Quantum relativistic oscillator. II. Nonrelativistic limit and phenomenological justification

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It is shown that the quantum relativistic oscillator with noncommuting internal momentum and position has as its nonrelativistic contraction limit the usual three-dimensional oscillator. The nonrelativistic limits for representations with integer and half-integer spin are compared with the quark-model picture. The spectrum predicted by the quantum relativistic oscillator with rotations is compared with experimental hadron spectra.

I. INTRODUCTION

This paper continues work on the idea—taken from molecular and nuclear physics—that low-energy (mass) hadron spectra and hadron structure can be described as collective phenomena in terms of relativistic vibrations and rotations. As for the quantum relativistic rotator[1] (QRR) we require also of our quantum relativistic oscillator[2] (QRO) three distinct correspondences: (a) the classical (Bohr) correspondence to the lowest mode of the classical relativistic string, (b) the elementary correspondence to the relativistic mass point, and (c) the nonrelativistic correspondence to the nonrelativistic quantum oscillator. (a) and (b) have been discusssed in I (Ref. 2); here we will discuss (c).

Whereas the QRR in the nonrelativistic and classical limit[1] could be interpreted as a rigid dumbbell (diquark dumbbell), the QRO is to be interpreted as a diquark vibrator in this limit. To come as close as possible to the conventional quark model the dumbbell angular momentum j, i.e., the spin of the extended object, must have a contribution from the quark spins. For the representations of the relativistic spectrum-generating group SO(3,2) considered in I, $s=j_{\min}$ was zero and the nonrelativistic limit was the usual spinless three-dimensional oscillator. In order to obtain relativistic oscillators with spin, we will study in Sec. II certain other representations of SO(3,2), for which $s=\frac{1}{2}$, $s=1$, and $s=\frac{3}{2}$. In Sec. III we derive the nonrelativistic limit of the QRO.

In the last section of the paper we give a comparison between the predicted spectra and the experimental hadron spectrum. For the $s=1$ case the predicted spin spectrum is different from the $s=0$ case fitted in I, we will get here different values for the ν unit $1/\alpha'$ and the j unit λ^2. Our best predictions from the representations with $s=1$ are very close to the linearly rising Regge trajectories (with $\alpha'\approx 1$ GeV^{-2}) for the "yrast" states[3] with $\nu=j$ plus small rotational corrections depending upon $j(j+1)$, whereas for the $s=0$ representations we had obtained[2] $1/\alpha'\approx 0.5$ GeV^2 and $\lambda^2\approx 0.12$ GeV^2. Baryons are treated on the same footing using the SO(3,2) representations with $s=\frac{1}{2}$.

II. SOME SINGLETON REPRESENTATIONS OF SO(3,2) AND THEIR GROUP CONTRACTION

The Casimir operators of $\mathrm{SO}(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ $(\mu,\nu=0,1,2,3)$ are

$$C_{(2)}=\Gamma_\mu\Gamma^\mu+\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}$$
$$=\Gamma_0{}^2-\Gamma_i\Gamma_i-S_{i0}S_{i0}+\tfrac{1}{2}S_{ij}S_{ij} \tag{2.1}$$

(summed over $\mu,\nu=0,1,2,3$ and $i,j=1,2,3$) and

$$C_{(4)}=-[W_\mu W^\mu+(\tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma})^2] , \tag{2.2}$$

where

$$W_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}S^{\nu\rho}\Gamma^\sigma . \tag{2.3}$$

For the singleton representations of SO(3,2) (Ref. 4),

$$\mu=\text{eigenvalue of }\Gamma_0 ,$$
$$j(j+1)=\text{eigenvalue of }\tfrac{1}{2}S_{ij}S_{ij} , \tag{2.4}$$
$$j_3=\text{eigenvalue of }S_{12} ,$$

are a complete set of labels (quantum numbers) for the basis vectors $|\mu j j_3\rangle$ in an irreducible-representation space. We will consider only unitary singleton representations for which Γ_0 is bounded from below (or from above). The representations in this class are characterized by the two numbers

$$\mu_{\min}\ (\mu_{\max})=\text{lowest (highest) value of }\mu ,$$
$$s=\text{lowest value of }j , \tag{2.5}$$

and the eigenvalues of the Casimir operators

$$-R=\text{eigenvalue of }C_{(2)} ,$$
$$P_1=\text{eigenvalue of }C_{(4)} , \tag{2.6}$$

are given in terms of $\mu_{\min}$ and s. There exist the following subclasses:

(a) $s=0$, $\mu_{\min}\geq\frac{1}{2}$, with

$$-R=(\mu_{\min}-\tfrac{3}{2})^2-\tfrac{9}{4} , \tag{2.7a}$$

(b) $s=\frac{1}{2}$, $\mu_{\min}\geq 1$, with

$$-R=(\mu_{\min}-\tfrac{3}{2})^2-\tfrac{3}{2} , \tag{2.7b}$$

(c) $s=1,\frac{3}{2},2,\frac{5}{2},\ldots$, $\mu_{\min}=s+1$, with

$$-R=2(\mu_{\min}-1)^2-2=2s^2-2 . \tag{2.7c}$$

For all of these representations the value of the fourth-order Casimir operator is given by

$$P_1=s(s+1)[-R-(s-1)(s+2)] . \tag{2.8}$$

The multiplicity pattern or K type [reduction with respect to the maximal compact subgroup SO(3)⊗SO(2)] for the representations of subclass (a) is given in Fig. 3 of I, except that for $\mu_{\min}=\frac{1}{2}$ the representation splits into two irreducible representations, one consisting only of the boxes along the first diagonal $\mu=j+\frac{1}{2}$ (a Majorana representation) and the other, consisting of the remaining boxes, which is equivalent to the representation $s=0$, $\mu_{\min}=\frac{5}{2}$. The multiplicity pattern for subclasses (b) and (c) is given in Fig. 1, except that for $s=\frac{1}{2}$ and $\mu_{\min}=1$ this pattern also splits into two irreducible representations, one consisting only of the first diagonal $\mu=j+\frac{1}{2}$ (again a Majorana representation) and the other being equivalent to the representation $s=\frac{1}{2}$, $\mu_{\min}=2$.

Inönü-Wigner group contraction of some SO(3,2) representations with respect to SO(3), in which SO(3,2) goes into the semidirect product of the rotation group SO(3) and two translation groups, has been considered before for a different class of representations.[5] That the oscillator algebra in n dimensions can be obtained from SO($n+2$) has been discussed in Ref. 6. The contraction of representations of subclass (a) through a sequence of representations has been described in detail in another paper.[7] Here we review it briefly together with the contraction of representations of subclass (b).

Before we can do this in a physically transparent way we have to introduce the Galilean mass obtained in the group contraction of the Poincaré group into the extended Galilei group.[8] It was summarized in Sec. I C of Ref. 1(c) and entails the relations

$$P_i=P_i , \tag{2.9a}$$

$$P_0=mc+\frac{1}{c}H+O\left[\frac{1}{c^3}\right] , \tag{2.9b}$$

$$\begin{aligned}P_\mu P^\mu&={P_0}^2-\mathbf{P}^2\\&=(mc)^2\left[1+\frac{1}{(mc)^2}(2mH-\mathbf{P}^2)+O\left[\frac{1}{c^4}\right]\right] .\end{aligned} \tag{2.9c}$$

Here m is the Galilean mass, H the nonrelativistic energy operator, and the nonrelativistic momentum on the right-hand side of (2.9a) has again been denoted by P_i.

We can now define the generators ξ_i^0, π_i^0, h, and S_{ij} ($i,j=1,2,3$) in terms of the generators of SO(3,2):

$$\xi_i^0=\frac{1}{mc}S_{0i} , \tag{2.10a}$$

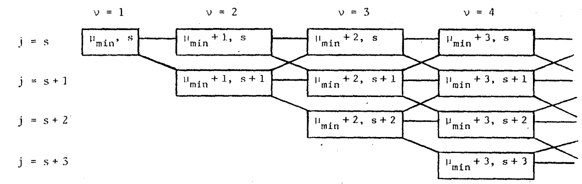

FIG. 1. Multiplicity pattern for the representations of subclasses (b) and (c), except that for $s=\frac{1}{2}$ and $\mu_{\min}=1$ this pattern splits into two irreducible representations. The two values in the boxes are [(eigenvalue $\hat{P}_\mu\Gamma^\mu$),j] where $j(j+1)=$(eigenvalue $\hat{W}$). Instead of (eigenvalue $\hat{P}_\mu\Gamma^\mu$) we use later in Fig. 4 $\nu=$eigenvalue $\hat{P}_\mu\Gamma^\mu-1$ and in Fig. 5 we use $\nu=$eigenvalue $\hat{P}_\mu\Gamma^\mu+1-\mu_{\min}$. Thus ν has always the values $\nu=1,2,3,\ldots$.

$$\pi_i^0=\frac{-1}{mc\alpha'}\Gamma_i , \tag{2.10b}$$

$$h=\frac{1}{(mc)^2\alpha'}\Gamma_0 , \tag{2.10c}$$

$$S_{ij}=S_{ij} . \tag{2.10d}$$

$1/c$ is the contraction parameter, and the factors m of dimension mass and α' of dimension $(mc)^{-2}$ [which is the parameter occurring in the Hamiltonian (2.33) of I] have been chosen so that h is an operator of dimension 1, ξ_i^0 an operator of dimension cm, and π_i^0 an operator of dimension momentum. The powers of c in (2.10) are such that SO(3,2) is contracted with respect to SO(3)$_{S_{ij}}$.

The contraction $1/c\to 0$ of the representation is now performed in the usual way by going through a sequence of irreducible representations in which the contracted generators Γ_i, S_{i0}, Γ_0 grow to infinity in such a way that

$$\xi_i^0\to\xi_i^{0(\infty)},\quad \pi_i^0\to\pi_i^{0(\infty)},\quad h\to h^{(\infty)}=\hbar I=I , \tag{2.11}$$

where $\xi^{0(\infty)},\pi^{0(\infty)},h^{(\infty)}$ are finite nonzero operators which together with S_{ij} fulfill the commutation relations of the algebra for the three-dimensional oscillator.[7] In order to achieve this, one has to let the eigenvalue $(-R)$ of the operator $C_{(2)}$ of Eq. (2.1) go to infinity such that

$$\text{eigenvalue of } \frac{1}{c^4}C_{(2)}=-\frac{1}{c^4}R\to\text{finite value as } \frac{1}{c}\to 0 . \tag{2.12}$$

Using (2.7a) or (2.7b) this means one has to let $\mu_{\min}\to\infty$ such that $-(1/c^4)R$ goes to a finite value, which we can choose and which characterizes the representation of the group obtained by the contraction.

Occurring simultaneously with the contraction of SO(3,2) is the contraction process of the Poincaré-group representations characterized by the eigenvalues of $P_\mu P^\mu\equiv p^2=p^2(\nu)$ into the extended Galilei group representations. This proceeds via the sequence of representations

$$p^2\to\infty \text{ such that } \frac{1}{c}p\to m=\text{Galilean mass} . \tag{2.13}$$

m is a finite value which we are free to choose and we choose one and the same value m for every $p^2(\nu)$.

We now couple the two contractions (2.12) and (2.13) by requiring that $(-R)$ be taken to infinity such that[9]

$$-R=\beta^2(mc)^4\alpha'^2\ , \tag{2.14}$$

where β^2 is an arbitrary dimensionless number which we will fix later. That the condition (2.14) is reasonable follows from the fact that $-R$ goes to infinity like c^4 and p goes to infinity like c.

From the second-order Casimir operator (2.1) we obtain using (2.10) and (2.14)

$$\Gamma_0{}^2=\beta^2m^4c^4\alpha'^2+(mc\alpha')^2(\pi^0)^2+(mc)^2(\xi^0)^2-\tfrac{1}{2}S_{ij}S_{ij}$$

or

$$\Gamma_0=\beta m^2c^2\alpha'\left[1+\frac{1}{c^2}\left[\frac{1}{\beta^2m^2}(\pi^0)^2+\frac{1}{\beta^2m^2\alpha'^2}(\xi^0)^2-\frac{1}{2}\frac{1}{\beta^2m^4c^2\alpha'^2}S^2\right]\right]^{1/2}.$$

Expanding the $[\ ]^{1/2}$ in powers of $1/c$ we obtain

$$\begin{aligned}\Gamma_0&=m\alpha'\left[1\times\beta mc^2+\frac{1}{2\beta m}(\pi^0)^2+\frac{1}{2\beta m\alpha'^2}(\xi^0)^2\right]+O\left[\frac{1}{c^2}\right]\\&=(mc)^2\alpha'\beta\left[1+\frac{1}{2\beta^2(mc)^2}(\pi^0)^2+\frac{1}{2\beta^2\alpha'^2(mc)^2}(\xi^0)^2\right]+O\left[\frac{1}{c^2}\right].\end{aligned} \tag{2.15}$$

III. THE NONRELATIVISTIC LIMIT OF THE QRO

We shall now consider the nonrelativistic limit of the relativistic observables $\xi_\mu=-d_\mu$, π_μ and their relativistic commutation relations. For d_μ and the commutation relation (CR)

$$[d_\mu,d_\nu]=-\frac{i}{(P_\mu P^\mu)}\Sigma_{\mu\nu}\ , \tag{3.1}$$

this was already done in detail in Ref. 1(c). One obtains

$$\begin{aligned}d_i&=S_{i\nu}\frac{P^\nu}{P_\mu P^\mu}=S_{i0}\frac{P^0}{P_\mu P^\mu}+S_{ij}\frac{P^j}{P_\mu P^\mu}\\&=-mc\xi_i^0\frac{mc}{(mc)^2}+O\left[\frac{1}{c}\right],\end{aligned}$$

where we have used (2.9) and (2.10). Thus

$$-d_i=\xi_i=\xi_i^0+O\left[\frac{1}{c}\right]\to\xi_i^{0(\infty)}=\xi_i^{(\infty)}\ , \tag{3.2}$$

where according to (4.3g) of Ref. 7 the $\xi_i^{(\infty)}$ commute. Similarly one shows

$$\xi_0=-d_0\to 0\ . \tag{3.3}$$

It was already shown in Ref. 1(c) that

$$\Sigma_{0i}\to 0,\quad \Sigma_{ij}\to S_{ij}+\xi_i^{(\infty)}P_j-\xi_j^{(\infty)}P_i\equiv\Sigma_{ij}^{(\infty)}\ . \tag{3.4}$$

Using (2.10), (3.2), (3.3), and (3.4) one sees therefore that (3.1) goes into the identity $0=0$ for μ or $\nu=0$ and into

$$[\xi_i^{(\infty)},\xi_j^{(\infty)}]=0\quad\text{for}\quad i,j=1,2,3\ . \tag{3.5}$$

For the intrinsic momenta, which according to (2.44) of I are given by

$$\pi_\mu=-\frac{1}{\alpha'}\frac{1}{(P_\mu P^\mu)^{1/2}}(g_\mu^\sigma-\hat P_\mu\hat P^\sigma)\Gamma_\sigma\ , \tag{3.6}$$

one obtains using (2.9) and (2.10)

$$\pi_i=\pi_i^0+P_i\frac{1}{\alpha'(mc)^3}\left[mc+\frac{H}{c}\right][\alpha'\beta m^2c^2+O(c^0)]+O\left[\frac{1}{c^2}\right]$$

or

$$\pi_i=\pi_i^0+\beta P_i+O\left[\frac{1}{c^2}\right]. \tag{3.7}$$

Thus in the nonrelativistic limit

$$\pi_i\to\pi_i^{0(\infty)}+\beta P_i\equiv\pi_i^{(\infty)}\ , \tag{3.8}$$

where according to (4.3a) of Ref. 7, and the CR of P_i, the $\pi_i^{(\infty)}$ commute. For the $\mu=0$ component we obtain using (2.9) and (2.10)

$$\pi_0=-\frac{1}{\alpha' mc}\left[\Gamma_0-\Gamma_0-\frac{mcP^i}{(mc)^2}mc\alpha'\pi_i\right]\to 0\ . \tag{3.9}$$

Using (3.8), (3.9), and (3.4) one thus sees that the CR [cf. (2.46) of I]

$$[\pi_\mu,\pi_\nu]=-i\frac{1}{\alpha'^2(P_\mu P^\mu)}\Sigma_{\mu\nu} \tag{3.10}$$

goes into the identity $0=0$ for μ or $\nu=0$ and into

$$[\pi_i^{(\infty)},\pi_j^{(\infty)}]=0\quad\text{for}\quad i,j=1,2,3\ . \tag{3.11}$$

For the CR [cf. (2.45) of I]

$$[\xi_\mu,\pi_\nu]\underset{C}{=}-i(g_{\mu\nu}-\hat P_\mu\hat P_\nu)\ , \tag{3.12}$$

one obtains in the nonrelativistic limit using (2.9), (2.10), (3.2), (3.3), (3.8), and (3.9)

$$[\xi_i^{(\infty)},\pi_j^{(\infty)}]=-ig_{ij}=i\delta_{ij}\ . \tag{3.13}$$

Therewith we have shown that in the nonrelativistic limit, the relativistic noncommuting intrinsic coordinates and momenta of the extended object go into operators that fulfill the conventional three-dimensional Heisenberg CR. The relativistic CR (3.1), (3.10), and (3.12) are therefore relativistic generalizations of the usual three-dimensional canonical commutation relations.

We shall now consider the relativistic Hamiltonian (2.33) of I,

$$\mathscr{H}=v(P_\mu P^\mu - \frac{1}{\alpha'}\hat{P}_\mu \Gamma^\mu) , \quad (3.14)$$

and the constraint relation [(2.34) or I] that follows from it,

$$(P_\mu P^\mu)^{3/2} - \frac{1}{\alpha'}P_\mu \Gamma^\mu \underset{c}{=} 0 . \quad (3.15)$$

From (2.9c) we obtain

$$(P_\mu P^\mu)^{3/2}=(mc)^3\left[1+\frac{1}{(mc)^2}(2mH-\mathbf{P}^2)+O\left[\frac{1}{c^4}\right]\right]^{3/2}$$
$$=(mc)^3\left[1+\frac{3}{2}\frac{1}{(mc)^2}(2mH-\mathbf{P}^2)\right]+O\left[\frac{1}{c}\right] . \quad (3.16)$$

Inserting this, (2.15), (2.10), and (2.9) into (3.15) we obtain

$$(mc)^3\left[1+\frac{3}{2}\frac{1}{(mc)^2}(2mH-\mathbf{P}^2)\right]-\left[mc+\frac{1}{c}H\right](mc)^2\beta\left[1+\frac{1}{2(mc)^2\beta^2}(\pi^0)^2+\frac{1}{2(mc)^2(\alpha'\beta)^2}(\xi^0)^2\right]-mcP_i\pi_i^0+O\left[\frac{1}{c}\right]\underset{c}{=}0 .$$

Rearranging terms this goes into

$$(mc)^3(1-\beta)+(mc)\left[\frac{3}{2}(2mH-\mathbf{P}^2)-\beta mH -\frac{1}{2\beta}(\pi^0)^2-\frac{1}{\beta\alpha'^2}(\xi^0)^2-P_i\pi_i^0\right]+O\left[\frac{1}{c}\right]\underset{c}{=}0 . \quad (3.17)$$

From this we conclude that the arbitrary factor in (2.14) has to be chosen $\beta=1$. Then the second term on the left-hand side of (3.17) gives

$$H\underset{c}{=}\frac{3\mathbf{P}^2}{4m}+\frac{1}{2m}\frac{1}{2}\left[(\pi^0)^2+\frac{1}{\alpha'^2}(\xi^0)^2+2P_i\pi_i^0\right] .$$

If we use (3.8) with $\beta=1$ and (2.2) this can be rewritten in the limit $c\to\infty$ as

$$H\underset{c}{=}\frac{\mathbf{P}^2}{2m}+\frac{1}{4m}\pi^{(\infty)2}+\frac{1}{4m\alpha'^2}\xi^{(\infty)2} . \quad (3.18)$$

We have thus obtained for the Galilean energy the energy operator of a harmonic oscillator with translational degrees of freedom that moves as a whole with momentum P_i and has a total mass m.

To make this more explicit we will now think of a vibrating dumbbell, i.e., two masses m_1 and m_2 that interact with a harmonic potential characterized by a constant k. Its reduced mass

$$\mu=\frac{m_1m_2}{m_1+m_2}=\frac{m_1m_2}{m}$$

is used to define a new momentum $\tilde{\pi}_i$ in terms of the $\pi_i^{(\infty)}$ by

$$\tilde{\pi}_i=\left[\frac{2\mu}{4m}\right]^{1/2}\pi_i^{(\infty)} \quad (3.19a)$$

and then a new position

$$\tilde{\xi}_i=\left[\frac{4m}{2\mu}\right]^{1/2}\xi_i^{(\infty)} \quad (3.19b)$$

such that $\tilde{\pi}_i$ and $\tilde{\xi}_i$ fulfill again the canonical CR (3.13). Inserting this into (3.18) we obtain

$$H=\frac{\mathbf{P}^2}{2m}+\frac{1}{2\mu}\tilde{\pi}^2+\frac{2\mu}{(4m\alpha')^2}\tilde{\xi}^2 . \quad (3.20)$$

This is the energy operator of the vibrating dumbbell with $\tilde{\xi}$ being the distance between the two masses and $k=\mu/(2m\alpha')^2$ is the potential constant. Thus the nonrelativistic limit of the QRO can accommodate the picture of a vibrating diquark system, but it does not predict it, because the $\pi_i^{(\infty)}$ and $\xi_i^{(\infty)}$ in (3.18) could also be interpreted differently.

To interpret the angular momentum of the QRO we consider the fourth-order Casimir operator in the nonrelativistic limit. If we insert (2.10) into (2.3), we obtain

$$W_0=-\tfrac{1}{2}\epsilon_{onrs}S^{nr}\pi^{0s}mc\alpha' , \quad (3.21)$$

and using also (2.15),

$$W_m=\epsilon_{mnro}(\tfrac{1}{2}S^{nr}\Gamma^0+S^{on}\Gamma^r) \quad (\epsilon_{mnro}=\epsilon_{mnr})$$
$$=\epsilon_{mnro}(\tfrac{1}{2}S^{nr}-\xi^{0n}\pi^{0r})(mc)^2\alpha'+O(c^0) . \quad (3.22)$$

Further,

$$\tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}=\tfrac{1}{2}\epsilon_{mnro}S^{mn}S^{ro}=-\tfrac{1}{2}\epsilon_{mnro}S_{mn}S_{ro}$$
$$=\tfrac{1}{2}\epsilon_{mnr}S_{mn}\xi_r^0mc=S_r\xi_r^0mc , \quad (3.23)$$

where

$$S_r=\tfrac{1}{2}\epsilon_{rmn}S_{mn} .$$

Inserting this into the fourth-order Casimir operator $C_{(4)}$ we obtain

$$\left[\frac{1}{(mc)^2\alpha'}\right]^2C_{(4)}=\left[\epsilon_{mnr}\left[\frac{1}{2}S_{nr}-\xi_n^0\pi_r^0\right]\right]^2+O\left[\frac{1}{c^2}\right] . \quad (3.24)$$

The eigenvalue P_1 of $C_{(4)}$ is given according to (2.8) and (2.14) by

$$P_1=s(s+1)[(mc)^4\alpha'^2-(s-1)(s+2)] .$$

Inserting this into (3.24) we obtain

$$\text{eigenvalue}[\epsilon_{mnr}(\tfrac{1}{2}S_{nr}-\xi_n^0\pi_r^0)]^2=s(s+1)+O\left[\frac{1}{c^2}\right]$$

$$s=0,\tfrac{1}{2},1,\frac{3}{2}\cdots \quad (3.25)$$

This result suggests that we define an operator

$$\tilde{S}_m=\epsilon_{mnr}(\tfrac{1}{2}S_{nr}-\xi_n^{0(\infty)}\pi_r^{0(\infty)})=S_m-\epsilon_{mnr}\xi_n^{0(\infty)}\pi_r^{0(\infty)} \,. \quad (3.26)$$

After contraction, $1/c\to 0$, we obtain from (3.25) that the square of this vector operator $\tilde{S}_m\tilde{S}_m=\tilde{\mathbf{S}}^2$ is an invariant and has the eigenvalue

$$\tilde{\mathbf{S}}^2=s(s+1)I \,. \quad (3.27)$$

It is an angular momentum operator, i.e., fulfills the CR

$$[\tilde{S}_m,\tilde{S}_n]=i\epsilon_{mnr}\tilde{S}_r \,, \quad (3.28)$$

which follows immediately from the CR of S_{nr} and from the CR of $\xi_n^{0(\infty)}$ and $\pi_m^{0(\infty)}$ obtained in (3.5), (3.11), and (3.13).

$\xi_i^{0(\infty)}$ and $\pi_j^{0(\infty)}$ are the intrinsic position and momentum of the three-dimensional oscillator at rest (i.e., when the momentum of the system as a whole is $P_j=0$). $\xi_i^{0(\infty)}\pi_j^{0(\infty)}-\xi_j^{0(\infty)}\pi_i^{0(\infty)}$ is therefore the orbital angular momentum of the oscillating constituents. The intrinsic orbital angular momentum of the oscillator, S_m, is—according to (2.26) of I—in the rest frame the spin of the extended object,

$$\Sigma_{mn}\underset{\text{rest}}{=}S_{mn} \,.$$

(3.26) therefore says that the spin of the extended object is the oscillatory orbital angular momentum $\epsilon_{mnr}\xi_n^{0(\infty)}\pi_r^{0(\infty)}$ plus the angular momentum $\tilde{S}_m$:

$$S_m=\epsilon_{mnr}\xi_n^{0(\infty)}\pi_r^{0(\infty)}+\tilde{S}_m \,. \quad (3.29)$$

This new angular momentum $\tilde{S}_m$ must therefore be the total spin of the constituents.

Using (3.8), (3.2), and (3.4), the relation (3.29) can also be expressed as

$$\begin{aligned}\tilde{S}_{nr}&=S_{nr}-(\xi_n^{(\infty)}\pi_r^{(\infty)}-\xi_r^{(\infty)}\pi_n^{(\infty)}-\xi_n^{(\infty)}P_r+\xi_r^{(\infty)}P_n)\\&=\Sigma_{nr}^{(\infty)}-\xi_n^{(\infty)}\pi_r^{(\infty)}+\xi_r^{(\infty)}\pi_n^{(\infty)} \,,\end{aligned} \quad (3.30)$$

which again says that (for $P_r\neq 0$) the spin of the extended object $\mathbf{\Sigma}^{(\infty)}$ is the sum of the orbital angular momentum and the total spin of the oscillating constituents:

$$\Sigma_{nr}^{(\infty)}=\xi_n^{(\infty)}\pi_r^{(\infty)}-\xi_r^{(\infty)}\pi_n^{(\infty)}+\tilde{S}_{nr} \,. \quad (3.31)$$

The value s of the angular momentum $\tilde{S}_i$ has a meaning already before contraction, i.e., for the relativistic model. It is the number which together with $(-R)$ characterizes the representation of SO(3,2); only for its interpretation in terms of the constituents of the extended object does one need the nonrelativistic limit. We will, therefore, call s the total constituent spin of the QRO also before contraction.

For the subclass of representations (a), $s=0$ and therefore the operator $\tilde{S}_m=0$. Thus the QRO model with representations given by the multiplicity pattern in Fig. 3 of paper I has as its nonrelativistic limit the ordinary three-dimensional oscillator with constituent spin equal to zero. The spin of the extended object, j, is equal to the oscillatory angular momentum and in the nonrelativistic limit $j(j+1)$ is the eigenvalue of $(\boldsymbol{\xi}^{0(\infty)}\times\boldsymbol{\pi}^{0(\infty)})^2$.

The multiplicity pattern in Fig. 1 with $s=\frac{1}{2}$ describes the QRO model with total constituent spin equal $\frac{1}{2}$ (spinning relativistic string). It would be a candidate—after parity doubling to include the antiparticle spaces—for the description of baryon resonances with the $j^P=\frac{1}{2}^+$ baryon as the oscillatory ground state. The contraction limit of the QRO does not tell us anything about the number of constituents, it says only that they are oscillating and that the sum of their spins is $\frac{1}{2}$.

For the vector mesons ρ, ω, $K^*\cdots$, the quark model requires that the total constituent spin be 1, with the masses m_1 and m_2 having spin $\frac{1}{2}$ each. Thus the ρ, $\omega,\ldots$ towers should be described by a QRO with multiplicity pattern given by Fig. 1 with $s=1$. The lowest state of this QRO, the ground state with oscillatory angular momentum $\boldsymbol{\xi}^{(\infty)}\times\boldsymbol{\pi}^{0(\infty)}$ equal to zero, is the state with $\mu=2$, $j=s=1$. ρ (or ω or $K^*\cdots$) is assigned to this state and there is no state in this representation with mass lower than the ρ mass. Negative values of m^2 cannot occur in this representation. We will see in Sec. IV that this representation gives an equally good fit to the normal j^P and positive C_nP resonances as the fit shown in Fig. 4 of I, but the values of the empirical parameters $1/\alpha'$ and λ^2 are now different with α' being about 1 GeV^{-2} without causing the problems of negative mass squared.

The physical intepretation of the nonrelativistic contraction process is not clear to us. In general one would always require of a new theory describing a new domain of physics to correspond in a certain sense to the old theory when the new domain of applicability is restricted to the old familiar domain. Thus the quantum oscillator should become the classical oscillator when the observables are commuting and the theory of the relativistic oscillator should become the theory of the nonrelativistic oscillator when the $1/c\to 0$ limit is taken. For the contraction limit of the symmetry group there exists a physical counterpart: one can arrange the velocities such that the motion of the mass point becomes nonrelativistic. However, this need not be true for the intrinsic dynamics. It is not clear whether one can by any manipulations make a relativistic oscillator into a nonrelativistic one. So the vibrating dumbbell is not an attainable limit, but just an approximate theoretical picture, and this picture is the better the less uncommuting the "intrinsic" position and momentum coordinates are. From Eqs. (2.10a) and (2.10b) or (3.1) and (3.10) one sees that this approximate picture is better for larger values of the hadron mass mc.

IV. EXPERIMENTAL EVIDENCE FOR THE QRO

We have derived two mass formulas for two different models. For the QRO model we obtained from the Hamiltonian[2]

$$\mathscr{H}=v\left(P_\mu P^\mu-\frac{1}{\alpha'}\hat{P}_\mu\Gamma^\mu-m_0{}^2\right) , \quad (4.1)$$

the mass spectrum[10]

$$m^2=\tilde{m}_0{}^2+\frac{1}{\alpha'}\nu,\quad \nu=0,1,2,\ldots,$$
$$j=0 \text{ or } 1,\ldots,\nu\,. \tag{4.2}$$

$m_0{}^2$ and $\tilde{m}_0{}^2$ are arbitrary constants with $\tilde{m}_0{}^2$ depending upon $m_0{}^2$ and the choice of the value $\mu_{\min}$ for the SO(3,2) representation in Fig. 3 of I or in Fig. 1. [In I we have omitted the irrelevant constant $m_0{}^2$ and absorbed it in $(1/\alpha')\mu_{\min}$.] ν and therewith m^2 is degenerate and the different values of spin j for which (4.2) gives the same mass are given by the columns of the pattern in Fig. 1 or Fig. 3 of I.

For the QRR model we obtained from the Hamiltonian[1]

$$\mathscr{H}=v(P_\mu P^\mu-\lambda^2\hat{W}-m_0{}^2)\,, \tag{4.3}$$

the mass spectrum

$$m^2=\tilde{m}_0{}^2+\lambda^2 j(j+1),\quad j=0,1,2,\ldots\,. \tag{4.4}$$

j is nondegenerate and the values of j are given by the pattern in Fig. 2 of I (and a corresponding pattern for half-integer spin with $j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$). One may combine both models in the same way as one combines the rotator and oscillator of molecular physics to obtain the vibrating rotator.

If one takes for this model the Hamiltonian

$$\mathscr{H}=v(P_\mu P^\mu-\frac{1}{\alpha'}\hat{P}_\mu\Gamma^\mu-\lambda^2\hat{W}-m_0{}^2)\,, \tag{4.5}$$

then one obtains the mass spectrum

$$m^2=\tilde{m}_0{}^2+\frac{1}{\alpha'}\nu+\lambda^2 j(j+1)\,. \tag{4.6}$$

We have not yet constructed such a model and do not know whether there would be theoretical difficulties which may prevent this. But we want to give here an empirical test of (4.6) and determine the j mass unit (j un) λ^2 and the ν mass unit (ν un) $1/\alpha'$ from the experimental data. (4.6) includes (4.2) as the special case $\lambda^2=0$.

In molecular physics j un is about three orders of magnitude smaller than ν un and in nuclear physics j un is more than one order of magnitude smaller. One may, therefore, expect that λ^2 will be much smaller than $1/\alpha'$. If the analogy with molecular physics prevails one may also expect to have correction terms to (4.6) of the form ν^2 (anharmonicity) and $\nu j(j+1)$ (vibration-rotation interaction); we will not include these terms in our fits.

In order to make fits of the meson and baryon resonances we have to group them into towers whose members are considered as different states of the QRO with different quantum numbers ν and j. Each tower is described by the reducible representation space of the Poincaré group,

$$\mathscr{H}=\sum_{\nu,j}\oplus\mathscr{H}^\nu(m(\nu,j),j)\,, \tag{4.7}$$

where $\sum_{\nu,j}$ runs over the values given by, e.g., the pattern in Fig. 1. In I we have made a fit associating a tower to the representation space (3.11) of I which belonged to the pattern of Fig. 3 of paper I. Whereas the value of j for each resonance is known from experiment, the problem is the assignment of the new vibrational quantum number ν to the observed resonances. Assigning the resonances in order of increasing mass to boxes of Fig. 3 in paper I, for which ν increases in steps of 2 for a fixed value of j, will lead to a different empirical value for $1/\alpha'$ than assigning them to boxes of Fig. 1 (with $s=1$), where ν increases in steps of 1. Here we will do the latter, expecting for $1/\alpha'$ a value approximately twice that of the value obtained for Fig. 3 of paper I.

We choose first the mesons with nonstrange quarks and with $CP=+1$ and normal j^P. As the $I=0$ masses (ω tower) are almost degenerate with the $I=1$ masses (ρ tower) we have fitted them jointly to (4.6) (exchange degeneracy). The result of this fit is given in Fig. 2. Each box gives on the left-hand side (LHS) the mass value (in GeV) predicted by the fit and on the right-hand side (RHS) the particle symbol with the mass that has been used as input for the fit. If the rhs of a box is empty then no input for the corresponding values of (j,ν) has been used. The values of the parameters determined by this fit are

$$\frac{1}{\alpha'}=1.03\pm0.05\ \text{GeV}^2\,,$$
$$\lambda^2=0.02\pm0.01\ \text{GeV}^2\,, \tag{4.8}$$
$$\tilde{m}_0{}^2=-0.45\pm0.04\ \text{GeV}^2\,.$$

The lowest state in this representation is the one with $j=1$, $\nu\equiv$ eigenvalue $\hat{P}_\mu\Gamma^\mu-1=1$ so that $m^2(1,1)=0.78\ \text{GeV}^2$ is the lowest mass predicted for this model. The "yrast" states[3] with $j=\nu$ lie on an almost linearly rising Regge trajectory because $\lambda^2\ll 1/\alpha'$. In addition there are "daughters" with $j=\nu-1,\ \nu-2,\ldots,1$ with only slightly lower masses. (For the representation of Fig. 3 of paper I every second daughter is missing, i.e., $j=\nu-2,\ \nu-4,\ldots,1$ for $\nu=$odd and $j=\nu-2,\ \nu-4,\ldots,0$ for $\nu=$even.) Every well established resonance with $I=0$ and $I=1$ which does not contain s, c, or b flavor and for which $CP=+1$ and j^P is natural has been included in Fig. 2, except those with $j^P=0^+$. On the other hand, there are places in Fig. 2 for which no resonance has been established experimentally, but none of these seem to be an embarrassment. The $\chi^2/n_D=2/16$ is better than the one for the representation of Fig. 3 in I (which is 8.6/18 for mesons only) but this is really not relevant. The important advantage of this fit over the one in Fig. 4 of I is that it assigns to ρ and ω the total constituent spin $s=1$ as required by the quark model.

The K tower is interesting because of the recently found 1^- resonances[11] at 1410 and 1790 MeV. Figure 3 gives a comparison between the experimental resonances and the masses predicted by (4.6) with the values

$$1/\alpha'=1.11\pm0.08\ \text{GeV}^2\,,$$
$$\lambda^2=0.02\ \text{GeV}^2\,, \tag{4.9}$$
$$\tilde{m}_0{}^2=-0.25\pm0.04\ \text{GeV}^2\,.$$

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$	$\nu=6$	$\nu=7$	$\nu=8$
j=8								3.00
j=7							2.78 M(2.75)	2.96
j=6						2.54 r(2.51) δ(2.45)	2.73	2.92
j=5					2.28 ρ(2.35)	2.50	2.70	2.88
j=4				2.00 h(2.03) δ(2.04)	2.24 ε(2.30)	2.46	2.66	2.85
j=3			1.69 ω(1.67) g(1.69)	1.97	2.21 ρ(2.20)	2.44	2.64	2.83
j=2		1.31 f(1.27) A_2(1.31)	1.66 Θ(1.69)	1.94	2.19 ε(2.18)	2.41	2.62	2.81
j=1	0.78 ω(0.78) ρ(0.77)	1.28 ρ(1.26)	1.64 ρ(1.60)	1.92	2.18 ρ(2.18)	2.40	2.61	2.80

FIG. 2. Comparison between experimental data (for the $CP=+1$, j^P=normal mesons) and predictions of the QRO model. The boxes are as in Fig. 1 except that j now increases upward whereas in Fig. 1 j increases downward. The particle symbol with the mass value (in GeV) gives the data that have been used as input in the fit to (4.6). The values of the parameters (4.8) have been obtained by minimizing χ^2 with m^2 fitted and with an error taken as $\Delta m^2=\Gamma m$, where Γ is the width of the resonance. When a particle with $I=0$ and a particle with $I=1$ are listed in one box corresponding to the pair of numbers (j,ν), both their masses have been used as input for the place (j,ν). The values of the masses predicted by the fit (in GeV) are given on the left side of the boxes. They should be compared with the values in brackets on the right side of the box. If there is no evidence for a particle with that mass *and spin* the right side of the boxes is left blank. Most experimental values are taken from Ref. 13, $M(2.75)$ is from Ref. 14.

We see that the new K^*'s fit in very well.

The comparison for the ϕ or $s\bar{s}$ tower is given in Fig. 4. There are not many meson resonances of this kind known and some of the assignments to this tower may be questionable. The values of the parameters for this tower are

$$1/\alpha'=1.00\pm0.09 \text{ GeV}^2 ,$$
$$\lambda^2=0.05\pm0.02 \text{ GeV}^2 , \qquad (4.10)$$
$$\tilde{m}_0{}^2=-0.06\pm0.04 \text{ GeV}^2 .$$

Compared with the ρ, ω, and K^* towers, we would have preferred a slightly larger value for $1/\alpha'$. Although we have no theory for the flavor dependence of α' we would expect from the nonrelativistic limit (3.20) that $1/\alpha'$ increases with the quark mass, in particular if the potential constant k is independent of flavor. This seems to be fulfilled for all other meson towers, except the ϕ.

There are not enough particles in the $c\bar{c}$ tower or the $b\bar{b}$ tower to provide a meaningful test of our model. But there are a sufficient number of 1^- states to check whether one can find an assignment for the ν quantum number such that the oscillator spectrum will result.

In (4.11) we give in the last line the experimental masses in GeV of the $c\bar{c}$ mesons used in the fit and in the first line the values of ν assigned to them. The second line gives the values for the masses predicted by the fit with the parameter $1/\alpha'=2.0007\pm0.0002$ GeV^2:

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
j=5					2.41
j=4				2.12 K*(2.06)	2.37
j=3			1.80 K*(1.78)	2.08	2.34
j=2		1.41 K*(1.43)	1.76	2.06	2.31
j=1	0.89 K*(0.89)	1.38 K*(1.41)	1.74 K*(1.79)	2.04	2.29

FIG. 3. Comparison between experimental data and predictions of the QRO model for the K^* mesons. $K^*(1.41)$ and $K^*(1.79)$ are from Ref. 12.

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
j=5					2.54
j=4				2.22	2.44
j=3			1.88 φ(1.85)	2.13	2.35
j=2		1.50 f'(1.53)	1.80 f(1.81)	2.06 g_T(2.16)	2.29
j=1	1.02 φ(1.02)	1.43	1.74 φ(1.69)	2.01	2.24

FIG. 4. Comparison between experimental data and predictions for the $CP=+1$, j^P=normal mesons which are suspected to be $s\bar{s}$. None of the resonances used here have been included in Fig. 2.

$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$	$\nu=6$	
3.0960	3.404	3.6860	3.948	4.194	4.425	
3.0960±0.0001		3.6860±0.0001	4.030±0.052	4.160±0.078	4.415±0.043	(4.11)

For the $b\,\bar{b}$ tower the comparison between experimental masses and the mass values predicted by the fit is the following (all masses in GeV):

$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$	
9.4825	9.771	10.051	10.323	10.585	
9.4600±0.0440		10.025±0.030	10.355±0.017	10.575±0.014	(4.12)

The νun obtained from this fit is $1/\alpha'=5.550\pm0.097$ GeV^2. In both cases there is no experimentally observed mass for the $\nu=2$ state. Were it not for this discrepancy one could consider the agreement excellent (in spite of large χ^2 due to the small errors). It is remarkable that the same place ($j=1,\nu=2$) is also vacant for the ϕ and ω towers; $\rho(1250)$ which is assigned to this place in the ρ tower also needs more experimental support.

For the baryon resonances one uses the representations of Fig. 1 with $s=\frac{1}{2}$ for the N and $s=\frac{3}{2}$ for the Δ. We have made fits to the N, Λ, Σ, and Δ towers and have obtained reasonable results. Here we shall only reproduce the fits for the N tower starting with $\frac{1}{2}^+$ and the Δ tower starting with $\frac{3}{2}^+$. The νun and $j$un values for the Λ and Σ towers are $1/\alpha'\approx0.7$ GeV^2, $\lambda^2\approx0.08$ GeV^2. The fit ($\chi^2/n_D=7.4/9$) of the N resonances is given in Fig. 5. The values of the fitted parameters are

$$1/\alpha'=1.052\pm0.061 \text{ GeV}^2\,,$$
$$\lambda^2=0.004\pm0.02 \text{ GeV}^2\,. \tag{4.13}$$

These values are essentially identical with the values for the nonstrange meson towers.

The fit of the Δ tower is displayed in Fig. 6. The values of the fitted parameters are

$$1/\alpha'=1.19\pm0.16 \text{ GeV}^2\,,$$
$$\lambda^2=0.00\pm0.02 \text{ GeV}^2\,, \tag{4.14}$$

and are nearly the same within errors as those for the N tower.

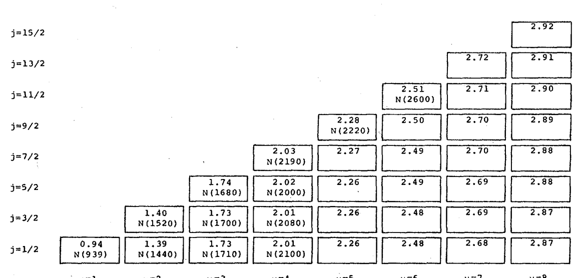

FIG. 5. Comparison between experimental data and predictions for the nucleon resonances. The resonances used in the fit have spin-parity $j^P=\frac{1}{2}^+,\frac{3}{2}^-,\frac{5}{2}^+,\frac{7}{2}^-,\ldots$. Most of them are degenerate in mass with a resonance of the same j and opposite parity.

In the fit of the N tower we have only used the resonances with $\frac{1}{2}^+,\frac{3}{2}^-,\frac{5}{2}^+,\ldots$. To all of these resonances (except the ground state) there is listed in the Particle Data Group table a partner with opposite parity: $\frac{1}{2}^-$, $\frac{3}{2}^+,\frac{5}{2}^-,\ldots$. In most cases the masses of partners with the same spin and opposite parity are almost degenerate. This is reminiscent of the l-type doubling in molecular physics which would suggest that the angular momentum is not only rotational but contains also—except in the vibrational ground state $N(939)$ which is nondegenerate—vibrational angular momentum. With this degeneracy all existing low-lying nucleon resonances are accommodated in the representation of Fig. 5. However, since the masses of the nucleon resonances, determined by phase shift analysis, are not well defined and have a large error (width), the correctness of this fit should not be overestimated.

If the slope parameters for the ρ, ω, N, and Δ tower agree as indicated by Eqs. (4.9), (4.13) and (4.14), then the mass-squared splittings within these four towers is the same (within the large errors), i.e., if we adjust the lowest m^2 of all tower to the same lowest level, then the νth excited states of all tower lie roughly also on the same excited levels. Thus we have one $j=\frac{1}{2}$, two $j=1$, and one $j=\frac{3}{2}$ state for which the m^2 levels have been adjusted to be the same and then we have for every $p=1,3,5,\ldots$, one $j=p/2$ state (N^*), two $j=(p/2+1/2)$ states (ρ^*,ω^*), and one $j=(p/2+1)$ state (Δ^*) for which the adjusted m^2 levels are the same. In nuclear physics such a picture, in which different nuclei (corresponding to different towers here) have the same level structure and level spacing, is taken as evidence for supersymmetries.[12] There the inter-

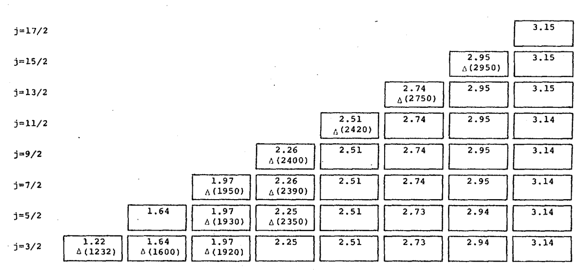

FIG. 6. Comparison between experimental data and predictions for the Δ resonances.

pretation for this feature is that the additional nucleon of the even-odd nucleus which is added to the core of the even-even nucleus couples only weakly to the core. This would correspond here to the interpretation that an additional quark is added with weak coupling to the diquark core to make the baryon, which is not a very plausible explanation.

We have presented in this section the best evidence in favor of our model. Except for the missing $\nu=2$ state the agreement is very nice. An important point is that for the classes of resonances that we fitted into towers we have essentially exhausted the existing experimental values and not just chosen a suitable subset of data. But there are other mesons, e.g., the π and K, which could not be explained as ground states of an oscillator tower. Masses and widths often have large errors and as we have total freedom in the assignment of the new quantum number ν all experimental evidence may be accidental. Still, that such a simple model can accommodate so much experimental data is remarkable.

So far we have not included electromagnetic or other external interactions. This one could introduce by minimal coupling

$$P_\mu \to P_\mu - eA_\mu$$

leading to an interaction term in the Hamiltonian (4.1) of the form

$$\frac{e}{\alpha' M} A_\mu \Gamma^\mu \ .$$

As this has the usual form, with Γ_μ corresponding to the current operator, one could hope to develop a program to calculate transitions and other interaction properties. They may provide further experimental tests of our model.

At the moment evidence can only be provided by the hadron spectrum, and there the model has not failed. The greatest point in favor of this model is however not the magnitude of experimental data that it can reproduce, but its theoretical beauty. It is simple and soluble and the mass that it predicts is really the relativistic mass by which a hadron is defined.

ACKNOWLEDGMENT

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[1](a) R. R. Aldinger *et al.*, Phys. Rev. D **28**, 3020 (1983); (b) A. Bohm *et al.*, **28**, 3032 (1983); (c) R. R. Aldinger *et al.*, **29**, 2828 (1984).

[2]A. Bohm, M. Loewe, and P. Magnollay, Phys. Rev. D **31**, 2304 (1985).

[3]States of lowest energy or mass for each value of angular momentum (spin of the extended object) are called yrast states. See A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969).

[4]J. B. Ehrman, Ph.D. thesis, Princeton University, 1954, Sec. VII i 3; L. Jaffe, J. Math. Phys. **12**, 882 (1971); E. Angelopoulos in *Quantum Theory, Groups, Fields, and Particles*, edited by A. O. Barut (Reidel, Boston, 1983), p. 101.

[5]A. Bohm and R. B. Teese, J. Math. Phys. **17**, 94 (1976).

[6]E. Celeghini and M. Tarlini, Nuovo Cimento **68B**, 133 (1982).

[7]A. Bohm *et al.*, University of Texas Report No. DOE-ER-40200-583, 1985 (unpublished).

[8]E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. USA **39**, 510 (1953).

[9]We could choose $-R$ to increase to infinity with any power of c as $c \to \infty$ and still obtain the oscillator algebra. We would then have to use different factors of $1/c$ in the definitions (2.10a)—(2.10c) to obtain finite, nonzero oscillator generators in the limit $c \to \infty$. Besides the leading term, we could allow $-R$ to also have terms of lower order in c, including negative orders. That we have chosen $-R$ to have a leading term of order c^4 is a consequence of the choice we will make for the relativistic Hamiltonian and of our desire that the Galilean mass be positive.

[10]In the previous sections we have denoted the eigenvalue of $P_\mu P^\mu$ by p^2 and by m^2 the Galilean mass $p^2/c^2 \to m^2$. In the present section only the relativistic mass will occur and we denote the eigenvalue of $P_\mu P^\mu$ again by m^2 and take $c=1$.

[11]D. Aston *et al.*, Phys. Lett. **149B**, 158 (1984).

[12]Hong-Zhou Sun *et al.*, Phys. Rev. C **29**, 352 (1984).

[13]Particle Data Group, Rev. Mod. Phys. **56**, S1 (1984).

[14]D. L. Denney *et al.*, Phys. Rev. D **28**, 2726 (1984).