Classical limit of the Korteweg-de Vries hierarchy of isospectral transformations

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The Korteweg-de Vries hierarchy of equations, whose solutions are quantum-mechanical isospectral potentials, is shown to reduce, in the classical limit, to a family of equations whose solutions are isoperiodic potentials. The quantum-mechanical spectra of several families of classically isoperiodic potentials are examined in the light of the above correspondence.

I. INTRODUCTION

The characteristic parameter of a quantum-mechanical bound state is its total energy. The corresponding characteristic of a classical bound trajectory is its period. The connection between the classical period and the quantal energy is well known in the context of the WKB approximation. Confining our attention to one-dimensional potentials, for which the trajectories are always regular, we shall be interested in the relation between transformations of the potential under which the classical period is invariant and transformations under which the quantum-mechanical energy is invariant. One case of this relation, namely, that between a quantum-mechanical equidistant spectrum and a classical energy-independent period, has recently been investigated.[1-3] It was established that three indenumerably infinite sets of potentials exist, each showing one of the following characteristics: (a) energy-independent classical period and equidistant quantal spectrum; (b) energy-independent classical period and nonequidistant quantal spectrum; or (c) energy-dependent classical period and equidistant quantal spectrum.

As is well known, the quantum-mechanical energy is invariant under any unitary transformation of the Hamiltonian. Similarly, transformations of the classical Hamiltonian (or Lagrangian) under which the equations of motion are invariant have been studied.[4] Clearly, invariance of the equations of motion is a sufficient, though certainly not a necessary condition for the invariance of the classical period. In general, quantal unitary transformations as well as classical period-preserving transformations do not retain the standard form of the Hamiltonian. However, certain classes of transformations, which only affect the potential, transforming a local potential into another local potential, are known. The simplest such transformation, which is trivially valid both classically and quantally, is the shift transformation

$$V(x) \rightarrow V(x+t) , \tag{1}$$

where t is an arbitrary parameter. Using the notation $V_t = \partial V/\partial t$, $V_x = \partial V/\partial x$ we can express this transformation by the partial differential equation

$$V_t = V_x , \tag{2}$$

whose solution $V(x,t)$ satisfying the initial condition $V(x,0) = V_0(x)$ is

$$V(x,t) = V_0(x+t) . \tag{3}$$

In quantum mechanics this transformation is known to be the first in an infinite hierarchy of transformations of the potential, each one of which preserves the bound-state energies.[5] The next transformation in this hierarchy is expressed by the Korteweg-de Vries (KdV) equation

$$V_t = \tfrac{3}{2} V V_x - \tfrac{1}{4} V_{xxx} . \tag{4}$$

Potentials which support identical sets of eigenvalues are known as isospectral potentials.

In classical mechanics a simple period-preserving transformation can be constructed by distorting the confining potential $V_0(x)$, which we assume to have a single minimum, into $V(x)$, in such a way that the classical turning points $X_{\pm}^0(E)$ and $X_{\pm}(E)$ satisfy

$$X_+(E) - X_-(E) = X_+^0(E) - X_-^0(E) \tag{5}$$

for all energies.[6] Here, X_+ (X_-) denotes the right (left) classical turning point. We shall refer to potentials related by a period-preserving transformation as isoperiodic potentials. We note in passing that the shift transformation, which is the first member of the KdV hierarchy, is also an isoperiodic transformation, satisfying Eq. (5) trivially.

Isospectral transformations in quantum mechanics are of interest in the context of the inverse problem, involving the determination of the potential from spectral data.[7] The corresponding isoperiodic transformations in classical mechanics have an analogous role with respect to the classical inverse problem.[6] The well-known Rydberg-Klein-Rees (RKR) procedure[8] enabling the determination of molecular potential surfaces from rotation-vibration spectra is one example in which the characterization of the possibility of nonuniqueness of the inverse problem is of practical interest.

II. THE CLASSICAL LIMIT OF THE KdV HIERARCHY OF TRANSFORMATIONS

By way of introduction, we derive the KdV hierarchy of isospectral transformations for the potential in the one-dimensional Schrödinger equation

$$\hbar^2 \psi_{xx} = (V - \lambda)\psi , \tag{6}$$

using a slight modification of the approach presented by Gardner *et al.*[9] The potential is a function of both the dynamical coordinate x and a parameter t, $V = V(x,t)$. We are interested in characterizing the types of dependence of V on t which do not affect the eigenvalues. From the Hellmann-Feynman theorem it follows that if the eigenvalues are independent of t, then

$$\lambda_t = \int_{-\infty}^{\infty} V_t \psi^2 dx = 0 . \tag{7}$$

Equation (7) is an identity satisfied for all values of t, so that it implies $\lambda(t) = \lambda(0)$. We shall assume that the t dependence of the potential can be specified by the partial differential (evolution) equation

$$V_t = P(V, V_x, V_{xx}, \dots, V_n) . \tag{8}$$

From Eq. (7) it follows that P should satisfy

$$\int_{-\infty}^{\infty} P \psi^2 dx = 0 . \tag{9}$$

It is easy to show by repeated integration by parts that Eq. (9) is satisfied if P is of the form

$$P = \frac{\hbar^2}{2} A_{xxx} - 2(V - \lambda) A_x - V_x A , \tag{10}$$

where A is an arbitrary function of x, t, and λ. Note that our A differs from that in Ref. 9 by a factor of $\hbar^{-2}$. Equation (9), with P as given by Eq. (10), can also be derived from the hypervirial theorem

$$\int_{-\infty}^{\infty} \psi^*[H,W]\psi \, dx = 0$$

with $W = 2Ap - (\hbar/i)A_x$, where $p = (\hbar/i)\partial/\partial x$ and $H = p^2 + V$.

We wish to determine the form of A so as to make P independent of λ. Writing A in the form

$$A = \lambda^n \left[A_0 + \frac{1}{\lambda} A_1 + \frac{1}{\lambda^2} A_2 + \cdots + \frac{1}{\lambda^n} A_n \right] , \tag{11}$$

where A_i $(i = 0, 1, \dots, n)$ are independent of λ, and requiring that all the λ-dependent terms in P vanish, we obtain

$$V_t = P = \frac{\hbar^2}{2} A_{n,xxx} - 2V A_{n,x} - V_x A_n \tag{12}$$

as well as the recursion formula

$$A_{m+1,x} = V A_{m,x} + \frac{1}{2} A_m V_x - \frac{\hbar^2}{4} A_{m,xxx} , \tag{13}$$

where $m = 0, 1, \dots, n-1$. Note that Eq. (13) is independent of n. It can be used to write Eq. (12) in the form

$$V_t = -2A_{n+1,x} . \tag{14}$$

For $n = 0$ Eq. (14) can obtain the form $V_t = V_x$, provided that we choose $A_0 = -1$. As pointed out in the Introduction, this equation, which is independent of $\hbar$, is equally valid in classical and quantum mechanics. A slight generalization can be achieved by choosing $A_0 = -\phi_0(t)$, where $\phi_0(t)$ is an arbitrary function of t. With this choice of A_0, Eq. (14) becomes $V_t = \phi_0(t)V_x$. The solution of this equation satisfying $V(x,0) = V_0(x)$ is $V(x,t) = V_0(x + \Phi_0(t))$, where $\Phi_0(t) = \int_0^t \phi_0(t)dt$. This is still an

origin shift transformation, which is both isoperiodic and isospectral. With the choice of A_0 just made, it follows from Eq. (13) that

$$A_{1,x} = -\frac{1}{2}\phi_0(t)V_x \tag{15}$$

or

$$A_1 = -\frac{1}{2}\phi_0(t)V + \phi_1(t) , \tag{16}$$

where $\phi_1(t)$ is arbitrary. Thus, for $n = 1$ Eq. (14) results in

$$V_t = -\frac{\hbar^2}{4}\phi_0(t)V_{xxx} + [\frac{3}{2}\phi_0(t)V - \phi_1(t)]V_x \tag{17}$$

which, for $\phi_0(t) = 1$ and $\phi_1(t) = $const, reduces to the KdV equation. The classical limit of this equation, obtained by setting $\hbar = 0$, is

$$V_t = [\frac{3}{2}\phi_0(t)V - \phi_1(t)]V_x . \tag{18}$$

A thorough mathematical investigation of this limit was carried out by Lax and Levermore,[10] who refer to it as the zero-dispersion limit. The following equation in the hierarchy can be derived in an entirely analogous manner. However, it is more interesting to note that although the complete hierarchy grows more and more complicated, the classical limits retain a simple form in the various orders. This can be seen as follows. Setting $\hbar = 0$ in Eq. (13) we obtain

$$A_{m+1,x} = V A_{m,x} + \frac{1}{2} A_m V_x . \tag{19}$$

Note that Eq. (14) does not contain $\hbar$ explicitly, so that its form remains unchanged in the classical limit. Using the expression for A_1 obtained above [Eq. (16)] the following classical-limit expressions result:

$$A_2 = -\frac{3}{8}\phi_0(t)V^2 + \frac{1}{2}\phi_1(t)V + \phi_2(t) , \tag{20}$$

$$A_3 = -\frac{5}{16}\phi_0(t)V^3 + \frac{3}{8}\phi_1(t)V^2 + \frac{1}{2}\phi_2(t)V + \phi_3(t) , \tag{21}$$

etc. Equations (16), (20), and (21) suggest that in general

$$A_n = \sum_{m=0}^{n} \alpha_{n,m} V^{n-m} \phi_m(t) , \tag{22}$$

where $\phi_m(t)$ are arbitrary functions of t. Substituting Eq. (22) in Eq. (19) it follows that

$$\alpha_{n+1,m} = \frac{n - m + \frac{1}{2}}{n - m + 1} \alpha_{n,m} ,$$

$$m = 0, 1, \dots, n, \quad n = 0, 1, 2, \dots . \tag{23}$$

Equation (23) determines $\alpha_{n,m}(n > m)$ in terms of $\alpha_{m,m}$ and it follows easily that

$$\alpha_{n,m} = \frac{[2(n-m)-1]!!}{2^{n-m}(n-m)!} \alpha_{m,m} . \tag{24}$$

The choice of $\alpha_{m,m}$ is arbitrary because it merely amounts to normalization of the arbitrary functions $\phi_m(t)$. We shall therefore set $\alpha_{m,m} = -1$ for all m.

Thus, it follows from Eq. (14) that

$$V_t = F_n(V,t)V_x , \tag{25}$$

where

$$F_n(V,t) = \sum_{m=0}^{n} \beta_{n,m} V^{n-m} \phi_m(t)$$

and

$$\beta_{n,m} = \frac{[2(n-m)+1]!!}{2^{n-m}(n-m)!} . \tag{26}$$

III. SOLUTION OF THE CLASSICAL-LIMIT EQUATIONS

Each one of the equations in the hierarchy (25) is quasi-linear,[11] i.e., they involve the derivatives linearly. The initial values are given along the line $t=0$,

$$V(x,0) = V_0(x) .$$

Along the tangent to an arbitrary curve C passing through the point (x,t) (Fig. 1), we have

$$dV = V_x dx + V_t dt .$$

Substituting Eq. (25) for V_t we obtain

$$dV = V_x[dx + F_n(V,t)dt] . \tag{27}$$

Choosing the curve C, which will be referred to as the characteristic curve of the partial differential equation (25), so that

$$dx + F_n(V,t)dt = 0 , \tag{28}$$

we obtain along C $dV=0$, i.e., $V(x,t)=V_0(x_0)$. Consequently, Eq. (28) can be integrated into

$$x = x_0 - \sum_{m=0}^{n} \beta_{n,m} V_0(x_0)^{n-m} \Phi_m(t) , \tag{29}$$

where

$$\Phi_m(t) = \int_0^t \phi_m(t')dt' .$$

Equation (29) is the equation of the characteristic passing through $(x_0,0)$. Let x_0' be the point satisfying $V_0(x_0')=V_0(x_0)$. It follows from Eq. (29) that the characteristic through x_0' is parallel to the characteristic through x_0, so that for all t the equation
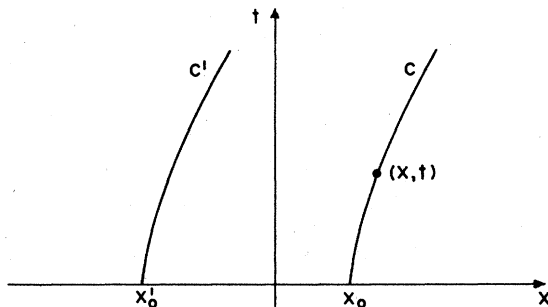
$x(t)-x'(t)=x_0-x_0'$ holds, as illustrated in Fig. 1. Since V is constant along the characteristics, the distortion of the potential specified by Eq. (25) satisfies the classical condition for isoperiodicity, Eq. (5). Each choice of $F_n(V,t)$ represents a different distortion, differing by the shapes of the characteristics passing through any initial point $(x,0)$, but they all belong to the indenumerably infinite class of isoperiodic distortions.

A simple geometrical interpretation may be helpful; it is presented in Fig. 2. Note that

$$\frac{tV_t(x_0,0)}{\Delta x_0} \simeq -V_x(x_0,0)$$

and

$$\frac{tV_t(x_0',0)}{\Delta x_0'} \simeq -V_x(x_0',0) .$$

Using Eq. (25) and noting that $V(x_0,0)=V(x_0',0)$, we obtain $\Delta x_0 = \Delta x_0'$.

One simple feature of the transformation of the potential defined by Eq. (25) is that the minimal value of $V(x,t)$ with respect to x is independent of t. Thus, the minima of all the potentials related by Eq. (25) are at the same height. This is certainly not the case for the quantum-mechanical transformations such as the KdV equation, as one can easily verify by considering a potential which has a positive third derivative at the minimum [say, $V_0 = x^4/4 - x^3/3$; $V_{0,x}(x_{min}=1)=0$; $V_{0,xxx}(x_{min}=1)=4$]. Clearly, from Eq. (17) with $\phi_0=1$, it follows that at the minimum of $V_0(x)$, $V_t = -(\hbar^2/4)V_{xxx}$, so that a positive V_{xxx} means a reduction in V, at x_{min}, for the distorted potential relative to the minimal value of $V_0(x)$. This rather trivial distinction between the quantum-mechanical transformation and its classical limit is relevant to the interpretation of some numerical results presented in Sec. V.

A special case of particular interest of Eq. (25) corresponds to the choice $\phi_m(t)=f_m=$const. In this case the characteristic is a straight line through x_0, whose slope is determined by $V_0(x_0)$. The two characteristics through x_0 and x_0' satisfying $V_0(x_0)=V_0(x_0')$ have equal slopes

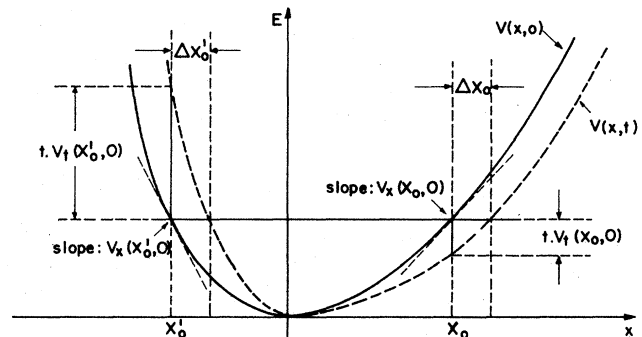FIG. 1. Parallel characteristic curves for $V_0(x_0)=V_0(x_0')$.

FIG. 2. The isoperiodic transformation.

and are therefore parallel to one another. Equation (25) can be written in the form

$$V_t = f(V)V_x , \tag{25'}$$

where $f(V)$ stands for $F_n(V,t)$ when all ϕ_m are independent of t.

Consider a potential $V_0(x)$ which is monotonically decreasing for $x < 0$ and monotonically increasing for $x > 0$. Let $V(x)$ be an arbitrary monotonically decreasing function of x for $x < 0$, such that $V(0) = V_0(0)$. In view of the monotonicity of both $V_0(x)$ and $V(x)$ for $x < 0$, we can invert them, obtaining $x = \phi_0^-(V)$ and $x = \phi^-(V)$, respectively. To define $V(x)$ for $x > 0$ we use the isoperiodicity condition as follows. First, inversion of $V_0(x)$ $(x > 0)$ results in $x = \phi_0^+(V)$. $\phi^+(V)$, the inverse of $V(x)$ for $x > 0$, is defined by means of the relation

$$\phi^+(V) - \phi_0^+(V) = \phi^-(V) - \phi_0^-(V) ,$$

i.e.,

$$x = \phi^+(V) = \phi_0^+(V) + \phi^-(V) - \phi_0^-(V) \tag{30}$$

which can be inverted into $V(x)$ $(x > 0)$. To determine the form of the function $f(V)$ generating the isoperiodic transformation $V_0(x) \rightarrow V(x)$ we note that V is constant along the characteristic a (Fig. 3), so that $V(x_1) = V_0(x_0)$, where

$$x_1 = x_0 + t_1 \frac{dx}{dt} = x_0 - t_1 f[V_0(x_0)] .$$

Assuming that $x_1 > 0$ we have also $x_1 = \phi^+(V)$ and $x_0 = \phi_0^+(V_0) = \phi_0^+(V)$ so that

$$\phi^+(V) - \phi_0^+(V) = -t_1 f(V) . \tag{31}$$

t_1 can now be chosen arbitrarily, say, $t_1 = 1$, so that $f(V)$ is finally given in the form

$$f(V) = \phi_0^+(V) - \phi^+(V) . \tag{32}$$

In view of the isoperiodicity condition this result is identical with $\phi_0^-(V) - \phi^-(V)$ which is the form $f(V)$ would obtain assuming that x_1 and x_0 are negative.

What Eq. (32) establishes is that a t-independent choice of the constants of integration ϕ_m in Eq. (25) can always be made, transforming an arbitrary single minimum potential $V_0(x)$ into any other potential $V(x)$ which is iso-
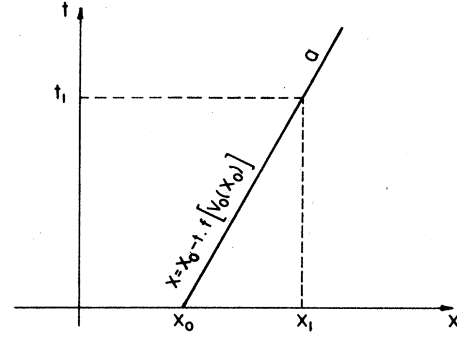

FIG. 3. The straight characteristic for t-independent constants of integration.

periodic with it. The corresponding $f(V)$ is determined by $V_0(x)$ and $V(x)$. However, the additional freedom available in Eq. (25) enables the choice of not only the initial and final forms of the potential, at $t_0 = 0$ and t_1, but also the complete scenario in between.

IV. THE POWER POTENTIALS

Before embarking on a quantitative study of one specific set of potentials, it will be of interest to point out a certain extension of the analysis of the properties of isoperiodic potentials presented in Ref. 6. Let $V(x)$ be a continuous potential which is monotonically decreasing for $x < 0$, monotonically increasing for $x > 0$, and which vanishes at $x = 0$. The first derivative of V with respect to x need not be continuous at $x = 0$, so that this point is not necessarily a minimum in the strict sense. Any integral of the form

$$I_F = \int_{x_-(E)}^{x_+(E)} F(E - V(x)) dx$$

$$= \int_{x_-(E)}^{0} F(E - V(x)) dx + \int_{0}^{x_+(E)} F(E - V(x)) dx \tag{33}$$

is invariant with respect to isoperiodic transformations. This follows by noting that I_F can be written as a sum of two integrals over V, i.e.,

TABLE I. Energies, i.e., eigenvalues of Eq. (40), for the isoperiodic linear potentials, specified by Eqs. (35) and (36) with $N = 1$. The first row is the ground-state energy and the following rows are consecutive energy differences.

$n \backslash A$	1	1.1	2	50	∞
0	0.808 62	0.822 22	0.833 74	1.029 86	1.169 05
1−0	1.047 13	1.034 14	0.986 26	0.875 89	0.874 93
2−1	0.722 34	0.724 44	0.773 74	0.719 95	0.716 30
3−2	0.666 53	0.661 13	0.650 71	0.636 45	0.633 07
4−3	0.581 09	0.586 86	0.581 63	0.581 69	0.578 71
5−4	0.555 96	0.549 61	0.548 97	0.541 99	0.539 27

TABLE II. Energies for the isoperiodic quadratic potentials.

$n \backslash A$	1	2	15	135	1215	∞
0	0.500 00	0.506 84	0.557 63	0.617 56	0.664 81	0.750 00
1−0	1.000 00	0.989 03	0.951 40	0.950 45	0.963 14	1.000 00
2−1	1.000 00	1.005 51	0.984 97	0.971 62	0.975 51	1.000 00
3−2	1.000 00	0.998 70	0.998 18	0.981 95	0.981 69	1.000 00
4−3	1.000 00	0.999 38	1.003 06	0.987 80	0.985 51	1.000 00
5−4	1.000 00	1.000 95	1.003 79	0.992 14	0.988 16	1.000 00

$$I_F = \int_E^0 F(E-V)\frac{dx_-}{dV}dV + \int_0^E F(E-V)\frac{dx_+}{dV}dV$$

$$= \int_0^E F(E-V)\frac{d(x_+ - x_-)}{dV}dV . \tag{34}$$

From the invariance of $x_+(V)-x_-(V)$ with respect to isoperiodic transformations it follows that I_F is also invariant. The special case $F=[E-V(x)]^{-1/2}$ corresponds to the classical period, discussed in Ref. 6. For potentials whose first derivative with respect to x is everywhere continuous, $F=[E-V(x)]^{1/2}$ corresponds to the WKB quantization condition. This means that within the WKB approximation isoperiodic potentials with a continuous first derivative are also isospectral.

As an illustration of the quantum properties of classically isoperiodic potentials we examine the power potentials

$$V_N(x) = \begin{cases} A|x|^N, & x < 0 \\ Bx^N, & x \geq 0 \end{cases} \tag{35}$$

which, for $N>1$, have a continuous first derivative. Noting that the classical turning points at an energy E are $x_-(E) = -(E/A)^{1/N}$ and $x_+(E) = (E/B)^{1/N}$ it is obvious that $V_N(x)$ is isoperiodic with $V_N^0(x) = |x|^N$, for which $-x_-^0 = x_+^0 = E^{1/N}$, provided that

$$x_+(E)-x_-(E) = x_+^0(E)-x_-^0(E) ,$$

i.e.,

$$B^{-1/N} + A^{-1/N} = 2 . \tag{36}$$

The form of $f(V)$ for the isoperiodic transformation of the power potentials is easily derived using Eq. (32). Thus, for $V_N(x)$ we have $V_0^+ = x^N$ and $V^+ = Bx^N$ so that $\phi_0^+ = V^{1/N}$ and $\phi^+ = (V/B)^{1/N}$. Hence,

$$f(V) = V^{1/N}(1 - B^{-1/N}) . \tag{37}$$

Let us point out that for $x < 0$ we could have obtained $V_0^- = |x|^N$, $V^- = A|x|^N$, i.e., $\phi_0^- = -V^{1/N}$, $\phi^- = -(V/A)^{1/N}$ so that $f(V) = V^{1/N}(A^{-1/N} - 1)$ which is identical with Eq. (37) because of the relation between A and B, Eq. (36). For $N=1$, Eq. (25′) with $f(V)$ given by (37) corresponds to the classical limit of the KdV transformation, Eq. (18).

The isoperiodic transformation from $|x|^N$ corresponds to the perturbation

$$\Delta V = \begin{cases} (A-1)|x|^N, & x < 0 \\ (B-1)x^N, & x > 0 . \end{cases} \tag{38}$$

For infinitesimal distortions $A = 1+\epsilon$ so that $A^{-1/N} \simeq 1-\epsilon/N$ and $B \simeq 1-\epsilon$. Hence

$$\delta V \simeq \begin{cases} \epsilon|x|^N, & x < 0 \\ -\epsilon x^N, & x > 0 \end{cases} \tag{39}$$

which means that δV is an odd function of x. Thus, in a perturbation theory treatment with $|x|^N$ as the zero-order potential, the first-order corrections to the eigenvalues vanish.

V. EIGENVALUES FOR THE POWER POTENTIALS

The Schrödinger equation for the power potentials

$$\left[-\frac{1}{2}\frac{d^2}{dx^2} + V_N(x) \right] \psi = E\psi \tag{40}$$

can only be solved analytically for the linear and quadratic potentials, $N=1,2$. In the former case the solutions can be written in terms of the Airy functions. Continuity of the logarithmic derivative at $x=0$ results in the quantization condition

TABLE III. Energies for the isoperiodic quartic potentials.

$n \backslash A$	1	10	50	1250	∞	WKB
0	0.667 99	0.691 14	0.719 99	0.781 37	0.949 92	0.546 27
1−0	1.725 65	1.729 24	1.740 70	1.782 42	1.961 27	1.817 30
2−1	2.303 16	2.279 00	2.261 74	2.260 98	2.398 41	2.306 96
3−2	2.638 93	2.638 85	2.623 80	2.605 60	2.715 06	2.644 29
4−3	2.908 58	2.910 37	2.904 91	2.881 85	2.970 64	2.911 74
5−4	3.135 03	3.134 75	3.135 81	3.115 36	3.188 27	3.137 24

TABLE IV. Energies for the isoperiodic X^{10} potentials.

$n \backslash A$	1	2	10	250	∞	WKB
0	0.728 96	0.729 38	0.732 89	0.746 42	0.901 18	0.413 35
1−0	2.132 12	2.132 88	2.139 40	2.164 91	2.490 96	2.166 07
2−1	3.399 08	3.399 37	3.402 16	3.415 26	3.739 50	3.463 81
3−2	4.509 20	4.508 97	4.507 02	4.503 75	4.789 87	4.544 82
4−3	5.490 33	5.490 00	5.487 16	5.478 08	5.732 40	5.508 16
5−4	6.381 86	6.381 66	6.380 12	6.372 53	6.602 08	6.392 98

$$A^{1/3} \frac{A_i'(x)}{A_i(x)} \bigg|_{x=-2E(2A)^{-2/3}}$$

$$+ B^{1/3} \frac{A_i'(y)}{A_i(y)} \bigg|_{y=-2E(2B)^{-2/3}} = 0 . \quad (41)$$

Similarly, for the quadratic case the quantization condition, which was also derived in Refs. 2 and 12, obtains the form

$$A^{1/4} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \bigg|_{a=1/4-E/2\sqrt{A}}$$

$$+ B^{1/4} \frac{\Gamma(b+\frac{1}{2})}{\Gamma(b)} \bigg|_{b=1/4-E/2\sqrt{B}} = 0 . \quad (42)$$

For higher values of N the Schrödinger equations can only be solved numerically. This was done using the Runge-Kutta method. Actually, numerical integration was also used for the linear potential because of some numerical difficulties we encountered in the evaluation of the Airy functions required for the solution of the analytic quantization condition, Eq. (41).

The results, for $N = 1$, 2, 4, and 10, are presented in Tables I—IV. The accuracy was monitored by changing the step size as well as the range of integration. Most of the computations were carried out with a step size $h = 0.05/(n+2)$, where n is the principal quantum number, $n = 0, 1, 2, \ldots$. This guarantees that the number of steps within each oscillation of the wave function is roughly constant. The range of integration was usually chosen as $1.6x_- < x < 1.6x_+$, where x_- and x_+ are the left and right classical turning points. The results are believed to be accurate to the number of digits presented.

In view of the fact that isoperiodic transformations correspond to the classical limit of isospectral transformations, one may expect that a comparison of the quantal spectra obtained for isoperiodic potentials would be of interest. However, the variations in the height of the potential minimum are purely quantum effects which are completely lost in the classical limit. Therefore, a uniform

shift of the spectrum corresponding to one potential relative to another one, isoperiodic with it, cannot be controlled within the classical framework. This suggests that the comparison of energy differences rather than the energies themselves should be applied as a test of the quantal significance of classical isoperiodicity.

The energy differences between consecutive levels, as presented in the tables, are rather remarkably stable with respect to the isoperiodic distortion of the potential until the limit $A \to \infty$ is approached fairly closely. This limiting behavior may be related to the difference in the WKB quantization condition between potentials with a finite and an infinite slope, discussed by Fröman and Fröman.[13] Note that the energies themselves are not as constant as the energy differences.

What the results strongly suggest is that the leading quantal correction to classical isoperiodicity is a vertical shift of the potential. A further readjustment of the shape of the bottom part of the potential curve, affecting mostly the low-lying energy levels, is also required. The case of the harmonic potentials, $N = 2$, which had already been studied by Ghosh and Hasse,[2] could have been expected to be unique because the two limiting cases $A = 1$ and $A = \infty$ are isospectral (apart from a trivial overall shift), and, furthermore, the WKB approximation is rigorously exact for the two limiting cases,[14,15] suggesting that the same might be expected to hold true in between, at least approximately. However, as the results indicate, the quadratic potentials are only approximately isospectral, to an extent which is not markedly different from the other potentials studied.

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