{\pi},T) = \theta(T)\exp\left[-\frac{iH_{CL}(\pi,\bar{\pi})T}{\hbar}\right] , \tag{3.26}$$

where $H_{CL}(\pi,\bar{\pi})$ is the classical Hamiltonian associated with the action of Eq. (3.13).

Thus the Feynman path-integral prescription for the Grassmann coordinates leads us to a familiar form for the propagator. Having now found the Feynman propagator in τ space we shall next obtain some further insight into its significance by rederiving it via canonical quantization.

IV. CANONICAL QUANTIZATION OF GRASSMANN SPINORS

In this section we canonically quantize the Grassmann spinors. In a canonical quantization we reinterpret the Majorana spinors and their conjugates as operators in Hilbert space, and by correspondence with the Poisson brackets associated with $(\psi_\alpha,\pi_\beta^\dagger)$ and $(\psi_\alpha^\dagger,\pi_\beta)$ postulate the equal-τ anticommutators

$$\{\hat{\psi}_\alpha(\tau),\hat{\pi}_\beta^\dagger(\tau)\} = i\hbar C_{\alpha\beta} , \quad \{\hat{\psi}_\alpha^\dagger(\tau),\hat{\pi}_\beta(\tau)\} = i\hbar C_{\alpha\beta} \tag{4.1}$$

with all other equal-τ anticommutators vanishing. The presence of the factor $C_{\alpha\beta}$ in the anticommutators is dictated by Lorentz covariance, so that just like the commutation relation $[\hat{x}_\mu,\hat{p}_\nu] = -i\hbar g_{\mu\nu}$ we see again that in the Grassmann space $C_{\alpha\beta}$ serves as the metric. Unlike the $\hat{\xi}_\mu,\hat{\xi}_\nu$ anticommutators of Eq. (1.10) which are symmetric in the μ,ν indices, the Majorana anticommutators are antisymmetric in the α,β indices. This will enable us below to construct an eigenspectrum for $\hat{\psi}_\alpha(\tau)$ and $\hat{\pi}_\alpha(\tau)$ analogous to that of Eq. (1.3). In terms of $\hat{\bar{\psi}}(\tau)$ and $\hat{\bar{\pi}}(\tau)$ we may rewrite the anticommutation relations of Eq. (4.1) as

$$\{\hat{\psi}_\alpha(\tau),\hat{\bar{\pi}}_\beta(\tau)\} = i\hbar\delta_{\alpha\beta} , \quad \{\hat{\bar{\psi}}_\alpha(\tau),\hat{\pi}_\beta(\tau)\} = -i\hbar\delta_{\alpha\beta} . \tag{4.2}$$

The quantum Hamiltonian associated with the action of Eq. (3.13) is

$$\hat{H} = \frac{\hat{\bar{\pi}}(\tau)\hat{\pi}(\tau)}{f} , \tag{4.3}$$

so that the Heisenberg equations of motion are

$$\frac{d\hat{\psi}_\alpha(\tau)}{d\tau} = \frac{\hat{\pi}_\alpha(\tau)}{f} , \quad \frac{d\hat{\pi}_\alpha(\tau)}{d\tau} = 0 , \quad \frac{d\hat{\bar{\psi}}_\alpha(\tau)}{d\tau} = \frac{\hat{\bar{\pi}}_\alpha(\tau)}{f} , \quad \frac{d\hat{\bar{\pi}}_\alpha(\tau)}{d\tau} = 0 . \tag{4.4}$$

We solve these equations in terms of initial $\tau=0$ values $\hat{\psi}_\alpha,\hat{\bar{\psi}}_\alpha,\hat{\pi}_\alpha,\hat{\bar{\pi}}_\alpha$ to obtain

$$\hat{\psi}_\alpha(\tau) = \frac{\hat{\pi}_\alpha\tau}{f}+\hat{\psi}_\alpha, \quad \hat{\pi}_\alpha(\tau) = \hat{\pi}_\alpha , \quad \hat{\bar{\psi}}_\alpha(\tau) = \frac{\hat{\bar{\pi}}_\alpha\tau}{f}+\hat{\bar{\psi}}_\alpha, \quad \hat{\bar{\pi}}_\alpha(\tau) = \hat{\bar{\pi}}_\alpha . \tag{4.5}$$

The initial values then satisfy

$$\{\hat{\psi}_\alpha,\hat{\bar{\pi}}_\beta\} = i\hbar\delta_{\alpha\beta} , \quad \{\hat{\bar{\psi}}_\alpha,\hat{\pi}_\beta\} = -i\hbar\delta_{\alpha\beta} , \tag{4.6}$$

as required of an equal-τ anticommutator.

Having removed the explicit τ dependence from the operators we next construct their eigenstates. In terms of mutually anticommuting Grassmann spinors η_α, $\bar{\eta}_\alpha$, σ_α, and $\bar{\sigma}_\alpha$ we seek states which obey

$$\hat{\psi}_\alpha|\eta,\bar{\eta}\rangle = \eta_\alpha|\eta,\bar{\eta}\rangle , \quad \hat{\bar{\psi}}_\alpha|\eta,\bar{\eta}\rangle = \bar{\eta}_\alpha|\eta,\bar{\eta}\rangle , \quad \hat{\pi}_\alpha|\sigma,\bar{\sigma}\rangle = \sigma_\alpha|\sigma,\bar{\sigma}\rangle , \quad \hat{\bar{\pi}}_\alpha|\sigma,\bar{\sigma}\rangle = \bar{\sigma}_\alpha|\sigma,\bar{\sigma}\rangle . \tag{4.7}$$

Such states can then be used to construct the $G_{if}(\eta,\bar{\eta},T)$ propagator of Eq. (3.19). Ignoring spinor indices we see that we are essentially trying to construct eigenstates of creation and annihilation operators $\alpha^\dagger$ and α which obey $\alpha\alpha^\dagger+\alpha^\dagger\alpha=1$. Since $\alpha^2=(\alpha^\dagger)^2=0$ the eigenvalues of α and $\alpha^\dagger$ must be Grassmann numbers with both α and $\alpha^\dagger$ being representable as Hilbert space operators of the generic form $|\rho\rangle\rho\langle\rho|$. Consequently the operators α and $\alpha^\dagger$ anticommute with all Grassmann numbers. Eigenstates of α and $\alpha^\dagger$ may be constructed by analogy with position and momentum coherent states. In terms of the Fock vacuum which α annihilates according to $\alpha|0\rangle=0$ we readily construct

$$\alpha|\rho\rangle=\alpha e^{\alpha^\dagger\rho}|0\rangle=\alpha(1+\alpha^\dagger\rho)|0\rangle=\rho|0\rangle=\rho(1+\alpha^\dagger\rho)|0\rangle=\rho|\rho\rangle \tag{4.8}$$

and

$$\alpha^\dagger|\rho^\dagger\rangle=\alpha^\dagger e^{-\rho^\dagger\alpha}\alpha^\dagger|0\rangle=\alpha^\dagger(1-\rho^\dagger\alpha)\alpha^\dagger|0\rangle=-\alpha^\dagger\rho^\dagger|0\rangle=\rho^\dagger\alpha^\dagger|0\rangle=\rho^\dagger(1-\rho^\dagger\alpha)\alpha^\dagger|0\rangle=\rho^\dagger|\rho^\dagger\rangle \tag{4.9}$$

using the Grassmann properties $\rho^2=(\rho^\dagger)^2=0$.

Given these eigenstates of α and $\alpha^\dagger$ the generalization to the Majorana spinors is straightforward. We define a vacuum via

$$\hat{\psi}_\alpha|0\rangle=0\ ,\quad \hat{\pi}_\alpha|0\rangle=0 \tag{4.10}$$

and eigenstates

$$|\eta,\bar\eta\rangle=\frac{1}{D^8}\exp\left[-\frac{i(\hat{\bar\pi}\eta+\bar\eta\hat\pi)}{\hbar}\right]\hat{\bar\psi}_1\hat{\bar\psi}_2\hat{\bar\psi}_3\hat{\bar\psi}_4|0\rangle\ ,$$
$$|\sigma,\bar\sigma\rangle=\frac{\hbar^4}{\tilde D^8}\exp\left[\frac{i(\hat{\bar\psi}\sigma+\bar\sigma\hat\psi)}{\hbar}\right]\hat{\bar\pi}_1\hat{\bar\pi}_2\hat{\bar\pi}_3\hat{\bar\pi}_4|0\rangle\ . \tag{4.11}$$

By use of the anticommutators of Eq. (4.6) it is then straightforward to show that these states satisfy Eq. (4.7). The states are normalized to Grassmann delta functions, viz.,

$$\langle\eta',\bar\eta'|\eta,\bar\eta\rangle=\frac{1}{D^{16}}\prod_\alpha(\bar\eta_\alpha-\bar\eta'_\alpha)(\eta_\alpha-\eta'_\alpha)\ ,$$
$$\langle\sigma',\bar\sigma'|\sigma,\bar\sigma\rangle=\frac{\hbar^8}{\tilde D^{16}}\prod_\alpha(\bar\sigma_\alpha-\bar\sigma'_\alpha)(\sigma_\alpha-\sigma'_\alpha)\ . \tag{4.12}$$

They satisfy

$$\langle\sigma,\bar\sigma|\eta,\bar\eta\rangle=\left[\frac{\hbar}{D\tilde D}\right]^8\exp\left[-\frac{i(\bar\eta\sigma+\bar\sigma\eta)}{\hbar}\right] \tag{4.13}$$

and are complete according to

$$\prod_\alpha\int|\eta,\bar\eta\rangle d\eta_\alpha d\bar\eta_\alpha\langle\eta,\bar\eta|=1\ ,$$
$$\frac{1}{\hbar^8}\prod_\alpha\int|\sigma,\bar\sigma\rangle d\sigma_\alpha d\bar\sigma_\alpha\langle\sigma,\bar\sigma|=1\ . \tag{4.14}$$

The above relations are completely analogous to the usual ones for the position and momentum eigenstates of Eq. (1.3) (which have Dirac delta-function normalization and plane waves for the $\langle p|q\rangle$ scalar products), and thus we see that we have constructed a canonical quantization analogous to the usual one for p and q.

To complete our discussion of the canonically quantized theory we calculate the propagator

$$G_{if}=\langle\eta_f,\bar\eta_f,T|\eta_i,\bar\eta_i,0\rangle=\theta(T)\langle\eta_f,\bar\eta_f|\exp(-i\hat HT/\hbar)|\eta_i,\bar\eta_i\rangle\ , \tag{4.15}$$

where $\hat H$ is given by

$$\hat H=\frac{\hat{\bar\pi}\hat\pi}{f} \tag{4.16}$$

according to Eqs. (4.3) and (4.5). Inserting complete sets of eigenstates of $\hat\pi_\alpha$ and $\hat{\bar\pi}_\alpha$ into Eq. (4.15) yields

$$G_{if}=\theta(T)\frac{\hbar^8}{D^{16}\tilde D^{32}}\prod_\alpha\prod_\beta\int d\sigma'_\alpha d\bar\sigma'_\alpha d\sigma_\beta d\bar\sigma_\beta(\bar\sigma_\alpha-\bar\sigma'_\alpha)(\sigma_\alpha-\sigma'_\alpha)\exp\left[\frac{i(\bar\eta_f\sigma'+\bar\sigma'\eta_f)}{\hbar}-\frac{i\bar\sigma\sigma T}{\hbar f}-\frac{i(\bar\eta_i\sigma+\bar\sigma\eta_i)}{\hbar}\right]$$
$$=\theta(T)\frac{\hbar^4T^4}{D^{16}f^4}\exp\left[\frac{if(\bar\eta_f-\bar\eta_i)(\eta_f-\eta_i)}{\hbar T}\right]\ , \tag{4.17}$$

which we recognize as Eq. (3.22). We thus check the consistency between Feynman path integration in Grassmann space and canonical quantization.

Further insight into the structure of the quantized theory may be obtained by rewriting the propagator of Eq. (3.22) as

$$G_{if}(\eta,\bar\eta,T)=\theta(T)\frac{\hbar^8}{\tilde D^{16}D^{16}}\int d^4\pi\int d^4\bar\pi\exp\left[\frac{i(\bar\eta\eta_f-\bar\eta_i)\pi}{\hbar}+\frac{i\bar\pi(\eta_f-\eta_i)}{\hbar}-\frac{iT\bar\pi\pi}{\hbar f}\right]\ . \tag{4.18}$$

This propagator satisfies the differential equation

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{1}{f}\left[i\hbar\frac{\partial}{\partial\eta^f_\alpha}\right]\left[-i\hbar\frac{\partial}{\partial\bar\eta^f_\alpha}\right]\right]G_{if}(\eta,\bar\eta,T)=i\hbar\delta(T)\prod_\alpha(\bar\eta^f_\alpha-\bar\eta^i_\alpha)(\eta^f_\alpha-\eta^i_\alpha)/D^{16}\ . \tag{4.19}$$

Recognizing a Grassmann-space delta function, we see that Eq. (4.19) is thus a Green's-function equation for Grassmann variables. Additionally, we may define a wave function $\psi(\eta,\bar\eta,T)$ which satisfies a Schrödinger equation

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{1}{f}\left[i\hbar\frac{\partial}{\partial\eta_\alpha}\right]\left[-i\hbar\frac{\partial}{\partial\bar\eta_\alpha}\right]\right]\psi(\eta,\bar\eta,T)=0\ . \tag{4.20}$$

Noting that the canonical anticommutators of Eqs. (4.6) admit of a differential operator representation

$$\hat{\pi}_\alpha=-i\hbar\frac{\partial}{\partial\bar{\psi}_\alpha}\ ,\quad \hat{\bar{\pi}}_\alpha=i\hbar\frac{\partial}{\partial\psi_\alpha}\ , \tag{4.21}$$

Eq. (4.20) may be reexpressed as

$$\left[i\hbar\frac{\partial}{\partial T}-\hat{H}\right]\psi(\eta,\bar{\eta},T)=0\ . \tag{4.22}$$

Thus Eq. (4.20) provides us with a wave mechanics interpretation of the Grassmann sector of the theory. The quantum mechanics in the Grassmann sector is therefore completely analogous to the usual one based on p and q.

V. THE COMPLETE SUPERSPACE PROPAGATOR

So far we have constructed the propagators associated with the separate space-time and Grassmann sectors of the theory. In this section we shall construct the propagator associated with the full $x^\mu(\tau)$, $\psi_\alpha(\tau)$, and $\bar{\psi}_\alpha(\tau)$ classical theory by using a Lagrangian which also contains cross terms between the space-time and Grassmann coordinates. Initially there is a great deal of freedom in choosing such a Lagrangian. In order to restrict the structure of the Lagrangian we will therefore impose some additional symmetries. Specifically, we shall extend the Galilean (in τ) invariance of the space-time sector of Sec. II to the Grassmann sector as well. Additionally, noting that we have been treating the space-time and Grassmann coordinates in our work in a completely analogous fashion, we shall introduce some symmetry between them. Thus we shall interpret the full-space-time, Grassmann manifold as a superspace and introduce some supersymmetry transformations which mix the space-time and Grassmann coordinates. As we shall see, the imposition of the Galilean symmetry and of the supersymmetry, both of which are natural in our model, will completely specify the structure of the full superspace Lagrangian.

We discuss first the Galilean invariance. In the space-time sector the coordinate-space form of the phase-space action of Eq. (2.4), i.e.,

$$S=\int d\tau\left[-\frac{m}{2}\frac{dx_\mu(\tau)}{d\tau}\frac{dx^\mu(\tau)}{d\tau}-\frac{m}{2}\right], \tag{5.1}$$

was constructed to be invariant under Lorentz transformations on the four $x^\mu(\tau)$ coordinates. However, our action is also nonrelativistic in the fifth coordinate τ, and is thus additionally Galilean invariant in the larger x^μ, τ 5-space. It is our wish to extend this Galilean invariance to the Grassmann Majorana spinor sector as well. As we shall see, this will have nontrivial implications for our theory.

To see how to implement Galilean invariance in the 5-space it is useful to recall the nonrelativistic limit of Lorentz invariance in the usual space-time 4-space. Under a typical ordinary Lorentz boost with velocity v ($=v_{01}$) between the space coordinate x and the ordinary time t, x and t transform as

$$x\rightarrow\frac{x-vt}{(1-v^2/c^2)^{1/2}}\ ,\quad t\rightarrow\frac{t-vx/c^2}{(1-v^2/c^2)^{1/2}}\ , \tag{5.2}$$

where c is the velocity of light, and hence transform as

$$x\rightarrow x-vt\ ,\quad t\rightarrow t \tag{5.3}$$

in the Galilean limit. For our purpose here we note that the Galilean transformations are not just obtainable from the Lorentz transformations in the limit in which c is taken to be infinite, but rather they are also in fact valid to the first nontrivial order in v/c. For our work here we shall define this latter v/c limit as the Galilean limit. To determine how any other vector (such as the Dirac current) then transforms in this Galilean limit we rewrite Eq. (5.2) in terms of the components of the Lorentz vector x^μ, viz., x and ct, to obtain

$$x\rightarrow\frac{x-(v/c)ct}{(1-v^2/c^2)^{1/2}}\ ,\quad ct\rightarrow\frac{ct-(v/c)x}{(1-v^2/c^2)^{1/2}}\ . \tag{5.4}$$

To lowest nontrivial order in v/c, Eq. (5.4) reduces to

$$x\rightarrow x-(v/c)ct\ ,\quad ct\rightarrow ct-(v/c)x\ . \tag{5.5}$$

Thus to lowest order in v/c the transformation properties of x and t in Eq. (5.3) differ from those of x and ct given in Eq. (5.5). Since x and ct are components of a Lorentz 4-vector, any other Lorentz 4-vector, j^μ say, then transforms according to Eq. (5.5), i.e., as

$$j^1\rightarrow j^1-(v/c)j^0\ ,\quad j^0\rightarrow j^0-(v/c)j^1 \tag{5.6}$$

in the Galilean limit. [In passing we note that in the limit in which the velocity of light is infinite the space components of j^μ would remain invariant even while the space components of x^μ itself would change according to Eq. (5.3). Hence taking the order v/c limit of the Lorentz transformations rather than the c equal infinity limit would appear to be the more sensible nonrelativistic limit of Lorentz invariance, with the implications for the space components of the arbitrary 4-vector in Eq. (5.6) then being nontrivial.] Further, from Eq. (5.6) we note that the time component of j^μ is also transformed nontrivially in the Galilean limit even while the time t itself remains unchanged in Eq. (5.3).

With these remarks in mind we proceed now to construct and transform spinors in the 5-space. To this end it is convenient to first discuss the full 5-space Minkowski invariance with 5-vector $x^\alpha=(x^\mu,\tau)$, where τ has the same dimension as x^μ, i.e., velocity times time, with O(4,1) metric $g_{\alpha\beta}=(g_{\mu\nu},-1)$ and with Minkowski invariant $x_\alpha x^\alpha=x_\mu x^\mu-\tau^2$. (With this convention when $dx_\alpha dx^\alpha=0$, $d\tau$ is equal to the ordinary 4-space proper time.) A Clifford algebra is next defined via $\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2g_{\alpha\beta}$. While there are now five mutually anticommuting gamma matrices in the 5-space we note that the Clifford algebra admits of a 4-dimensional representation $\gamma_\alpha=(\gamma_\mu,i\gamma_5)$. Hence in O(4,1) the appropriate spinors still have only four components. To construct an O(4,1) invariant we must momentarily replace our Majorana spinors $\psi_\alpha(\tau)$ and $\bar{\psi}_\alpha(\tau)$, which only depend on τ by spinors $\psi_\alpha(x^\alpha)$ and $\bar{\psi}_\alpha(x^\alpha)$ which depend on all five components of x^α and are thus fields on the O(4,1) manifold. In terms of these fields we construct an O(4,1) Minkowski invariant cross term between the fields $\psi_\alpha(x^\alpha)$ and the coordinates x^α, viz.,

$$\mathscr{L}_I=\bar{\psi}(x^\mu,\tau)(\gamma_\mu x^\mu+i\gamma_5\tau)\psi(x^\mu,\tau) \tag{5.7}$$

to thus connect the ordinary Lorentz vector and Lorentz pseudoscalar terms.

In the Galilean limit in O(4,1) the 5-space velocity c_5 becomes large but not infinite, and, with $\tau'=\tau/c_5$ being the analog of the ordinary time, Eqs. (5.3) and (5.6) are replaced by relations such as

$$\begin{aligned}&x^0\to x^0-v_{05}\tau'\ ,\quad \tau'\to\tau'\\&j^0\to j^0-(v_{05}/c_5)j^5\ ,\quad j^5\to j^5-(v_{05}/c_5)j^0\ ,\end{aligned} \tag{5.8}$$

where v_{05} is a typical O(4,1) boost velocity. Under these transformations the action of Eq. (5.1) is left invariant [it being just a linear combination of the nonrelativistic limit of $\int(dx_\alpha dx^\alpha)^{1/2}$ and the Galilean invariant $\int d\tau'$]. Additionally, the cross-term action

$$S_I=\int d\tau'\frac{d\bar{\psi}(\tau')}{d\tau'}\left[\frac{\gamma_\mu}{c_5}\frac{dx^\mu}{d\tau'}+i\gamma_5\right]\frac{d\psi(\tau')}{d\tau'} \tag{5.9}$$

and the Grassmann sector kinetic energy of Eq. (3.8), viz.,

$$S=\frac{f}{c_5}\int d\tau'\frac{d\bar{\psi}(\tau')}{d\tau'}\frac{d\psi(\tau')}{d\tau'}\ , \tag{5.10}$$

are also left invariant under the transformations of Eq. (5.8) provided that ψ is restricted to depend only on τ' and not on x^μ at all. Thus the actions of Eqs. (5.9) and (5.10) are completely O(4,1) Galilean invariant for a specific class of spinors ψ_α which are none other than the ones of interest for this entire work, namely the class of spinors ψ_α which are τ-dependent coordinates and not x^μ-dependent fields. We thus recognize our Majorana spinor coordinates $\psi_\alpha(\tau)$ and $\bar{\psi}_\alpha(\tau)$ not only as O(3,1) Lorentz spinors but also as O(4,1) Galilean spinors. Having now explored the implications of Galilean invariance in our 5-space, we see finally that we should build our general superspace action out of the actions of Eqs. (5.1), (5.9), and (5.10). Further restrictions on the structure of our superspace action are obtained by imposing some supersymmetry on our theory, a point to which we now turn.

To implement the supersymmetry in our five-dimensional formalism we note that since we have four coordinates x^μ and a parameter τ in the space-time sector of our theory we should have an analogous structure in the Grassmann sector. We thus introduce two parameters θ and $\bar{\theta}$ [one for $\psi_\alpha(\tau)$ and the other for $\bar{\psi}_\alpha(\tau)$] which are new τ-independent Lorentz scalar Grassmann numbers which are taken to be dimensionless for simplicity and which obey

$$\begin{aligned}&\bar{\theta}=\theta^\dagger\ ,\quad \theta^2=\bar{\theta}^2=0\ ,\\&\theta\bar{\theta}+\bar{\theta}\theta=0\ ,\\&\int d\theta\,\theta=\int d\bar{\theta}\,\bar{\theta}=1\ ,\\&\int d\theta\,\bar{\theta}=\int d\bar{\theta}\,\theta=0\ .\end{aligned} \tag{5.11}$$

For interpretative purposes only we may think of $(\psi_\alpha(\tau),\theta)$ and $(\bar{\psi}_\alpha(\tau),\bar{\theta})$ as 5-spinors in the Grassmann sector of O(4,1). Specifically, we already noted that the $\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2g_{\alpha\beta}$ Clifford algebra has a 4-dimensional representation. It also of course has a trivial reducible five-dimensional one which is block-diagonal. The quantities θ and $\bar{\theta}$ may then be associated with the trivial part of this five-dimensional representation and are hence decoupled from $\psi_\alpha(\tau)$ and $\bar{\psi}_\alpha(\tau)$ under Minkowski or Galilean O(4,1) transformations. They are thus only parameters and not dynamical degrees of freedom. Thus in our theory τ, θ, and $\bar{\theta}$ are the parameters in the superspace, and hence, as with T, we will also integrate out θ and $\bar{\theta}$ at the end.

To implement explicit supersymmetry transformations in our superspace we introduce two independent sets of Grassmann parameters ϵ_α^R and ϵ_α^I which transform as Majorana spinors under the Lorentz group and construct the following eight supersymmetry transformations:

$$\begin{aligned}&x^\mu\to x^\mu-\frac{i\bar{\theta}\theta}{m}(\bar{\epsilon}\gamma^\mu\psi-\bar{\psi}\gamma^\mu\epsilon)\ ,\\&\psi_\alpha\to\psi_\alpha+\frac{i\bar{\theta}\theta}{f}x_\mu\gamma^\mu_{\alpha\beta}\epsilon_\beta\ ,\\&\psi_\alpha^\dagger\to\psi_\alpha^\dagger+\frac{i\bar{\theta}\theta}{f}x_\mu\gamma^\mu_{\alpha\beta}\epsilon_\beta^\dagger\ ,\\&\bar{\psi}_\alpha\to\bar{\psi}_\alpha-\frac{i\bar{\theta}\theta}{f}\bar{\epsilon}_\beta\gamma^\mu_{\beta\alpha}x_\mu\ ,\\&\tau\to\tau\ ,\quad\theta\to\theta\ ,\quad\bar{\theta}\to\bar{\theta}\ ,\end{aligned} \tag{5.12}$$

with the eight Grassmann numbers $\bar{\theta}\theta\epsilon_\alpha^R$ and $\bar{\theta}\theta\epsilon_\alpha^I$ being the eight associated transformation parameters. [In the above transformations ϵ_α denotes $\epsilon_\alpha^R+i\epsilon_\alpha^I$, $\psi_\alpha^\dagger$ is the Hermitian conjugate of ψ_α with respect to Grassmann space only, and the Dirac gamma matrices are purely imaginary as in Eq. (3.2).] In terms of Hermitian Majorana spinors we may reexpress Eq. (5.12) in the form

$$\begin{aligned}&x^\mu\to x^\mu-\frac{2i\bar{\theta}\theta}{m}(\epsilon^R C\gamma^\mu\psi^R+\epsilon^I C\gamma^\mu\psi^I)\ ,\\&\psi_\alpha^R\to\psi_\alpha^R+\frac{i\bar{\theta}\theta}{f}x_\mu\gamma^\mu_{\alpha\beta}\epsilon_\beta^R\ ,\\&\psi_\alpha^I\to\psi_\alpha^I+\frac{i\bar{\theta}\theta}{f}x_\mu\gamma^\mu_{\alpha\beta}\epsilon_\beta^I\ ,\\&\tau\to\tau\ ,\quad\theta\to\theta\ ,\quad\bar{\theta}\to\bar{\theta}\ ,\end{aligned} \tag{5.13}$$

and thus see that x^μ and ψ_α^R are superpartners under $\bar{\theta}\theta\epsilon_\alpha^R$ transformations while x^μ and ψ_α^I are superpartners under $\bar{\theta}\theta\epsilon_\alpha^I$ transformations. (In passing we thus note that unlike the situation in standard supersymmetry theories, our supersymmetry is not between independent fermion and boson fields, i.e., between different species of fields, but rather it is a symmetry between the space-time and Grassmann coordinates of the same particle. Thus in our work the fermion with its x^μ, ψ_α^R, and ψ_α^I coordinates produces the supersymmetry all on its own and needs no further companion bosonic partner.) As can be seen, the supertransformations of Eq. (5.12) have been constructed to leave invariant the quadratic form $mx^2-2f\bar{\psi}\psi$. With τ,

θ, and $\bar{\theta}$ only being parameters the supersymmetry transformations also leave invariant the quadratic form $m\dot{x}^2-2f\bar{\psi}\dot{\psi}$ of the dynamical variables x^μ, ψ^α, and $\bar{\psi}^\alpha$. Additionally our supertransformations leave invariant any term which is proportional to $\bar{\theta}\theta$. Consequently, requiring that our final superspace Lagrangian be both supersymmetric and O(4,1) Galilean invariant leads us to our desired 5-space Lagrangian (written more conveniently in terms of our original variable τ rather than τ', with $\dot{x}$ denoting $dx/d\tau$, etc.)

$$\mathscr{L}=-\frac{m\dot{x}^2}{2}-\frac{m}{2}+f\bar{\psi}\dot{\psi}+g\bar{\theta}\theta\dot{\bar{\psi}}(\gamma_\mu\dot{x}^\mu+i\gamma_5)\dot{\psi} . \quad (5.14)$$

Thus we only need to introduce one new quantity, namely the coefficient g which is another ordinary commuting number with the same dimension as f. In the following we shall explore the dynamics associated with the Lagrangian of Eq. (5.14).

Before doing this, however, it is instructive to discuss the superalgebraic structure of our theory and compare it with the previous superalgebra studies in the literature. In the literature the question of how to combine supersymmetry with Poincaré invariance has been much studied. (A recent review is given by van Nieuwenhuizen, see Ref. 2.) The objective is to grade the Poincaré algebra with a set of spinor generators into a superalgebra which is closed under commutation and anticommutation. A typical example is the superalgebra considered by Salam and Strathdee in Ref. 2, viz.,

$$\begin{aligned}&[M_{\mu\nu},M_{\rho\sigma}]=-g_{\mu\rho}M_{\nu\sigma}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\rho\nu}+g_{\nu\sigma}M_{\rho\mu} ,\\&[M_{\mu\nu},P_\sigma]=g_{\nu\sigma}P_\mu-g_{\mu\sigma}P_\nu ,\quad [P_\mu,P_\nu]=0 ,\\&[S_\alpha,P_\mu]=0 ,\quad [S_\alpha,M_{\mu\nu}]=\tfrac{1}{4}[\gamma_\mu,\gamma_\nu]_{\alpha\beta}S_\beta ,\\&\{S_\alpha,S_\beta\}=-(\gamma_\mu C)_{\alpha\beta}P^\mu ,\end{aligned} \quad (5.15)$$

where $M_{\mu\nu}$ and P_μ are the Poincaré generators and S_α is a Majorana spinor generator. With just one Majorana spinor this is the smallest possible grading of the Poincaré group. The algebra of Eq. (5.15) admits of a differential representation in the x_μ,ψ_α superspace (ψ_α is a Majorana spinor)

$$\begin{aligned}&M_{\mu\nu}=x_\mu\partial_\nu-x_\nu\partial_\mu+\tfrac{1}{4}\psi^TC[\gamma_\mu,\gamma_\nu]C^{-1}\partial/\partial\psi ,\\&P_\mu=\partial_\mu ,\quad S_\alpha=\frac{i}{2}\gamma^\mu_{\alpha\beta}\psi_\beta\partial_\mu-iC_{\alpha\beta}\partial/\partial\psi_\beta .\end{aligned} \quad (5.16)$$

Thus S_α transforms the superspace coordinates according to

$$\begin{aligned}&x_\mu\to x_\mu-\tfrac{1}{2}\epsilon^TC\gamma_\mu\psi ,\\&\psi_\alpha\to\psi_\alpha+\epsilon_\alpha ,\end{aligned} \quad (5.17)$$

where ϵ_α is also a Majorana spinor. The transformations of Eq. (5.17) are somewhat asymmetric since though x_μ transforms into ψ_α,ψ_α does not transform into x_μ. This stands in contrast to our transformations of Eq. (5.12) where x_μ and ψ_α do mix with each other. For our purposes here we note that the transformations of Eq. (5.17) do not leave $mx^2-2f\bar{\psi}\psi$ (or any other linear combination of x^2 and $\bar{\psi}\psi$) invariant.

A generalization of the superalgebra of Eq. (5.15) to a case of more interest to us here, namely to two Majorana spinor generators $S_{\alpha i}$ $(i=1,2)$ has also been given by Salam and Strathdee[9] with a modified anticommutator (the commutation relations remain unchanged)

$$\{S_{\alpha i},S_{\beta j}\}=i\epsilon_{ij}(\gamma_\mu\gamma_5C)_{\alpha\beta}P^\mu . \quad (5.18)$$

In the superspace consisting of x_μ and two Majorana spinors $\psi_{\alpha i}$ $(i=1,2)$ the spinor generators have a differential representation

$$S_{\alpha i}=\tfrac{1}{2}(\gamma_\mu\gamma_5)_{\alpha\beta}\psi_{\beta i}\partial^\mu+i\epsilon_{ij}C_{\alpha\beta}\partial/\partial\psi_{\beta j} \quad (5.19)$$

and transform the superspace coordinates according to

$$\begin{aligned}&x_\mu\to x_\mu+\frac{i}{2}\epsilon_i^TC\gamma_\mu\gamma_5\psi_i ,\\&\psi_{\alpha i}\to\psi_{\alpha i}+\epsilon_{ij}\epsilon_{\alpha j} .\end{aligned} \quad (5.20)$$

Thus again $mx^2-2f\bar{\psi}\psi$ is not left invariant. Since the algebra of Eq. (5.18) is the only possible grading of the Poincaré algebra with two Majorana spinors[10] we thus see that it is impossible to grade the Poincaré algebra with supertransformations which leave $mx^2-2f\bar{\psi}\psi$ invariant.

To illustrate the point further we note that if we do want to keep $mx^2-2f\bar{\psi}\psi$ invariant we must seek transformations which mix x_μ and ψ_α with each other. Thus at first we might consider the transformations [viz., Eq. (5.13) without the $i\bar{\theta}\theta$ factor]

$$\begin{aligned}&x_\mu\to x_\mu-\frac{2}{m}\epsilon_i^TC\gamma_\mu\psi_i ,\\&\psi_{\alpha i}\to\psi_{\alpha i}+\frac{1}{f}x_\mu\gamma^\mu_{\alpha\beta}\epsilon_{\beta i} .\end{aligned} \quad (5.21)$$

However then, under anticommutation the associated spinor generators

$$S_{\alpha i}=\frac{2i}{m}\gamma^\mu_{\alpha\beta}\psi_{\beta i}\partial_\mu+\frac{i}{f}x_\mu(\gamma^\mu C)_{\alpha\beta}\partial/\partial\psi_{\beta i} \quad (5.22)$$

do not close on P_μ (or $M_{\mu\nu}$ either for that matter). Thus we confirm the general result of Ref. 10, with the generators of Eq. (5.22) simply not belonging to a grading of the Poincaré algebra.

The only apparent way left to satisfy the general analysis of Ref. 10 is to introduce the additional $\bar{\theta}\theta$ factor (i.e., a Grassmann even number whose square is zero) and generators

$$S_{\alpha i}=\bar{\theta}\theta\left[-\frac{2}{m}\gamma^\mu_{\alpha\beta}\psi_{\beta i}\partial_\mu-\frac{1}{f}x_\mu(\gamma^\mu C)_{\alpha\beta}\partial/\partial\psi_{\beta i}\right] . \quad (5.23)$$

These generators then do indeed generate the transformations of Eq. (5.13) while closing under anticommutation according to

$$\{S_{\alpha i},S_{\beta j}\}=0 . \quad (5.24)$$

Thus our chosen superalgebra is simply the trivial direct product of the algebra of the $S_{\alpha i}$ with the Poincaré algebra and there is no mixing of the respective generators under anticommutation. Thus we see that if we want to preserve the length of $mx^2-2f\bar{\psi}\psi$ in the superspace we must not look for a grading of the Poincaré algebra at all, but rather we must introduce the additional Grassmann numbers θ and $\bar{\theta}$ and use the generators of Eq. (5.23). Hence in our work supersymmetry is indeed useful but no grading of the Poincaré algebra can be considered. Having now discussed the algebraic structure of our supersymmetry transformations and motivated the use of Eq. (5.13) and the need for the $\bar{\theta}\theta$ factor, we turn at last to an analysis of the dynamics associated with our supersymmetric Lagrangian.

For the Lagrangian of Eq. (5.14) canonical conjugates are defined via

$$\begin{aligned} p^{\mu}&=-\frac{\partial\mathscr{L}}{\partial\dot{x}_{\mu}}=m\dot{x}^{\mu}-g\bar{\theta}\theta\dot{\bar{\psi}}\gamma^{\mu}\dot{\psi}\ ,\\ \pi_{\alpha}&=\frac{\partial\mathscr{L}}{\partial\dot{\bar{\psi}}_{\alpha}}=f\dot{\psi}_{\alpha}+g\bar{\theta}\theta(\gamma_{\mu}\dot{x}^{\mu}+i\gamma_5)_{\alpha\beta}\dot{\psi}_{\beta}\ ,\\ \bar{\pi}_{\alpha}&=\frac{\partial\mathscr{L}}{\partial\dot{\psi}_{\alpha}}=f\dot{\bar{\psi}}_{\alpha}+g\bar{\theta}\theta\dot{\bar{\psi}}_{\beta}(\gamma_{\mu}\dot{x}^{\mu}+i\gamma_5)_{\beta\alpha}\ . \end{aligned} \tag{5.25}$$

Using the Grassmann properties of θ and $\bar{\theta}$ we find that the Hamiltonian form of the Lagrangian is given as

$$\begin{aligned} \mathscr{L}=&-p\cdot\dot{x}+\dot{\bar{\psi}}\pi+\bar{\pi}\dot{\psi}+\frac{(p^2-m^2)}{2m}\\ &-\frac{\bar{\pi}\pi}{f}+\frac{g}{mf^2}\bar{\theta}\theta\bar{\pi}(\not{p}+im\gamma_5)\pi\ . \end{aligned} \tag{5.26}$$

The classical equations of motion are

$$\begin{aligned} &p^{\mu}-m\dot{x}^{\mu}+\frac{g}{f^2}\bar{\theta}\theta\bar{\pi}\gamma^{\mu}\pi=0\ ,\\ &\pi_{\alpha}-f\dot{\psi}_{\alpha}-\frac{g}{mf}\bar{\theta}\theta(\gamma^{\mu}p_{\mu}+im\gamma_5)_{\alpha\beta}\pi_{\beta}=0\ ,\\ &\bar{\pi}_{\alpha}-f\dot{\bar{\psi}}_{\alpha}-\frac{g}{mf}\bar{\theta}\theta\bar{\pi}_{\beta}(\gamma^{\mu}p_{\mu}+im\gamma_5)_{\beta\alpha}=0\ ,\\ &\dot{p}_{\mu}=0\ ,\quad \dot{\pi}_{\alpha}=0\ ,\quad \dot{\bar{\pi}}_{\alpha}=0\ , \end{aligned} \tag{5.27}$$

and yield

$$\ddot{x}_{\mu}=0\ ,\quad \ddot{\psi}_{\alpha}=0\ ,\quad \ddot{\bar{\psi}}_{\alpha}=0\ . \tag{5.28}$$

We thus see that because of the Grassmann properties of θ and $\bar{\theta}$ the space-time and Grassmann coordinates decouple in the classical equations of motion with $x_{\mu}^{\mathrm{ST}}(\tau)$, $\psi_{\alpha}^{\mathrm{ST}}(\tau)$, and $\bar{\psi}_{\alpha}^{\mathrm{ST}}(\tau)$ still being given by Eqs. (2.5) and (3.14). The stationary space-time and Grassmann momenta are given by

$$\begin{aligned} p_{\mu}^{\mathrm{ST}}(\tau)&=\frac{m(x_{\mu}^{f}-x_{\mu}^{i})}{T}-\frac{g}{T^2}\bar{\theta}\theta(\bar{\eta}_f-\bar{\eta}_i)\gamma_{\mu}(\eta_f-\eta_i)\ ,\\ \pi_{\alpha}^{\mathrm{ST}}(\tau)&=\frac{f(\eta_{\alpha}^{f}-\eta_{\alpha}^{i})}{T}\\ &\quad+\frac{g}{T}\bar{\theta}\theta\left[\frac{\gamma^{\mu}(x_{\mu}^{f}-x_{\mu}^{i})}{T}+i\gamma_5\right]_{\alpha\beta}(\eta_{\beta}^{f}-\eta_{\beta}^{i})\ ,\\ \bar{\pi}_{\alpha}^{\mathrm{ST}}(\tau)&=\frac{f(\bar{\eta}_{\alpha}^{f}-\bar{\eta}_{\alpha}^{i})}{T}\\ &\quad+\frac{g}{T}\bar{\theta}\theta(\bar{\eta}_{\beta}^{f}-\bar{\eta}_{\beta}^{i})\left[\frac{\gamma^{\mu}(x_{\mu}^{f}-x_{\mu}^{i})}{T}+i\gamma_5\right]_{\beta\alpha}\ , \end{aligned} \tag{5.29}$$

while the stationary classical action is

$$\begin{aligned} S_{\mathrm{ST}}=&-\frac{m(x_{\mu}^{f}-x_{\mu}^{i})^2}{2T}-\frac{mT}{2}+\frac{f}{T}(\bar{\eta}_f-\bar{\eta}_i)(\eta_f-\eta_i)\\ &+\frac{g}{T}\bar{\theta}\theta(\bar{\eta}_f-\bar{\eta}_i)\left[\frac{\gamma^{\mu}(x_{\mu}^{f}-x_{\mu}^{i})}{T}+i\gamma_5\right](\eta_f-\eta_i)\ . \end{aligned} \tag{5.30}$$

To specify the measure for the path integration we make the same expansion for the arbitrary $x_{\mu}(\tau)$, $p_{\mu}(\tau)$, $\psi_{\alpha}(\tau)$, $\bar{\psi}_{\alpha}(\tau)$, $\pi_{\alpha}(\tau)$, and $\bar{\pi}_{\alpha}(\tau)$ paths as in Eqs. (2.9), (2.10), and (3.16) except that now we expand around the solutions of Eqs. (5.27). After a little algebra we find that in the arbitrary path the action is given by

$$\begin{aligned} S=&S_{\mathrm{ST}}+\frac{Tb_{\mu}^{02}}{2m}-\frac{T\bar{\gamma}^{0}\gamma^{0}}{f}+\frac{Tg}{mf^2}\bar{\theta}\theta b_{\mu}^{0}J_{0}^{\mu}+\frac{Tg}{mf^2}\bar{\theta}\theta\bar{\gamma}^{0}(\not{p}_{\mathrm{ST}}+im\gamma_5)\gamma^0\\ &+\sum_{n=1}^{\infty}\left[\frac{Tb_{\mu}^{n2}}{4m}-\frac{T\bar{\gamma}^{n}\gamma^{n}}{2f}+\frac{n\pi}{2}(-a_{\mu}^{n}b_{n}^{\mu}+\bar{\beta}^{n}\gamma^{n}+\bar{\gamma}^{n}\beta^{n})+\frac{g}{mf^2}\bar{\theta}\theta b_{\mu}^{n}J_{n}^{\mu}+\frac{Tg}{2mf^2}\bar{\theta}\theta\bar{\gamma}^{n}(\not{p}_{\mathrm{ST}}+im\gamma_5)\gamma^{n}\right]\ , \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} J_{\mu}^{0}&=\bar{\pi}_{\mathrm{ST}}\gamma_{\mu}\gamma^{0}+\bar{\gamma}^{0}\gamma_{\mu}\pi_{\mathrm{ST}}+\bar{\gamma}^{0}\gamma_{\mu}\gamma^{0}+\tfrac{1}{2}\sum_{n=1}^{\infty}\bar{\gamma}^{n}\gamma_{\mu}\gamma^{n}\ ,\\ J_{\mu}^{n}&=\frac{T}{2}(\bar{\pi}_{\mathrm{ST}}+\bar{\gamma}^{0})\gamma_{\mu}\gamma^{n}+\frac{T}{2}\bar{\gamma}^{n}\gamma_{\mu}(\pi_{\mathrm{ST}}+\gamma^{0})+\sum_{k,m=1}^{\infty}\bar{\gamma}^{k}\gamma_{\mu}\gamma^{m}\int d\tau\cos\frac{n\pi\tau}{T}\cos\frac{k\pi\tau}{T}\cos\frac{m\pi\tau}{T}\ . \end{aligned} \tag{5.32}$$

Variation of Eq. (5.31) then confirms that S_{ST} is the stationary classical action.

Having now specified the underlying classical theory completely we can perform the path integration

$$G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})=G(x^f_\mu,\eta^f_\alpha,\bar{\eta}^f_\alpha,T;x^i_\mu,\eta^i_\alpha,\bar{\eta}^i_\alpha,0)$$
$$=\theta(T)\int[dx][dp][d\psi][d\bar{\psi}][d\pi][d\bar{\pi}]\exp(iS/\hbar)\ , \quad (5.33)$$

where the action is given in Eq. (5.31) and the measure is the product of the measures given in Eqs. (2.12) and (3.20). The actual integration can be performed readily since shifting the b^n_μ and a^n_μ variables eliminates the dependence on J^0_μ and J^n_μ, and yields

$$G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})$$
$$=-i\theta(T)\left[\frac{m}{2\pi i\hbar T}\right]^2\frac{\hbar^4T^4}{D^{16}f^4}\exp\left[\frac{iS_{\rm ST}}{\hbar}\right], \quad (5.34)$$

where $S_{\rm ST}$ is given in Eq. (5.30). Finally, a Fourier transform to the conjugate momentum space yields

$$\tilde{G}(p,\pi,\bar{\pi},T,\theta,\bar{\theta})=\theta(T)\exp\left[-\frac{iH_{\rm CL}(p,\pi,\bar{\pi})T}{\hbar}\right], \quad (5.35)$$

where $H_{\rm CL}(p,\pi,\bar{\pi})$ is the classical Hamiltonian associated with the Lagrangian of Eq. (5.26), viz.,

$$H_{\rm CL}(p,\pi,\bar{\pi})=-\frac{(p^2-m^2)}{2m}+\frac{\bar{\pi}\pi}{f}-\frac{g}{mf^2}\bar{\theta}\theta\bar{\pi}(\not{p}+im\gamma_5)\pi\ . \quad (5.36)$$

Thus despite the trilinear coupling g term which couples the various nonstationary paths in the path integration we are able to integrate the propagator right out to obtain the conventional $\exp(-iH_{\rm CL}T/\hbar)$ form.

As a check on our 5-space propagator we calculate it alternately via canonical quantization by evaluating

$$G_{if}=\theta(T)\langle x^f_\mu,\eta^f_\alpha,\bar{\eta}^f_\alpha\,|\exp(-i\hat{H}T/\hbar)\,|\,x^i_\mu,\eta^i_\alpha,\bar{\eta}^i_\alpha\rangle\ . \quad (5.37)$$

Using the quantum-mechanical Hamiltonian $\hat{H}$,

$$\hat{H}=-\frac{(\hat{p}^2-m^2)}{2m}+\frac{\hat{\bar{\pi}}\hat{\pi}}{f}-\frac{g}{mf^2}\bar{\theta}\theta\hat{\bar{\pi}}(\hat{\not{p}}+im\gamma_5)\hat{\pi}\ , \quad (5.38)$$

and the explicit sets of eigenstates given in Eqs. (4.11) then recovers Eq. (5.34) exactly, to thus confirm the complete consistency between the two quantization procedures.

To obtain the differential equation that the propagator obeys we rewrite Eq. (5.34) as

$$G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})=\theta(T)\frac{\hbar^8}{\tilde{D}^{16}D^{16}}\int\frac{d^4p}{(2\pi\hbar)^4}\int d^4\pi\int d^4\bar{\pi}\exp\left[-\frac{ip\cdot(x^f-x^i)}{\hbar}+\frac{i\bar{\pi}(\eta_f-\eta_i)}{\hbar}+\frac{i(\bar{\eta}_f-\bar{\eta}_i)\pi}{\hbar}-\frac{iH_{\rm CL}(p,\pi,\bar{\pi})T}{\hbar}\right] \quad (5.39)$$

and obtain

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{\hbar^2}{2m}\partial^f_\mu\partial^\mu_f-\frac{m}{2}-\frac{1}{f}\left[i\hbar\frac{\partial}{\partial\eta^f_\alpha}\right]\left[-i\hbar\frac{\partial}{\partial\bar{\eta}^f_\alpha}\right]+\frac{g}{mf^2}\bar{\theta}\theta\left[i\hbar\frac{\partial}{\partial\eta^f_\alpha}\right](i\hbar\gamma^\mu\partial^f_\mu+im\gamma_5)_{\alpha\beta}\left[-i\hbar\frac{\partial}{\partial\bar{\eta}^f_\beta}\right]\right]$$
$$\times G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})=i\hbar\delta(T)\delta^4(x^f-x^i)\prod_\alpha(\bar{\eta}^f_\alpha-\bar{\eta}^i_\alpha)(\eta^f_\alpha-\eta^i_\alpha)/D^{16}\ . \quad (5.40)$$

Thus $G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})$ is indeed a Green's function with Eq. (5.40) being completely analogous to Eq. (4.19).

The propagator of Eq. (5.35) is the propagator in the five-dimensional superspace. To reduce it to the usual four-dimensional space we define

$$\tilde{G}(p,\pi,\bar{\pi})=-\frac{im}{2\hbar}\int_{-\infty}^{\infty}dT(-i)\int d\theta\,d\bar{\theta}\,\tilde{G}(p,\pi,\bar{\pi},T,\theta,\bar{\theta}) \quad (5.41)$$

by a supersymmetric analogy with Eqs. (2.16) and (2.25). This yields

$$\tilde{G}(p,\pi,\bar{\pi})=\frac{2igm^2}{f^2}\frac{\bar{\pi}(\not{p}+im\gamma_5)\pi}{(p^2-m^2+i\epsilon)^2}\left[1+\frac{4m\bar{\pi}\pi}{f(p^2-m^2+i\epsilon)}+\frac{12m^2(\bar{\pi}\pi)^2}{f^2(p^2-m^2+i\epsilon)^2}+\frac{32m^3(\bar{\pi}\pi)^3}{f^3(p^2-m^2+i\epsilon)^3}\right] \quad (5.42)$$

with the $\bar{\pi}\pi$ series terminating since the four components of π_α and $\bar{\pi}_\alpha$ are Grassmann numbers. This interesting propagator has the correct covariant form of $(\not p+im\gamma_5)$ multiplied by powers of (p^2-m^2). Moreover, the leading $(\not p+im\gamma_5)/(p^2-m^2)^2$ term is more convergent in the ultraviolet than the standard Dirac propagator. However, since it behaves like the derivative of an ordinary pole term, it would appear to have interpretational difficulties near the mass shell, and so we shall not consider it further here.

To obtain a more readily interpretable propagator we recall that the standard Dirac propagator has only one parameter, the mass m, and is built from spinors of dimension $\hbar^{1/2}$. If we therefore restrict our theory by requiring that $\psi_\alpha(\tau)$ and (consequently) $\pi_\alpha(\tau)$ both have dimension $\hbar^{1/2}$ we find that the parameters f and g in the Lagrangian then both have the same dimension as T. We can thus eliminate all unknown parameters and reduce to the standard Dirac case if we set both f and g equal to T. This yields a superspace Lagrangian

$$\mathscr{L}=-\frac{m\dot{x}^2}{2}-\frac{m}{2}+T\dot{\bar{\psi}}\dot{\psi}+T\bar{\theta}\theta\dot{\bar{\psi}}(\gamma^\mu\dot{x}_\mu+i\gamma_5)\dot{\psi} \tag{5.43}$$

with classical Hamiltonian

$$H_{\rm CL}=-\frac{(p^2-m^2)}{2m}+\frac{\bar{\pi}\pi}{T}-\frac{1}{mT}\bar{\theta}\theta\bar{\pi}(\not p+im\gamma_5)\pi\ . \tag{5.44}$$

The presence of T in the Lagrangian is then a supersymmetric reflection of the presence of $\bar{\theta}\theta$ in the Lagrangian. Since T multiplies all $\dot{\bar{\psi}}\dot{\psi}$ type terms universally in the Lagrangian, any change in the 5-space T interval whenever the end points of the classical motion are changed can be absorbed in a renormalization of $\psi_\alpha(\tau)$. For the classical Hamiltonian of Eq. (5.44) the five-dimensional superspace propagator obeys the Klein-Gordon-like

$$\left[i\hbar\frac{\partial}{\partial T}-\frac{\hbar^2}{2m}\partial_\mu^f\partial_f^\mu-\frac{m}{2}\right]G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})$$
$$=i\hbar\delta(T)\exp\left[\frac{i(\bar{\eta}_f-\bar{\eta}_i)(\eta_f-\eta_i)}{\hbar}+\frac{i\bar{\theta}\theta}{\hbar m}(\bar{\eta}_f-\bar{\eta}_i)\left[i\hbar\gamma^\mu\frac{\partial}{\partial x_f^\mu}+im\gamma_5\right](\eta_f-\eta_i)-4\bar{\theta}\theta\right]\frac{\delta^4(x^f-x^i)}{\hbar^4} \tag{5.45}$$

and the Dirac-like

$$\bar{\theta}\theta\left[i\hbar\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}-\left[i\hbar\frac{\partial}{\partial\eta_\alpha^f}\right]\left[\frac{i\hbar\gamma^\mu\partial_\mu^f+im\gamma_5)}{m}\right]_{\alpha\beta}\left[-i\hbar\frac{\partial}{\partial\bar{\eta}_\beta^f}\right]\right]G_{if}(x,\eta,\bar{\eta},T,\theta,\bar{\theta})=0 \tag{5.46}$$

differential equations. Equation (5.46) is a superspace analog of the Dirac equation. [There is no Grassmann delta function on the right-hand side of Eq. (5.46) since there is no apparent meaning to forward propagation in θ.] Thus $(\partial/\partial\bar{\theta})(\partial/\partial\theta)$ acts as a Grassmann "time."[11]

Before integrating out θ and $\bar{\theta}$ it is instructive to first integrate out T to obtain

$$\frac{-im}{2\hbar}\int_{-\infty}^{\infty}dT\,\tilde{G}(p,\pi,\bar{\pi},T,\theta,\bar{\theta})$$
$$=\frac{m^2}{(p^2-m^2+i\epsilon)}\left[1+\frac{i\bar{\theta}\theta}{m\hbar}\bar{\pi}(\not p+im\gamma_5)\pi\right]\times\exp\left[-\frac{i\bar{\pi}\pi}{\hbar}\right], \tag{5.47}$$

which has the nice form of a Klein-Gordon propagator plus a Dirac propagator. By means of Eq. (5.41) we now integrate out θ and $\bar{\theta}$ to reduce the theory to ordinary four-dimensional space-time and obtain[12]

$$\tilde{G}(p,\pi,\bar{\pi})=\frac{m}{\hbar}\frac{\bar{\pi}(\not p+im\gamma_5)\pi}{(p^2-m^2+i\epsilon)}\exp\left[-\frac{i\bar{\pi}\pi}{\hbar}\right]. \tag{5.48}$$

Finally, a simple chiral rotation

$$\pi\to\exp\left[-\frac{i\pi\gamma_5}{4}\right]\pi \tag{5.49}$$

brings our propagator to the form

$$\tilde{G}(p,\pi,\bar{\pi})=\frac{m}{\hbar}\frac{\bar{\pi}(\not p+m)\pi}{(p^2-m^2+i\epsilon)}\exp\left[-\frac{\bar{\pi}\gamma_5\pi}{\hbar}\right], \tag{5.50}$$

which we recognize as the standard Dirac propagator with its familiar pole structure, to give us our desired objective.

VI. GENERAL COMMENTS

In this paper we have developed a new formulation of the quantum mechanics of a relativistic fermion based on the existence of an extended set of underlying classical coordinates. We have enlarged the classical theory to a superspace built out of both space-time and Grassmann coordinates. The quantization of the motion of these coordinates in the superspace then gives rise to quantized fermions with a standard Dirac structure. We believe that our work thus gives some new insight into the basic structure of fermions.

While our work stresses the primacy of particles over fields we should point out that there is one possible exception, namely the massless case. Specifically, in that case the action of Eq. (2.1) vanishes identically, and there is no other mass parameter available to give the action the correct dimension. Consequently, our formalism cannot be readily extended to massless particles or gauge fields. While gauge fields may thus exist as explicit fundamental classical fields, we can at this stage only speculate on a few alternative possibilities which require further study. First, gauge fields may exist at the classical level but may only arise via a local extension of the superspace supersymmetry. Second, gauge fields may only exist at the quantum level being therefore first-quantized *a priori*, with classical light then being described by macroscopically occupied quantum states whose dynamics is independent of $\hbar$ because of destructive interference. Finally, gauge fields may not be fundamental at all but may be fermion composites.

As well as providing a new approach to the quantum theory of a single species of fermion, our work has an interesting implication for the quantum theory of different types of fermions. Specifically, since electrons and protons, say, separately obey the Pauli principle, pairs of electrons and pairs of protons are separately antisymmetric under interchange. However, there is no Pauli principle between an electron and a proton leaving their interchange structure initially unclear. In our formalism we now note that both electrons and protons are associated with sets of Grassmann coordinates, which then mutually anticommute already in the underlying classical theory, even prior to quantization, to thus establish an antisymmetry between electrons and protons in the quantum theory.

We conclude this work with one final comment on the Grassmann coordinates. If Grassmann coordinates exist in the underlying classical theory we must ask why they have not therefore been observed experimentally. The reason for this is because the observable classical limit of quantum mechanics is actually a double limit, namely the limit of vanishing $\hbar$ in the equations of motion and also the limit of macroscopic occupation of quantum states. Because of the Pauli principle the Grassmann quantum states of Eqs. (4.11) cannot ordinarily be macroscopically occupied and thus play no role in the classical limit. The only apparent exception to this is in a superconductor where the ground state is macroscopically occupied. It would thus be of some interest to see whether there is some observable quantity in a superconductor (perhaps an electron trapped on a vortex line) which would reveal the existence of Grassmann coordinates.

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[1]There is a fairly extensive literature on the use of ξ_μ, which includes F.A. Berezin and M. S. Marinov, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 678 (1975) [JETP Lett. **21**, 320 (1975)]; A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento **35A**, 377 (1976); F. A. Berezin and M. S. Marinov, Ann. Phys. (N.Y.) **104**, 336 (1977); P. DiVecchia and F. Ravndal, Phys. Lett. **73A**, 371 (1979); A. Barducci, F. Bordi, and R. Casalbuoni, Nuovo Cimento **64B**, 287 (1981), and references therein.

[2]With the Majorana spinor ψ_α thus being another coordinate, the coordinates x_μ and ψ_α (which is usually called θ_α in the superspace literature) form a classical superspace of the type introduced by Salam and Strathdee [A. Salam and J. Strathdee, Nucl. Phys. **B76**, 477 (1974)]. Our work differs from theirs and its subsequent applications in supersymmetry and supergravity theories [see, e.g., P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981) for a recent review] in two essential ways. First, for Salam and Strathdee the superspace coordinates are purely kinematical classical coordinates with the superfields (which are functions of the superspace coordinates) then describing the dynamics; for us the classical superspace coordinates are the dynamical variables themselves with the quantization of their motion in superspace giving rise to the quantum theory. Second, as will be seen below in Sec. V, even though we will use the same set of superspace coordinates as Salam and Strathdee we will impose a different superalgebra on the dynamics.

[3]P. D. Mannheim, Phys. Lett. **137B**, 385 (1984).

[4]R. P. Feynman, Phys. Rev. **80**, 440 (1950); see especially Appendix A.

[5]Y. Nambu, Prog. Theor. Phys. **5**, 82 (1950).

[6]For an approach which is partway between ours and Feynman's see R. Casalbuoni, J. Gomis, and G. Longhi, Nuovo Cimento **24A**, 249 (1974), which uses the classical action of Eq. (2.4) and Nambu's quantum projection condition of Eqs. (2.24) and (2.25) below.

[7]A recent pedagogical review of these aspects of Majorana spinors may be found in P. D. Mannheim, Int. J. Theor. Phys. **23**, 643 (1984).

[8]There is an analog to this result in standard Dirac theory when restricted to real space. The only real solutions to the real Dirac equation $(i\hbar\gamma^\mu\partial_\mu - m)\psi(x)=0$ [in the Majorana basis of Eq. (3.2) each $i\gamma^\mu$ is purely real] are of the form $\exp(p\cdot x/\hbar)$. These solutions satisfy $(i\not{p}-m)\psi=0$ and consequently have a $p^2+m^2=0$ mass shell.

[9]A. Salam and J. Strathdee, Nucl. Phys. **B80**, 499 (1974).

[10]R. Haag, J. T. Lopuszanski, and M. Sohnius, Nucl. Phys. **B88**, 257 (1975).

[11]Thus just as the inverse of $i\hbar\partial/\partial T$ is used as the integrating T factor for the measure in Eq. (5.41), Eq. (5.46) implies that we should use $-i\int d\theta \int d\bar{\theta}$ as the analogous dimensionless $\theta,\bar{\theta}$ measure for the $\theta,\bar{\theta}$ projection in Eq. (5.41).

[12]The new factor $\exp(-i\bar{\pi}\pi/\hbar)$ is fairly innocuous since it is just the Grassmann Fourier transform of products of the form

$$\prod_\alpha [1+i(\bar{\eta}^f_\alpha-\bar{\eta}^i_\alpha)(\eta^f_\alpha-\eta^i_\alpha)/\hbar] ,$$

where $\alpha=1,2,3,4$ is not summed, to give a string of Grassmann delta functions, each of which would serve to project out appropriate eigenstates of the operators $\hat{\psi}_\alpha$ and $\hat{\bar{\psi}}_\alpha$ of Eq. (4.1) in a Dyson Wick expansion for an interacting theory in τ space.