Three-dimensional conformal supergravity and Chern-Simons terms

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The supersymmetric extension of the gravitational Chern-Simons term in three-dimensional spacetime coincides with three-dimensional conformal supergravity. The action reads $I=\gamma_{AB}R^{B}\omega^{A}+\frac{1}{6}f_{ABC}\omega^{C}\omega^{B}\omega^{A}$, with γ_{AB} the Killing supermetric and f_{ABC} the structure constants of Osp(1/4). The constraints read $R^{m}_{\mu\nu}(P)=0$, $R^{\alpha}_{\mu\nu}(Q)=0$, and $R_{\mu\nu}{}^{mn}(M)=0$. Even when auxiliary fields close the super-Poincaré gauge algebra, I is invariant under local Poincaré sypersymmetry and independent of auxiliary fields.

I. INTRODUCTION

Chern-Simons terms play an increasingly important role in supergravity. For example, in the recently obtained action of maximal seven-dimensional supergravity[1] one finds a linear combination of two Chern-Simons forms, the particular linear combination being determined by local supersymmetry (for notational convenience we will omit wedge symbols),

$$\Omega_5=\mathrm{tr}(FFFB-\tfrac{4}{5}FFBBB-\tfrac{2}{5}BBFBF+\tfrac{4}{5}BBBBBF-\tfrac{8}{35}BBBBBBB)\,,$$
$$\Omega_3=\mathrm{tr}(FB-\tfrac{2}{3}BBB)(\mathrm{tr}FF)\,. \qquad (1.1)$$

In models of this kind one might say that supergravity is simply a supersymmetric extension of Chern-Simons terms.

In eleven-dimensional supergravity one finds a Chern-Simons-type term $dA\wedge dA\wedge A$ where A is an Abelian three-form.[2] Thus eleven-dimensional supergravity is the supersymmetric extension of a Chern-Simons-type term involving a three-form. Differential algebras with p-forms ($p>1$) are known to be the group-theoretical origin of supergravity theories, but so far the role of Chern-Simons terms in these algebras has not been understood. In $d=11$ dimensions there exist four purely gravitational Chern-Simons terms constructed from the spin connection one-form

$$I=\mathrm{tr}(RRRRR\omega+\text{more})\,,$$
$$I=\mathrm{tr}(RRRR)\,\mathrm{tr}(R\omega+\text{more})\,,$$
$$I=(\mathrm{tr}RR)\,\mathrm{tr}(RRR\omega+\text{more})\,,$$
$$I=(\mathrm{tr}RR)\,(\mathrm{tr}RR)\,\mathrm{tr}(R\omega+\text{more})\,. \qquad (1.2)$$

It is at this point an open and interesting question whether supersymmetric extensions of (a particular linear combination of) these terms exist. If so, the vast amount of work done on Kaluza-Klein supergravity would have to be redone, since the classical field equations would be modified.

The first time Chern-Simons terms surfaced in supergravity was in the coupling of matter to simple ($N=1$) $d=10$ supergravity. Coupling Maxwell matter, supersymmetry dictated a particular linear combination of the two-form A of the gauge action and the Maxwell field B (Ref. 3):

$$F=dA-B\,dB\,. \qquad (1.3)$$

It was noted that F is Maxwell invariant if $\delta B=d\Lambda$ is accompanied by $\delta A=(dB)\Lambda$, and this was called "modified Maxwell invariance." The extension to Yang-Mills coupling[4] was achieved by replacing $B\,dB$ by the gauge Chern-Simons form $\mathrm{tr}(B\,dB+\frac{2}{3}BBB)$. Recently it was noted that if one also adds to F a purely gravitational Chern-Simons form

$$F=dA-\mathrm{tr}(B\,dB+\tfrac{2}{3}BBB)-\mathrm{tr}(R\omega-\tfrac{1}{3}\omega\omega\omega) \qquad (1.4)$$

and if the gauge group of B is SO(32) or $E_8\times E_8$, then all one-loop gauge and gravitational anomalies cancel.[5] However, supersymmetry is broken, and it is unknown whether a supersymmetric extension in $d=10$ dimensions exists. It may be that this extension can only be obtained in a superstring context.

In $d=4k+3$ dimensions one has purely gravitational Chern-Simons terms. In the simplest case, $d=3$, even a supersymmetric extension is known.[6] It is given by

$$I=\epsilon^{\mu\nu\rho}\,\mathrm{tr}(R_{\mu\nu}\omega_\rho-\tfrac{2}{3}\omega_\mu\omega_\nu\omega_\rho)+\overline{f^{\mu}}\tau_\nu\tau_\mu f^{\nu}\,,$$
$$f^{\mu}=\epsilon^{\mu\nu\rho}D_\nu\psi_\rho\,, \qquad (1.5)$$

where $\omega(e,\psi)$ is the spin connection with gravitino-induced torsion and $D(\omega(e,\psi))$ is the gravitationally covariant derivative. This result was obtained in Ref. 6 by explicitly checking that I is invariant under local supersymmetry. As one of the minor results of this paper we will extend these results to the case that the super-Poincaré gauge algebra is closed due to the presence of an auxiliary field S. Of course, there is also the question of Chern-Simons terms in ordinary gravity models in $d=3$ (Ref. 7), but we will not discuss that issue here.

Our main concern is to find a geometrical origin of the supersymmetric Chern-Simons terms, and through this to obtain a method to compute the supersymmetric extensions. For simplicity we shall consider the $d=3$ model, although our ultimate aim is the $d=10$ and $d=11$ models. Our basic observation is that the higher-derivative nature of (1.5) suggests that this invariant is, in fact, a conformal supergravity model. Our reasoning goes as follows. The fermionic equivalent of the Chern-Simons gravitational $R\omega$ term is something like $R(Q)\phi$ where $R(Q)$ is the gravitino curvature and ϕ a fermionic connection. However, we need two derivatives in the fermionic sector, hence we need a constraint which equates ϕ to the derivative of the gravitino. Such constraints are well known from $d=4$ simple conformal supergravity.[8]

In $d=4$ dimensions there exist other, parity-preserving, R^2 theories besides conformal supergravity, but they always involve propagating (Poincaré auxiliary) fields S and P, in addition to a propagating chiral (Poincaré auxiliary) field A_m.[9] In $d=3$ dimensions we expect and in $d=4$ we know[8] that conformal supergravity models do not contain propagating S and P fields, while in $d=3$ dimensions we do not expect a chiral gauge field A_μ to be present. Thus we expect that conformal supergravity in $d=3$ will also be a Poincaré supergravity theory.

In Sec. II we will discuss the gauge algebra and derive the constraints of $d=3$ simple conformal supergravity. This is interesting in itself. The theory of $d=4$ simple conformal supergravity was established in a series of papers by Kaku, Townsend, and the author in 1979 (Ref. 8) and we will closely follow that approach. In the meantime, however, new insights have simplified the original treatment[10–12] and we shall use these simplifications to our advantage.

In Sec. III we shall show that the $d=3$ conformal supergravity action can be written in the simple and elegant form

$$I=\int d^3x\,(\gamma_{AB}R^B\omega^A+\tfrac{1}{6}f_{ABC}\omega^C\omega^B\omega^A)\,, \tag{1.6}$$

where γ_{AB} is the Killing metric of Osp(1/4) and f_{ABC} its structure constants. The problem here is to prove invariance under *all* local superconformal symmetries *in the presence* of constraints. The proof will be extremely simple by using some properties of γ_{AB}.

Finally, in Sec. IV we shall show that (1.6) can indeed be reinterpreted as a Poincaré supergravity. In $d=4$, the connection between conformal and Poincaré supergravity was worked out by Ferrara and the author.[13] The main result of that work which we will use here is that a Poincaré supersymmetry transformation δ_Q^P is a linear combination of ordinary and conformal supergravity transformations. Namely,

$$\delta_a^P(\epsilon)=\delta_Q^C(\epsilon)+\delta_S^C(S\epsilon+\tfrac{1}{2}\tau^\mu b_\mu\epsilon)\,, \tag{1.7}$$

where S is the auxiliary field of simple Poincaré supergravity and b_μ the dilaton. We shall end with a few comments on future developments.

II. THE GAUGE ALGEBRA OF $d=3$ CONFORMAL SUPERGRAVITY

The conformal algebra in $d=3$ dimensions is SO(3,2). Hence there is in $d=3$ dimensions a superconformal algebra, namely, Osp(1/4). (We consider only simple conformal supergravity and leave the extended theories for later.) By decomposing the generators M_{KL} ($K,L=1,\ldots,5$) into $d=3$ Lorentz generators M_{mn} ($m,n=1,\ldots,3$), dilatations $D=M_{45}$ and translations and boosts P_m and K_m, and decomposing the odd generators Q^A ($A=1,\ldots,4$) into ordinary and conformal supersymmetry generators Q^α and S^α ($\alpha=1,2$), one obtains the $d=3$ conformal superalgebra. It is given in Table I.

A striking difference with the $d=4$ conformal superalgebra is the absence of a chiral charge A. Of course, in $d=3$ there are no chiral transformations but in $d=4$ one needs A to satisfy the (QQS) Jacobi identity. In $d=3$, τ^{mn} is not independent of τ^m, and one needs to satisfy only one relation, namely,

$$\tau^{mn}O_I\tau^{mn}+2O_I=2^{[d/2]+1}\tau_m \quad \text{for } O_I=\tau_r\,, \tag{2.1}$$

where τ^r are Pauli matrices. In $d=4$, one must satisfy two relations, namely,[8]

$$\gamma^{mn}O_I\gamma^{mn}+2O_I-6\gamma_5O_I\gamma_5=\begin{cases}0 & \text{for } O_I=\gamma_{rs}\,,\\ 2^{[d/2]+1}\gamma_r & \text{for } O_I=\gamma_r\,,\end{cases} \tag{2.2}$$

where γ^m are Dirac matrices. The first two terms on the left-hand side of (2.1) and (2.2) are fixed by the (QSP) Jacobi identity and come from the M and D contribution to the $\{Q,S\}$ anticommutator. The term with γ_5 in (2.2) is due to the A contribution and can be normalized such that both relations in (2.2) are satisfied simultaneously.

Having obtained the structure constants of the conformal $d=3$ superalgebra, we next construct the curvature two-forms. We associate to each generator X_A a connection one-form ω^A and curvature R^A

$$R^A=d\omega^A-\tfrac{1}{2}f^A{}_{BC}\omega^C\omega^B\,. \tag{2.3}$$

These curvature two-forms are given in Table II.

We will now first discuss the local gauge algebra; afterwards we will turn to the construction of the action.

It is well known that under local gauge transformations connections transform into covariant derivatives of the gauge parameters. The latter are defined by arbitrary variations of the curvatures

$$\delta_g\omega^A=d\epsilon^A+f^A{}_{BC}\epsilon^C\omega^B=d\epsilon^A-f^A{}_{BC}\omega^C\epsilon^B\,, \tag{2.4}$$

$$\delta_gR^A=f^A{}_{BC}\epsilon^CR^B=-f^A{}_{BC}R^C\epsilon^B\,. \tag{2.5}$$

On the right-hand side of the $\{Q,Q\}$ anticommutator, one

TABLE I. (Anti-) commutation relations of the conformal superalgebra in $d=3$ dimensions. Note that $f^{M^{mn}}{}_{P^rK^s}=-2(\delta_r^m\delta_s^n-\delta_r^n\delta_s^m)$ and not $-2\delta_r^{[m}\delta_s^{n]}$.

$[M_{mn},M_{kl}]=\eta_{nk}M_{ml}+3$ terms, $m<n$ and $k<l$
$[M_{mn},P_k]=\eta_{nk}P_m-\eta_{mk}P_n$, idem K_k
$[D,P_k]=-P_k$, $[D,K_k]=K_k$
$[P_m,K_n]=-2M_{mn}+2\eta_{mn}D$, $m<n$ in M_{mn}
$[Q^\alpha,M_{mn}]=\frac{1}{2}(\tau_{mn})^\alpha{}_\beta Q^\beta$, idem S^α
$[Q^\alpha,K_m]=-(\tau_m)^\alpha{}_\beta S^\beta$
$[S^\alpha,P_m]=(\tau_m)^\alpha{}_\beta Q^\beta$
$[Q^\alpha,D]=\frac{1}{2}Q^\alpha$, $[S^\alpha,D]=-\frac{1}{2}S^\alpha$
$\{Q^\alpha,Q^\beta\}=-2(\tau^mC_3{}^{-1})^{\alpha\beta}P_m$
$\{S^\alpha,S^\beta\}=+2(\tau^mC_3{}^{-1})^{\alpha\beta}K_m$
$\{Q^\alpha,S^\beta)=2(\tau^{mn}C_3{}^{-1})^{\alpha\beta}M_{mn}-2(C_3{}^{-1})^{\alpha\beta}D$, $m<n$

TABLE II. Curvature two-forms of the $d=3$ super conformal algebra.

$R(M)^{mn}=d\omega^{mn}+\omega^{mk}\omega^{kn}-2(e^m f^n-e^n f^m)+2\bar{\psi}\tau^{mn}\phi$
$R(D)=db+2e^m f^m-2\bar{\psi}\phi$
$R(P)^m=De^m-\bar{\psi}\tau^m\psi+e^m b\ ,\quad De^m=de^m+\omega^{mn}e^n$
$R(K)^m=Df^m+\bar{\phi}\tau^m\phi-f^m b$
$\overline{R(Q)}=D\bar{\psi}+\bar{\phi}\tau_m e^m+\frac{1}{2}\bar{\psi}b\ ,\quad D\bar{\psi}=d\bar{\psi}+\frac{1}{4}\bar{\psi}\omega^{mn}\tau_{mn}$
$\overline{R(S)}=D\bar{\phi}-\bar{\psi}\tau_m f^m-\frac{1}{2}\bar{\phi}b$

produces P-gauge transformations instead of general coordinate transformations. In order to turn these P-gauge transformations into general coordinate transformations, one needs extra curvature-dependent terms as follows:

$$[\delta_Q(\epsilon_2),\delta_Q(\epsilon_1)]\omega_\mu^A=\delta_P(2\bar{\epsilon}_1\tau^m\epsilon_2)\omega_\mu^A+2\bar{\epsilon}_1\tau^\lambda\epsilon_2 R^A_{\lambda\mu}\ . \tag{2.6}$$

With these extra curvature terms in (2.6) one finds on the right-hand side a sum of local invariances of the theory. To see this, note the relation between general coordinate transformations and gauge transformations

$$\delta_{\text{gen coord}}(\xi^\mu)\omega_\mu^A=(D_\mu\epsilon)^A+\xi^\lambda R^A_{\lambda\mu}\ ,\quad \epsilon^A\equiv\xi^\lambda\omega_\lambda^A\ , \tag{2.7}$$

where $R^A_{\mu\nu}$ has strength two: $dx^\lambda dx^\mu R^A_{\lambda\mu}=2R^A$.

The extra curvature-dependent terms in (2.6) can be obtained as follows.[8] By imposing constraints on curvatures which can be solved algebraically for some fields $\hat{\omega}^A$ it follows that the transformation rules of the solved fields $\hat{\omega}^A$ contain extra terms proportional to curvatures,

$$\delta(\text{total})\hat{\omega}^A_\mu=\delta_g\hat{\omega}^A_\mu+\delta(\text{extra})\hat{\omega}^A_\mu\ . \tag{2.8}$$

The $\delta(\text{extra})\hat{\omega}^A_\mu$ can be computed from the property that the total variation of a constraint vanish identically, and yield in (2.6) the extra curvature terms.

Inspection of Table II reveals that only $R(P)$, $R(Q)$, $R(M)$, and $R(D)$ contain products of a vielbein and another connection, namely, products of a vielbein and a dilation b, a conformon f^m (=conformal boost field), a spin connection ω^{mn}, or a conformino ϕ (=conformal gravitino). Hence the vielbein e^m and gravitino ψ must be independent fields and transform as $\delta_g\omega^A$ in (2.4), while f^m, ω^{mn}, ϕ, and b may or may not have to be eliminated as independent fields by imposing constraints on (some or all of) the curvatures $R(P)$, $R(Q)$, $R(M)$, and $R(D)$.

Counting field components, one discovers that, as in $d=4$ simple conformal supergravity,[8] there are equal numbers of bosonic and fermionic components if one eliminates ω^{mn} and f^m:

$$\begin{aligned}&9(e_\mu^m)-3\text{ (gen coord)}-3\text{ (local Lor)}-1\text{ (scale)}=2\text{ Bose}\ ,\\&6(\psi_\mu^\alpha)-2\text{ (ord sups)}-2\text{ (conf sups)}=2\text{ Fermi}\ ,\\&3(b_\mu)-3\text{ (local conformal boosts)}=0\text{ Bose}\ .\end{aligned} \tag{2.9}$$

One therefore might expect that, as in $d=4$, the gauge algebra closes without adding auxiliary fields. This is indeed the case, as we shall show.

Since P is produced only in the $\{Q,Q\}$ anticommutator, only $\delta_Q(\text{extra})$ may and must be nonvanishing. Hence all constraints must be K, S, M, and D invariant. (Invariance under general coordinate transformations is obvious.)

Let us begin by considering the vielbein. Since it transforms into the gravitino, while both are independent fields, there is no $\delta(\text{extra})$ contribution to the $\{Q,Q\}$ anticommutator when evaluated on the vielbein, and one must require the constraint

$$R^m_{\mu\nu}(P)=0\ . \tag{2.10}$$

Next consider the gravitino. Since $\delta(\text{total})\delta(\text{total})\psi=\delta_g\delta(\text{total})\psi+\frac{1}{4}[\delta(\text{extra})\omega^{mn}\tau_{mn}\epsilon]e^n$, we have a $\delta(\text{extra})$ term in the $\{Q,Q\}\psi$ relation. One can compute $\delta(\text{extra})\omega^{mn}$ from

$$\begin{aligned}&\delta(\text{total})R(P)=\delta_g R(P)+[\delta(\text{extra})\omega^{mn}]e^n\equiv 0\ ,\\&\delta_g R(P)^m=2\bar{\epsilon}\tau^m R(Q)\ .\end{aligned} \tag{2.11}$$

The result is

$$\delta_Q(\text{extra})\omega_{\mu mn}=-\bar{\epsilon}\tau_\mu R_{mn}(Q)+\bar{\epsilon}\tau_m R_{n\mu}(Q)+\bar{\epsilon}\tau_n R_{\mu m}(Q)\ . \tag{2.12}$$

However, from (2.6) it follows that one needs

$$\frac{1}{4}[\delta_Q(\text{extra},\epsilon_2)\omega^{mn}](\tau_{mn}\epsilon_1)-(\epsilon_1\leftrightarrow\epsilon_2)=2(\bar{\epsilon}_1\tau^\lambda\epsilon_2)R_{\lambda\mu}(Q)\ . \tag{2.13}$$

If $\delta(\text{extra})\omega_{\mu mn}=-\bar{\epsilon}\tau_\mu R_{mn}(Q)$, then (2.13) is satisfied. Therefore, we must use the constraint $\tau_m R_{n\mu}(Q)-\tau_n R_{m\mu}(Q)=0$. This is equivalent to $R_{\mu\nu}(Q)=0$. Hence we have derived a second constraint,

$$R^\alpha_{\mu\nu}(Q)=0\ . \tag{2.14}$$

[In $d=4$, one needs $\delta(\text{extra})\ \omega_{\mu mn}=-2\bar{\epsilon}\gamma_\mu R_{mn}(Q)$ and this leads to the constraint $\gamma^\mu R_{\mu\nu}(Q)=0$. In $d=3$, $R_{\mu\nu}(Q)=0$ is equivalent to $\tau^\mu R_{\mu\nu}(Q)=0$ as one may verify by solving both constraints explicitly, or by counting degrees of freedom. Thus, although the $R(Q)$ constraints in $d=4$ and $d=3$ differ by a factor of 2, they can both be written in a similar way.]

Since we solved $R(P)=0$, there is no constraint left from which one can solve b, hence b must be an independent field. In Sec. IV we shall solve $R(Q)$ explicitly starting from

$$(D_\mu-\tfrac{1}{2}b_\mu)\psi_\nu-(D_\nu-\tfrac{1}{2}b_\nu)\psi_\mu=\tau_\mu\phi_\nu-\tau_\nu\phi_\mu\ . \tag{2.15}$$

In this section we shall only need $\delta(\text{extra})\phi_\nu$.

From $\delta(\text{total})R(Q)=0$ we deduce $\delta(\text{extra})\phi$ as follows. We begin with

$$\delta_Q(\text{total})R_{\mu\nu}(Q)=-\tfrac{1}{2}R_{\mu\nu}(D)\epsilon+\tfrac{1}{4}(\tau_{mn}\epsilon)R_{\mu\nu}(M)^{mn}+\tau_\mu\delta(\text{extra})\phi_\nu-\tau_\nu\delta(\text{extra})\phi_\mu=0\ . \tag{2.16}$$

Hence

$$\delta(\text{extra})\phi_\nu=\tfrac{1}{4}(\tau_\mu\epsilon)R_{\mu\nu}(D)+\tfrac{1}{8}(\tau_{\nu\rho\sigma}\epsilon)R_{\rho\sigma}(D)-\tfrac{1}{8}(\tau_\mu\tau_{mn}\epsilon)R_{\mu\nu}(M)^{mn}-\tfrac{1}{16}(\tau_{\nu\rho\sigma}\tau_{mn}\epsilon)R^{\rho\sigma mn}(M)\ . \tag{2.17}$$

From the Bianchi identities

$$DR^A = dR^A + f^A{}_{BC} R^C \omega^B = 0 \ , \qquad (2.18)$$

one can find expressions for the cyclic identity and pair exchange of the M curvature. Namely, from $R(P)=0$ one finds

$$R(M)_{[\nu\rho\sigma]}{}^m = -R_{[\nu\rho}(D) e_{\sigma]}{}^m \ . \qquad (2.19)$$

By using the identity

$$3[R_{[\nu\rho\sigma]\tau}(M) + R_{[\rho\sigma\tau]\nu}(M) - R_{[\sigma\tau\nu]\rho}(M) - R_{[\tau\nu\rho]\sigma}(M)] = 2[R_{\nu\rho\sigma\tau}(M) - R_{\sigma\tau\nu\rho}(M)] \ , \qquad (2.20)$$

we then deduce that

$$R_{\nu\rho\sigma\tau}(M) - R_{\sigma\tau\nu\rho}(M) = [R_{\rho\tau}(D) g_{\nu\sigma} + 3 \text{ terms}] \ , \qquad (2.21)$$

and after contraction over $\nu\tau$ we find the antisymmetric part of the Ricci tensor

$$R_{\rho\sigma}(M) - R_{\sigma\rho}(M) = -R_{\rho\sigma}(D) \ . \qquad (2.22)$$

[In $d=4$, one finds $-2R_{\rho\sigma}(D)$ at this point.] As a check one may verify that the f-dependent terms in this last result match.

By inserting these results into the expression for $\delta(\text{extra})\phi_\nu$, one finds the rather simple result

$$\delta(\text{extra})\phi_\nu = \tfrac{1}{4}\tau^\mu \epsilon [R_{\mu\nu}(D) + R_{\mu\nu}(M) + R_{\nu\mu}(M) - \tfrac{1}{2} g_{\mu\nu} R(M)] \ . \qquad (2.23)$$

In particular, the $\tau_{\nu\rho\sigma} R(D)^{\rho\sigma}$ terms cancel, as in $d=4$ (Ref. 8). Since the factors of $\frac{1}{4}$ in this result will be important, let us discuss a good check. Since $\phi=\phi(e,\psi,b)$ is independent of f while $\delta_Q(\text{total}) = \delta_Q(\text{gauge})$ on e,ψ,b is also independent of f, the f-dependent terms in $\delta(\text{extra})\phi_\nu$ must be minus those in $\delta(\text{gauge})\phi_\nu$, i.e., $\delta(\text{extra})\phi_\nu = \tau^m f_{m\nu}\epsilon + \text{more}$. This is indeed the case: $\frac{1}{4}R_{\mu\nu}(D)$ contains $f_{[\mu\nu]}$, while $\frac{1}{2}R_{(\mu\nu)} - \frac{1}{8} g_{\mu\nu} R(M)$ contains $f_{(\mu\nu)}$ (note: not the traceless part of $f_{(\mu\nu)}$, since we are in $d=3$ instead of $d=4$).

Let us now return to the $\{Q,Q\}$ relation evaluated on the dilaton. We have

$$\begin{aligned}
[\delta_Q(\epsilon_2), \delta_Q(\epsilon_1)] b_\mu &= \delta_Q(\text{total}, \epsilon_2) 2\bar{\epsilon}_1 \phi_\mu - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \delta_P(2\bar{\epsilon}_1 \tau^m \epsilon_2) b_\mu + [2\bar{\epsilon}_1 \delta(\text{extra}, \epsilon_2)\phi_\mu - \epsilon_1 \leftrightarrow \epsilon_2] \\
&= -4(\bar{\epsilon}_1 \tau^m \epsilon_2) f_\mu^m + (\bar{\epsilon}_1 \tau^\mu \epsilon_2)[R_{\mu\nu}(D) + R_{\mu\nu}(M) + R_{\nu\mu}(M) - \tfrac{1}{2} g_{\mu\nu} R(M)] \ . \qquad (2.24)
\end{aligned}$$

It is clear that the $R_{\mu\nu}(D)$ term is off by a factor of $\frac{1}{2}$ compared to (2.6). In the case of $d=4$, one has two independent fields on which to close the gauge algebra, the axion A as well as the dilaton b, and there one finds[12]

$$\delta_Q(\text{extra in } d=4)\phi_\nu = \gamma^\mu \left[\tfrac{1}{6}\epsilon R_{\mu\nu}(D) - \frac{i}{4}\gamma_5 \epsilon R_{\mu\nu}(A) - \frac{i}{16}\epsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}(A) + \tfrac{1}{6} R_{\nu\mu}(M) + \tfrac{1}{12} R_{\mu\nu}(M) - \tfrac{1}{12} g_{\mu\nu} R(M) + R(Q) \text{ terms} \right] , \qquad (2.25)$$

and again all f-dependent terms sum up to $\gamma^m f_{m\mu}\epsilon$. From the Bianchi identity on $R(P)$ one deduces[14,15] that

$$R_{\mu\nu}(D) = -\frac{i}{4}\epsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}(A) \quad \text{in } d=4 \ . \qquad (2.26)$$

Requiring closure on b_μ as well as A_μ leads to the constraint

$$R_{\mu\nu}(M) + R_{\mu\nu}(D) + \tfrac{1}{2}\bar{\psi}^\lambda \gamma_\mu R_{\nu\lambda}(Q) = 0 \quad \text{in } d=4 \qquad (2.27)$$

which is equivalent (and simpler) than the constraint obtained in Ref. 11. Thus, in $d=4$, $\delta(\text{extra})\phi_\nu$ contains both $R_{(\mu\nu)}(M)$ and $R_{[\mu\nu]}(M)$. In $d=3$, on the other hand, $\delta(\text{extra})\phi_\nu$ contains only $R_{(\mu\nu)}(M)$.

In $d=4$ one can impose the full (2.27) as a constraint; this eliminates f_μ^m completely and yields a closed gauge algebra, of the minimal form as given in (2.6). We note parenthetically that if one only requires closure on the $d=4$ axial-vector field, it is sufficient to require that the symmetric part of (2.27) vanish; this then eliminates only $f_{(m\mu)}$. However, in this case $f_{[m\mu]}$ remains as an independent field in the theory, and since in this case one would not have equal numbers of bosonics and fermionic field components, the algebra would not close on $f_{[m\mu]}$. One must thus eliminate in $d=4$ also $f_{[m\mu]}$, and any constraint which can do this is allowed. The algebra on the dilaton always closes because one can always add to the (Q,Q) gauge commutator a conformal boost term (K acts only on the dilaton). The simplest choice is to have no extra term at all, and this is achieved in $d=4$ by imposing (2.27). The $\delta(\text{extra}) f_\mu^m$ which results from (2.27) does not affect the results for $\delta(\text{extra})\phi_\mu$ and $\delta(\text{extra})\omega_\mu^{mn}$ because ϕ_μ and ω_μ^{mn} do not depend on f_μ^m.

In $d=3$ we must also eliminate the whole f_μ^m in order to have equal numbers of bosonic and fermionic degrees of freedom [see (2.9)]. Since $R_{\mu\nu}{}^{mn}(M)$ has as many components as f_μ^m, we choose the constraint

$$R_{\mu\nu}{}^{mn}(M) = 0 \ . \qquad (2.28)$$

From (2.22) we then also find

$$R_{\mu\nu}(D) = 0 \qquad (2.29)$$

so that, with (2.23), $\delta_Q(\text{extra})\phi_\nu = 0$. It follows that, as in $d=4$, the gauge commutator is given by (2.6), and that, in particular, no extra local symmetries on the right-hand side of (2.6), such as local K transformations, are needed (although one can always add them).

We close by observing that in $d=4$ the constraint (2.27) is preferred in the sense that with this constraint the action becomes extremely simple [see (3.1)]. Any other constraint would lead to a less simple form of the $d=4$ action. In $d=3$ the action will be independent of f_μ^m, and therefore any constraint from which f_μ^m can be solved algebraically is as good as any other.

III. THE ACTION OF $d=3$ CONFORMAL SUPERGRAVITY

The gauge action of conformal supergravity has a very simple form[16] in $d=4$,

$$I=\int d^4x\,[R(M)^{mn}R(M)^{rs}\epsilon_{mnrs}-8\bar{R}(Q)\gamma_5R(S)$$
$$-4iR(A)R(D)]\;. \tag{3.1}$$

Originally, this form of the action resulted by requiring invariance under all local symmetries, and in this way the various constraints were found to be needed. We now prefer to first analyze the gauge algebra and deduce the various constraints, and afterwards to check that the action is invariant.

Of course, the $d=4$ Pontryagin invariant

$$I=\int d^4x\,\gamma_{AB}R^BR^A \tag{3.2}$$

with γ_{AB} the Killing metric of Osp(1/4), is a total derivative since its variation vanishes. To prove this, it suffices to use $\delta R^A=(D\delta\omega)^A$ and the Bianchi identities of (2.18). Hence, by peeling off an exterior derivative, one gets a reasonable candidate for the action of $d=3$ conformal supergravity,

$$I=\int d^3x\,[\gamma_{AB}R^B\omega^A+\tfrac{1}{6}f_{ABC}\omega^C\omega^B\omega^A] \tag{3.3}$$

with $f_{ABC}=\gamma_{AD}f^D{}_{BC}$ the totally super-antisymmetric structure constants of Osp(1/4). Indeed, under arbitrary variations, $\delta R^A=(D\delta\omega)^A$, hence $\delta I=\int d^3x\,2R_A\delta\omega^A$ vanishes under gauge variations $\delta\omega^A=(D\epsilon)^A$ in $d=3$ dimensions. Moreover, the terms containing only the spin connection and $R(M)$ reproduce the ordinary gravitational Chern-Simons terms. However, in order that the Q transformations of Sec. II are really supersymmetry transformations we had to impose constraints. With these constraints, the $\{Q,Q\}$ commutator produced a general coordinate transformation on the vielbein and gravitino as required on physical grounds; without constraints one would find that $\{Q,Q\}\omega^{mn}=0$, which is to be rejected. We must therefore investigate whether the action in (3.3) is invariant in the presence of the constraints. Moreover, we want to obtain invariance not only under Poincaré (=ordinary) local supersymmetry, but rather *all* superconformal symmetries, including ordinary and conformal supersymmetry. This will automatically lead to invariance under Poincaré supersymmetry, and extend the supersymmetric Chern-Simons terms of Ref. 6 to the case that auxiliary fields close the Poincaré gauge algebra. Our result will contain the special subcase that a cosmological constant is present, just as in ordinary Poincaré supergravity the formulation with auxiliary fields contains the case when a cosmological constant is present.

In fact, since the fermionic terms need two derivatives because the bosonic Chern-Simons term has three derivatives when $\omega=\omega(e,\psi)$, it is clear that (3.3) cannot be the whole result, because one would find a first-derivative fermionic kinetic term. The constraints $R(Q)=0$, on the other hand, will relate the conformino ϕ to the *derivative* of the gravitino, and in this way one will obtain a quadratic derivative gravitino term. This reasoning led to our conjecture that the supersymmetric $d=3$ Chern-Simons term is nothing but $d=3$ conformal supergravity. This we shall now prove.

There is one possible loophole in the above arguments. Namely, it might be that γ_{AB} vanishes identically. Indeed, for the super-Poincaré algebra, $\gamma_{AB}\equiv 0$, but, as we shall show explicitly, for the simple orthosymplectic algebra, the Killing form is nonvanishing. We recall its definition

$$\gamma_{AB}=f^P{}_{AQ}f^Q{}_{BP}(-)^{\sigma(P)}\;, \tag{3.4}$$

where $\sigma(P)=0$ if X_p is an even generator while $\sigma(P)=1$ if X_p is odd.

Without any work we can at once prove that the action remains invariant under the modified Q-gauge transformations. On general grounds it is clear that γ_{AB} is only nonzero for the *MM*, *PK*, *QS*, and *DD* entries. Moreover, δ(extra) is only nonvanishing for the K, S, and M entries since these are the only fields which have been eliminated. Also $\delta(\text{extra})\omega^{mn}=0$, see (2.13) and (2.14). This leaves only $\delta(\text{extra})f^m$ and $\delta(\text{extra})\phi$. However, these modified variations are multiplied by $R(P)$ and $R(Q)$, respectively, which vanish. Hence the action is indeed invariant under all local gauge invariances of the super conformal group, with P gauges replaced by general coordinate transformations.

To find the explicit form of the action we will now evaluate the entries of the Killing supermetric γ_{AB}. The easiest way is to use the total superantisymmetry of f_{ABC}. For example, from

$$f_{P^mS^\alpha S^\beta}=\gamma_{P^mK^n}f^{K^n}{}_{S^\alpha S^\beta}=\gamma_{P^mK^n}(2\tau^nC_3{}^{-1})^{\alpha\beta}$$
$$=-f_{S^\alpha P^m S^\beta}=-\gamma_{S^\alpha Q^\gamma}f^{Q^\gamma}{}_{P^M S^\beta}$$
$$=\gamma_{S^\alpha Q^\gamma}(\tau^mC_3{}^{-1})^{\beta\gamma} \tag{3.5}$$

we deduce that $\gamma_{P^mK^n}=a\delta_{mn}$ and that $\gamma_{S^\alpha Q^\beta}=-a(C_3{}^{-1})^{\alpha\beta}$. As a check, and in order to fix the overall scale a, we computed γ_{AB} directly from its definition in (3.4). The result reads

$$\gamma_{M^{mn}M^{rs}}=-5\delta_{mr}\delta_{ns}(m<n,r<s)\,,\quad \gamma_{DD}=5\;,$$
$$\gamma_{P^mK^n}=\gamma_{K^mP^n}=-10\eta_{mn}\;, \tag{3.6}$$
$$\gamma_{Q^\alpha S^\beta}=\gamma_{S^\alpha Q^\beta}=20(C_3{}^{-1})^{\alpha\beta}\;.$$

In $d=4$, one finds similar results, except that $\gamma_{DD}=0$ since Q^α and P^m (and S^α and K^m) have the same number of components in $d=4$.

The $d=3$ action (after extraction of an overall constant $-\frac{5}{2}$) thus reads

$$I=\int d^3x\,[R^{mn}\omega^{mn}+4R(K)^me^m-8R(\bar{\phi})\psi-2R(D)b$$
$$+\tfrac{1}{6}(-\tfrac{2}{5})f_{ABC}\omega^C\omega^B\omega^A]\;,$$

where we only used that $R(P)=R(Q)=0$. Let us determine the f-field equation. Direct evaluation shows that its field equation vanishes identically. In other words, I is independent of f. This can immediately be understood from the fact that $\delta I = R(P)\delta f = 0$, since $R(P)=0$, and since the expressions for ω^{mn} and ϕ which solve $R(P)=0$ and $R(Q)$ do not depend on f. Thus, no matter what constraint one imposes to solve for f^m, the action remains invariant. In $d=4$, the f-field equation was equivalent to the M constraint in (2.7), but here we can in principle leave f as an independent field in the theory without violating invariance of the action.

We eliminate ω^{mn} and ϕ as independent fields by solving the constraints $R(P)=0$ and $R(Q)=0$ and find

$$\omega_{\mu mn} = \omega_{\mu mn}(e) - (\bar{\psi}_\mu \tau_m \psi_n - \bar{\psi}_\mu \tau_n \psi_m + \bar{\psi}_m \tau_\mu \psi_n) + e_{\mu m} b_n - e_{\mu n} b_m , \tag{3.8}$$

$$\phi_\mu = -\tfrac{1}{2}\tau^\nu(\mathscr{D}_\nu \psi_\mu - \mathscr{D}_\mu \psi_\nu) - \frac{i}{2}\epsilon_{\mu\rho\sigma}\mathscr{D}_\rho \psi_\sigma , \tag{3.9}$$

where $\mathscr{D}_\rho \psi_\sigma = D_\rho \psi_\sigma - \frac{1}{2} b_\rho \psi_\sigma$.

Substituting these results into the action, we discover that the b-dependent terms cancel. This result, which is independent whether or not one has imposed and/or solved the remaining M constraint (since f^m cancels), can also readily be understood as follows. Since the remaining fields are e^m, ψ^m, and b while only b transforms under local conformal boosts, it follows from the invariance of the action under local boosts that it must be b-independent.[8]

The action thus becomes

$$I = \int d^3x \, [R(\omega)^{mn}\omega^{mn} + 2\bar{\psi}\tau^{mn}\phi\omega^{mn} + 4\bar{\phi}\tau^m\phi e^m - 8\bar{\phi}D\psi - \tfrac{1}{15} f_{ABC}\omega^C\omega^B\omega^A] , \tag{3.10}$$

where

$$\begin{aligned} -\tfrac{1}{2} f_{ABC}\omega^C\omega^B\omega^A &= \gamma_{AD}(R^D - d\omega^D)\omega^A \\ &= (-\tfrac{5}{2}\omega^{mn})(\omega^{mk}\omega^{kn} + 2\bar{\psi}\gamma^{mn}\phi) \\ &\quad - 10 e^m \bar{\phi}\gamma^m \phi - 20\psi(\tfrac{1}{4}\bar{\phi}\omega^{mn}\tau_{mn}) \\ &\quad - 20\phi(\tfrac{1}{4}\bar{\psi}\omega^{mn}\tau_{mn} + \bar{\phi}\tau^m e_m) . \end{aligned} \tag{3.11}$$

In this result all $\bar{\phi}_m \phi e^m$ terms cancel. [Although the ϕ dependence of I is given by $\delta I = R(Q)\delta\phi = 0$, this does not mean that one may omit all ϕ terms, because $\phi = \phi(e,\psi,b)$. The field equation, $R(Q)$, does vanish identically, but in the action the composite fields $\phi(e,\psi,b)$ do not cancel]. Also the explicit $\omega\psi\phi$ terms cancel. One is left with

$$I = \int d^3x \, [R^{mn}(\omega(e,\psi))\omega^{mn}(e,\psi) + \tfrac{1}{3}\omega^{mn}\omega^{nk}\omega^{km} - 8\bar{\phi}(e,\psi)D\psi] . \tag{3.12}$$

Inserting the solution for ϕ given in (3.9) one obtains

$$I = \int d^3x \, [(R^{mn}\omega^{mn} + \tfrac{1}{3}\omega^{mn}\omega^{nk}\omega^{km}) + 4i(\epsilon^{\mu\alpha\beta}D_\alpha\bar{\psi}_\beta)\tau_\nu\tau_\mu(\epsilon^{\nu\gamma\delta}D_\gamma\psi_\delta)] . \tag{3.13}$$

This is our final result for the action of conformal supergravity in $d=3$ dimensional spacetime.

IV. CONFORMAL SUPERGRAVITY AS A CHERN-SIMONS TERM

Since the action of the last section is invariant under local Q and S conformal supersymmetry transformations, it is also invariant under the particular linear combination[13]

$$\delta_Q^P(\epsilon) = \delta_Q(\epsilon) + \delta_S(S\epsilon + \tfrac{1}{2}\tau^\mu b_\mu \epsilon) . \tag{4.1}$$

Since the vielbein is S inert, we have

$$\delta_Q^P(\epsilon) = 2\bar{\epsilon}\tau^m\psi_\mu$$

which is the transformation law of the vielbein in Poincaré supergravity. For the gravitino, on the other hand,

$$\begin{aligned} \delta_Q \psi_\mu &= D_\mu[\omega(e,\psi,b)]\epsilon - \tfrac{1}{2} b_\mu \epsilon \\ &= D_\mu[\omega(e,\psi)]\epsilon - \tfrac{1}{2}\tau_\mu \not{b}\epsilon , \end{aligned} \tag{4.2}$$

$$\delta_S(\epsilon)\psi_\mu = \tau_\mu \epsilon . \tag{4.3}$$

Hence in δ_Q^P the dilaton terms cancel

$$\delta_Q^P \psi_\mu = D_\mu \epsilon + \tau_\mu S \epsilon \tag{4.4}$$

and this is the transformation rule of the gravitino in Poincaré $d=3$ supergravity.

It follows that

(i) the conformal $d=3$ supergravity action is independent of the auxiliary field S. It contains only the vielbein and gravitino.

(ii) it is invariant under Poincaré supersymmetry, even when the gravitino law $\delta\psi_\mu$ contains the auxiliary field S. We have checked these results explicitly, using

$$\delta\omega_{\mu mn} = \tfrac{1}{4}(e_m^\nu \bar{\epsilon}\tau_n - e_n^\nu \bar{\epsilon}\tau_m)\psi_{\mu\nu} - \tfrac{1}{4} e_m^\rho e_n^\sigma \bar{\epsilon}\tau_\mu \psi_{\rho\sigma} + \frac{S}{4}(\bar{\psi}_\mu g_{mn}\epsilon + \bar{\psi}_m \epsilon e_{n\mu} - \bar{\psi}_n \epsilon e_{m\mu}) ,$$

where $\psi_{\mu\nu} = D_\mu \psi_\nu - D_\nu \psi_\mu$.

Because the supersymmetric Chern-Simons form is invariant under supersymmetry laws *with auxiliary fields,* one may add it to other invariant actions, such as the ordinary $d=3$ supergravity action, and/or the supersymmetric cosmological term, without destroying local supersymmetry. This at once shows that one can add a cosmological term to the supersymmetric Chern-Simons terms. It also allows adding matter without losing supersymmetry.

We close with a few remarks. It would be interesting to extend our results to extended $d=3$ conformal supergravity, and to study its relation to Osp(N/4). Extended conformal supergravities in $d=4$ were obtained in Ref. 14 but do not seem to be relevant for this purpose. In higher dimensions we intend to study conformal supergravities in $d=10$ (Ref. 17).

Note added in proof. A tensor calculus for $N=1$ conformal supergravity in $1+1$ and $2+1$ dimensions has been given by T. Uematsu [Z. Phys. C (to be published)]. The extension of our results to $N>1$ has recently been obtained by M. Roček, C. S. Zhang, and the author (unpublished).

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