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Bound states of heavy $Q^2\bar{Q}^2$ systems

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The Born-Oppenheimer approximation for heavy quarks in the MIT bag is extended to the $Q^2\bar{Q}^2$ system, where the glue plus volume energy becomes an operator in color space. Taking its lowest eigenvalue as the approximate potential energy, the four-body Schrödinger equation is solved variationally for the ground-state energy. With this approximation, it is found that heavy-quark systems such as $c^2\bar{c}^2$ are stable against breakup into two ψ ($c\bar{c}$) mesons. It is necessary to solve the coupled-channel (in color space) problem to confirm this result.

I. INTRODUCTION

For a system composed of slowly moving quarks and antiquarks, it seems reasonable to use a Born-Oppenheimer approximation in which the quarks are treated, at first, as static, localized sources of the glue field. If one were able to calculate the energy of the ground state of QCD as a function of the positions of the heavy quarks, that would become the potential energy in the Schrödinger equation for the quark motion. If there are excited states of QCD that are nearby in energy and strongly coupled to the ground state via the quark motion, then the Schrödinger equation becomes a coupled-channel problem.

From lattice-gauge-theory calculations,[1] the ground-state energy in the presence of one (each) static quark and antiquark has been found as a function of the separation of the quarks, and it varies like r^{-1} at small distances and linearly with r at large distances. Unfortunately, at the present time there does not exist corresponding ground-state information for more complicated systems such as $Q^2\bar{Q}^2$, and certainly nothing about excited states.

The $Q\bar{Q}$ system has also been studied in the MIT bag model using the very same Born-Oppenheimer approximation.[2,3] For fixed positions of the quarks the linearized (classical) Yang-Mills equations are solved for the glue field and simultaneously for the bag boundary. The energy found in this way also varies like r^{-1} at small distances and linearly with r at large distances, and when the Schrödinger equation is solved with this potential good agreement is obtained[3] with the (spin-averaged) spectrum of ψ ($c\bar{c}$) and Υ ($b\bar{b}$). The same approximation has been applied to the s quark, and even though its motion is becoming somewhat relativistic fair agreement is obtained[4] for the masses of ϕ ($s\bar{s}$) and Ω^- (s^3) (with just one new parameter, m_s). When the same procedure is applied to the u and d quarks their motion becomes very relativistic and this raises a serious question about the applicability of a static potential to light quarks.

The success of this Born-Oppenheimer approximation when applied to the bag model for heavy quarks encourages us to try it on the $Q^2\bar{Q}^2$ system. There is brand new physics here, due to the possibility of one $Q\bar{Q}$ cluster separating from the other, and this requires the ability to deal with highly deformed bags. We have only handled that aspect of the problem crudely, but expect that it is not too serious in the search for true four-body bound states, which is the subject of this paper. Provided the bound state (if it exists) is such that all interparticle separations are comparable, our treatment should be adequate.

In Sec. II the bag-model potential energy is derived for an arbitrary number of quarks and antiquarks and then specialized to the $Q\bar{Q}$ and $Q^2\bar{Q}^2$ systems. The defect at large separations resulting from the use of a spherical bag is discussed and a simple modification proposed. The results are presented in Sec. III, first for the mesons and then for the dimesons. Section IV compares the potential energy derived in this paper with some others that have been used in the $Q^2\bar{Q}^2$ system, and the results are summarized in Sec. V.

II. THE POTENTIAL ENERGY

A. The bag-model potential

For a set of static color charges F_i^a located at positions $\mathbf{r}_i$, the energy for a bag of arbitrary shape described by parameters R_α, is[5]

$$W(F_i,\mathbf{r}_i,R_\alpha)=\int dV\left[\tfrac{1}{2}\sum_a(\nabla\phi^a)^2+B\right]$$
$$=\frac{g^2}{2}\sum_{i,j}F_i\cdot F_j G_N(\mathbf{r}_i,\mathbf{r}_j,R_\alpha)+BV(R_\alpha)\,, \quad (1)$$

where

$$\nabla^2\phi^a(\mathbf{r})=-\rho^a(\mathbf{r})=-g\sum_i F_i^a\delta^{(3)}(\mathbf{r}-\mathbf{r}_i) \quad (2)$$

and the boundary condition is

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$$\hat{\mathbf{n}}\cdot\nabla\phi^a(\mathbf{r})=0\ . \tag{3}$$

G_N is the Neumann Green's function for the surface, B is the bag constant, and V is the volume enclosed within the surface. As in most applications of the bag model, the nonlinear terms in the Yang-Mills equations have been neglected.

Since there is no kinetic-energy term associated with the motion of the surface, this is a system with constraints, $P_\alpha=0$, where P_α is conjugate to R_α. To preserve this constraint in time requires

$$\frac{\partial W}{\partial R_\alpha}=0\ , \tag{4}$$

which, when solved for the R_α as functions of F_i and $\mathbf{r}_i$ gives the Hamiltonian

$$H=\sum_i \frac{{p_i}^2}{2m_i}+V\ , \tag{5}$$

where

$$V\equiv W(F_i,\mathbf{r}_i,R_\alpha(F_i,\mathbf{r}_i))\ . \tag{6}$$

As a result of the dependence of R_α on $\mathbf{r}_i$, this potential is a *many-body operator.* This will be seen explicitly below. From Eq. (1) it is seen that the F_i occur in V only as the dot products $F_i\cdot F_j$. The potential is translation invariant since the bag develops around the quarks. Solving Eq. (4) is equivalent to finding the surface on which

$$\tfrac{1}{2}\sum_a(\nabla\phi_s^a)^2=B\ . \tag{7}$$

So far, the entire discussion has been classical, that is, the F_i are classical variables [eight-dimensional vectors for SU(3)]. Once the Hamiltonian is known, the system can be quantized by replacing the Poisson brackets by commutators. For the color charges these are

$$[F_i^a,F_j^b]=i\hbar\delta_{ij}f_{abc}F_i^c\ , \tag{8}$$

where the f_{abc} are the structure constants for SU(3).

When the number of quarks (and antiquarks) is 2 or 3, only one color-singlet state is possible and the $F_i\cdot F_j$ become numbers. For $N=4$ there are two independent color-singlet states and therefore V as well as the R_α are 2×2 matrices in color space. When it is necessary to work in a particular representation we choose the basis states to be

$$\psi_1=|\,[(12)^1(34)^1]^1\rangle$$

and (9)

$$\psi_8=|\,[(12)^8(34)^8]^1\rangle\ ,$$

where particles 1 and 3 are quarks, and 2 and 4 are antiquarks. In this representation

$$F_1\cdot F_2=F_3\cdot F_4=\tfrac{1}{3}\begin{bmatrix}-4 & 0\\ 0 & \frac{1}{2}\end{bmatrix},$$

$$F_1\cdot F_3=F_2\cdot F_4=\tfrac{1}{3}\begin{bmatrix}0 & -\sqrt{2}\\ -\sqrt{2} & -1\end{bmatrix}, \tag{10}$$

and

$$F_1\cdot F_4=F_2\cdot F_3=\tfrac{1}{3}\begin{bmatrix}0 & \sqrt{2}\\ \sqrt{2} & -\frac{7}{2}\end{bmatrix}.$$

The equality $F_i\cdot F_j=F_k\cdot F_l$ when all four indices are different is independent of the representation and follows from

$$\sum_i F_i=0\ , \tag{11}$$

together with

$${F_i}^2=\tfrac{4}{3}\ . \tag{12}$$

In general, the solution of the set of Eqs. (4) for the surface can only be accomplished numerically. When all the interparticle separations are comparable, and small on the scale of $B^{-1/4}$, with the consequence that the surface is far away from the quarks, it is expected that a spherical approximation to the bag shape should be good. Although the Neumann Green's function for a sphere is known analytically, retaining only the dipole term from the homogeneous part is quite accurate at small separations, and much *less* in error than the full spherical result at large separations. Making this approximation

$$G_N(\mathbf{r}_i,\mathbf{r}_j,R)\cong\frac{1}{4\pi r_{ij}}+\frac{1}{2\pi R^3}\mathbf{r}_i\cdot\mathbf{r}_j\ , \tag{13}$$

and Eq. (1) becomes[6]

$$W(F_i,\mathbf{r}_i,R)=\tfrac{1}{2}\alpha_s\sum_{i\neq j}\frac{F_i\cdot F_j}{r_{ij}}+\frac{\alpha_s}{R^3}\left[\sum_i F_i\mathbf{r}_i\right]^2+\tfrac{4}{3}\pi R^3B \tag{14}$$

with $\alpha_s=g^2/4\pi$. Imposing the constraint, Eq. (4), gives

$$R^3=8^{1/2}\frac{\alpha_s}{k}A^{1/2} \tag{15}$$

and

$$V(F_i,\mathbf{r}_i)=\tfrac{1}{2}\alpha_s\sum_{i\neq j}\frac{F_i\cdot F_j}{r_{ij}}+\frac{k}{\sqrt{2}}A^{1/2}\ , \tag{16}$$

where

$$A\equiv\left[\sum_i F_i\mathbf{r}_i\right]^2=-\tfrac{1}{2}\sum_{i\neq j}F_i\cdot F_j{r_{ij}}^2 \tag{17}$$

and

$$k\equiv(8\pi B\alpha_s\tfrac{4}{3})^{1/2}\ . \tag{18}$$

Use has been made of Eq. (11) to obtain the final form of A, which is the square of the color-dipole-moment operator. It is seen that the potential consists of a sum of two-body Coulomb energies plus a confining term; the latter, due to the square-root operation, is a *many-body potential.*

In the two-body ($Q\bar{Q}$) system, Eq. (16) becomes

$$V^{(2)}(r)=-\tfrac{4}{3}\frac{\alpha_s}{r}+(\tfrac{2}{3})^{1/2}kr \tag{19}$$

and agrees with the numerical result, obtained from bags of more complicated shapes [including the cusps that are

required by Eq. (3)], to within 20 MeV for $r \lesssim 1$ fm.[3,6] It is only at still larger distances that the confining potential becomes

$$V(r) \to kr + V_0 ,$$

where k is called the string tension. Such very large distances are not relevant in the low-lying states of heavy-quark systems such as charmonium, and are probably not important in any hadron. Note, however, that the slope of the confining term in Eq. (19) is already 80% of its ultimate value.

As mentioned above, when the number of particles $N \geq 4$, A becomes a matrix, which is clearly positive definite. Using the idempotent formula yields a particularly simple expression for its square root for the four-body ($Q^2\bar{Q}^2$) system, where the matrix is 2×2:

$$A^{1/2} = \frac{1}{d_1 + d_2}(A + d_1 d_2 I) . \tag{20}$$

The d_α are the square roots of the eigenvalues of A, and are the color-dipole moments of the system. Combining Eqs. (16), (17), and (20) gives

$$V^{(4)}(F_i, \mathbf{r}_i) = -\tfrac{1}{2}\sum_{i\neq j} F_i \cdot F_j w_{ij} + \frac{kd}{\sqrt{2}} I$$

with

$$w_{ij} = \frac{k}{\sqrt{2}} \frac{r_{ij}^2}{d_1 + d_2} - \frac{\alpha_s}{r_{ij}} \tag{21}$$

and

$$d = \frac{d_1 d_2}{d_1 + d_2} .$$

In our calculations of spectra, no arbitrary constants are added on to $V^{(2)}$ nor $V^{(4)}$.

Due to the structure of these equations, it is useful to study a general matrix of the form

$$X = -\tfrac{1}{2}\sum_{i\neq j} F_i \cdot F_j x_{ij} \tag{22}$$

with $x_{ij} = x_{ji}$.

In the ψ_1–ψ_8 representation of Eq. (10) this can be written

$$X = \tfrac{1}{3}\begin{bmatrix} 4y_{12} & \sqrt{2}(y_{13} - y_{14}) \\ \sqrt{2}(y_{13} - y_{14}) & -\frac{1}{2}y_{12} + y_{13} + \frac{7}{2}y_{14} \end{bmatrix} , \tag{23}$$

where

$$y_{ij} = x_{ij} + x_{kl} = y_{kl}$$

with *all four indices unequal.* The matrix X depends, therefore, on only three combinations of the six x_{ij}. The eigenvalues of X, $e_\alpha(y_{ij})$, $\alpha = 1,2$, are found to be

$$e_\alpha(y_{ij}) = \tfrac{1}{12}\{7(y_{12} + y_{14}) + 2y_{13} \mp [81(y_{12} - y_{14})^2 + 36(y_{12} - y_{13})(y_{14} - y_{13})]^{1/2}\} . \tag{24}$$

Note that the eigenvalues are symmetric under the interchange of the spatial coordinates of any pair of identical particles, $1 \leftrightarrow 3$ or $2 \leftrightarrow 4$. The eigenvectors of X are also invariant under the simultaneous interchange of the spatial and color coordinates of any pair of identical particles. (A continuity argument eliminates the possibility that the eigenvectors might be antisymmetric under this interchange.)

From Eq. (17), the color dipole moments are

$$d_\alpha = [e_\alpha(s_{ij})]^{1/2}$$

with (25)

$$s_{ij} = r_{ij}^2 + r_{kl}^2 ,$$

and they are plotted in Fig. 1 for two geometric arrangements of the four particles. The smaller moment approaches a limit as the separation of the quark-antiquark pairs increases, and the larger moment increases linearly.

For these same geometries the elements of the potential matrix $V^{(4)}$ in the ψ_1–ψ_8 representation are plotted on Figs. 2 and 3. The eigenvalues and the mixing angle, defined by

$$V^{(4)}\chi_\alpha = v_\alpha \chi_\alpha ,$$
$$\chi_1 = \cos\theta\,\psi_1 + \sin\theta\,\psi_8 , \tag{26}$$
$$\chi_2 = -\sin\theta\,\psi_1 + \cos\theta\,\psi_8 ,$$

are also shown in these figures. From Eqs. (21)–(24) the v_α are given by

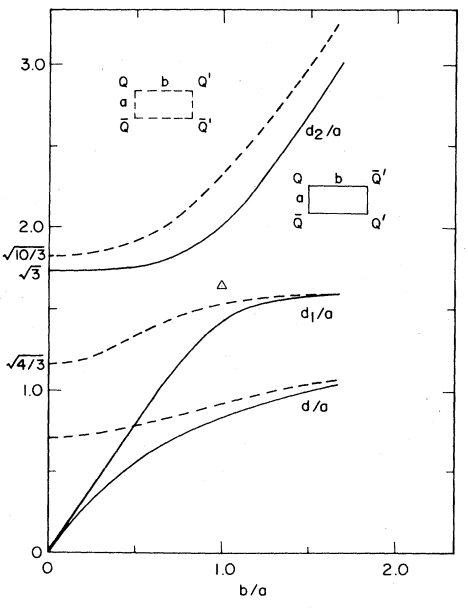

FIG. 1. The color dipole moments of $Q^2\bar{Q}^2$ for two rectangular geometries. d_1 and d_2 are the eigenvalues of $A^{1/2}$, Eq. (17), and $d \equiv d_1 d_2/d_1 + d_2$. The solid curves correspond to the case in which the identical particles are located at opposite vertices of the rectangle, and the dashed curves to the case in which the identical particles are at adjacent vertices. The point Δ shows the degenerate value of $d_1/a = d_2/a$ for the tetrahedral geometry.

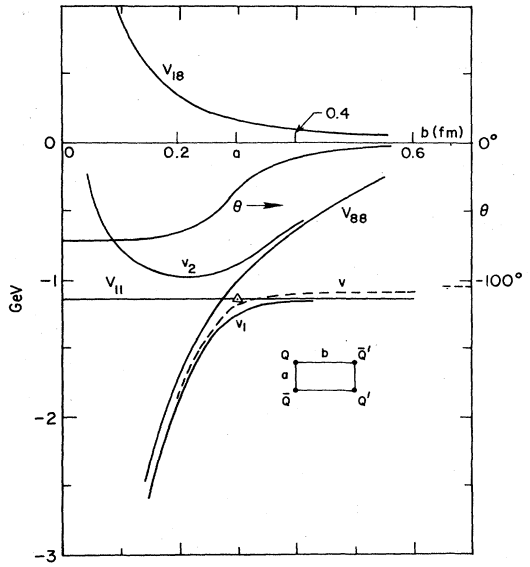

FIG. 2. The elements of the potential matrix V in the ψ_1–ψ_8 representation, and the eigenvalues v_α and mixing angle θ are shown for $Q^2\bar{Q}^2$ from the dipole approximation to the spherical bag model. The scale for θ is on the right-hand side. The rectangular geometry of the four particles is shown with the identical particles at opposite vertices. The point Δ shows the degenerate value of $v_1=v_2$ for the particles at the vertices of a regular tetrahedron with side 0.3 fm. The parameters are $\alpha_s=0.747$, $B^{1/4}=0.145$ GeV, and $a=0.3$ fm, appropriate for the charmed quark. The dashed curve labeled v shows the modified potential from Eq. (32) with $\gamma=1$, and its limiting value at $b=\infty$.

$$v_\alpha = e_\alpha(u_{ij}) + \frac{kd}{\sqrt{2}}$$

with (27)

$$u_{ij} = w_{ij} + w_{kl} \ .$$

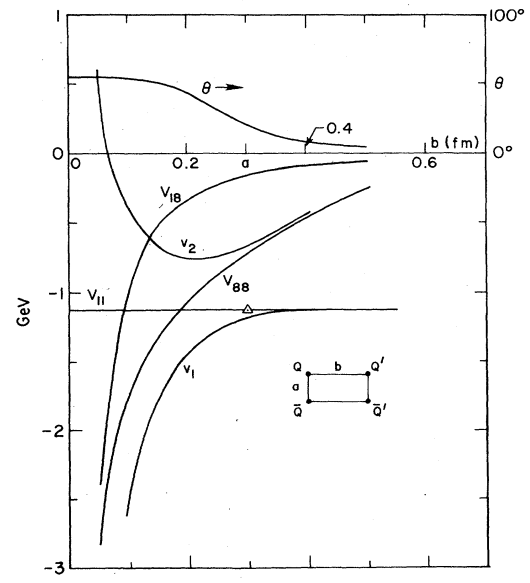

FIG. 3. Same as Fig. 2 with the identical particles at adjacent vertices.

At very small separation of the two quark-antiquark pairs ($b\to 0$) the lower eigenvalue corresponds to a state in which the two close particles have a definite total color. For Fig. 2 this is the state $|[(Q\bar{Q}')^1(Q'\bar{Q})^1]^1\rangle$, for which $\tan\theta=-\sqrt{8}$; and for Fig. 3 it is the state $|[(QQ')^{\bar{3}}(\bar{Q}\bar{Q}')^3]^1\rangle$, for which $\tan\theta=\sqrt{2}$. As the pairs separate the mixing angle θ undergoes a transition to the state in which $\chi_1=\psi_1$, with the change occurring fairly rapidly in the region where all the interparticle separations are comparable.

One procedure for solving the Schrödinger equation that follows from Eqs. (5) and (21) is to expand the solution in terms of the eigenstates of V,

$$\Psi = \sum_{\alpha=1}^{2} \phi_\alpha(\mathbf{r}_i)\chi_{\alpha,\mathbf{r}_i} \ , \tag{28}$$

where we have indicated explicitly that the χ_α depend parametrically on the coordinates of the particles. In the present paper we employ the strict Born-Oppenheimer approximation by neglecting the derivatives of χ that are induced by the kinetic-energy operator. According to the discussion in the Introduction, this corresponds to the assumption that only the ground state of QCD needs to be taken into account. When this approximation is made the Schrödinger equation becomes

$$\left[\sum_i \frac{{p_i}^2}{2m_i} + v_1\right]\phi_1 = E_1\phi_1 \ , \tag{29}$$

where v_1 is the lower eigenvalue of $V^{(4)}$.

B. Modification of the $Q^2\bar{Q}^2$ potential at large distances

The bag-model potential derived above is valid provided the bag is approximately spherical in shape. This breaks down whenever the particles begin to cluster into distinct groups and the bag becomes deformed. Although we do not have a solution of the bag-model equations at intermediate separations of the clusters,[7] the physics at large separations appears to be straightforward, at least insofar as the *lower* eigenvalue of the potential energy is concerned.[8] As we now show that the error (at large distances) due to the use of the spherical approximation is quite modest, a simple patching together of the theoretical potentials at small and large separations is more than adequate.

If one quark-antiquark cluster (ij) is well separated from the other (kl), then we expect that the *lower* eigenstate of V corresponds to the situation in which each cluster is in a singlet state with no interaction between them. That is to say, as $R_{ij,kl}$, the separation of the two clusters, gets large, the lower eigenvalue, v_1, should become

$$v_1' \equiv V^{(2)}(r_{ij}) + V^{(2)}(r_{kl}) = -\tfrac{4}{3}\alpha_s\left[\frac{1}{r_{ij}}+\frac{1}{r_{kl}}\right] + (\tfrac{2}{3})^{1/2}k(r_{ij}+r_{kl}) \ , \tag{30}$$

where $V^{(2)}$ is the quark-antiquark potential given in Eq. (19), and

$$\mathbf{R}_{ij,kl} = \tfrac{1}{2}(\mathbf{r}_i+\mathbf{r}_j-\mathbf{r}_k-\mathbf{r}_l) \ . \tag{31}$$

Since the bag must be highly deformed in this region one could not have expected a single spherical bag to yield the correct energy. The surprising thing is that the dipole approximation to a spherical bag yields an energy that is not so far wrong. It can be shown from Eqs. (21) and (27) that the confining term of v_1 is just $kd_1/\sqrt{2}$, and from Eqs. (24) and (25) $d_1 \to 2(r_{ij}^2+r_{kl}^2)^{1/2}/\sqrt{3}$ in this region. Consequently, v_1 does not decouple into a sum of terms referring to the separate $Q\bar{Q}$ systems. While the Coulomb terms in v_1 are correct, as in Eq. (30), in place of the factor $(r_{ij}+r_{kl})$ appearing in the last term of that equation, v_1 acquires a factor $(r_{ij}^2+r_{kl}^2)^{1/2}$. For the parameters used in Fig. 2 this leads to a discrepancy, $v_1'-v_1$, of only 80 MeV.

The other clusterings, such as $(QQ')(\bar{Q}\bar{Q}')$ or $(Q)(Q'\bar{Q}\bar{Q}')$, are expected to be less important because the potential energy becomes large. Furthermore, the dipole approximation to the spherical bag gives the correct *slope* for the confining potential between these clusters. To make an estimate of the importance of the small defect mentioned above concerning the clusters that can separate without being confined, we take the potential energy to be a weighted average of v_1 and v_1' that reduces to $v_1(v_1')$ at small (large) intercluster separation. For each configuration of the four particles, all possible clusterings into two groups are examined and the *largest* of the intercluster coordinates, R, is found. Whenever R corresponds to a confined configuration, the four-body potential energy is simply taken to be

$$v^{(4)}=v_1 \ . \tag{32a}$$

If R corresponds to either (12)(34) or (14)(23), that is, to separating quark-antiquark pairs, then the potential is taken to be

$$v^{(4)}=v_1 e^{-R/\gamma R_1}+v_1'(1-e^{-R/\gamma R_1}) \ , \tag{32b}$$

where R_1 is the *minimum* eigenvalue of the bag radius from Eq. (15),

$$R_1=8^{1/6}\left[\frac{\alpha_s d_1}{k}\right]^{1/3} ,$$

and γ is an adjustable parameter. $v^{(4)}$ is plotted on Fig. 2 for the rectangular geometry shown with $\gamma=1$. There is nothing fundamental about Eq. (32), but by examining the results of the next section as γ is varied we obtain an estimate of the uncertainty in the binding energy. As expected from the discussion above, it is quite small.

III. RESULTS

We want to compare the energy $E^{(4)}$ of the ground state of the Hamiltonian

$$H^{(4)}=\sum_{i=1}^{4}\frac{p_i^2}{2m_i}+v^{(4)} \ , \tag{33}$$

with the sum of the energies of two separated mesons. For equal-mass quarks, the latter energy is $2E^{(2)}$, with $E^{(2)}$ the energy of the ground state of the Hamiltonian

$$H^{(2)}=\sum_{i=1}^{2}\frac{p_i^2}{2m_i}+V^{(2)} \ . \tag{34}$$

$V^{(2)}$ is the quark-antiquark potential obtained from the *very same* bag model, Eq. (19). With $E^{(2)}$ obtained from an exact numerical solution of the Schrödinger equation, a variational calculation of $E^{(4)}$, if it shows the existence of a four-body bound state, is sufficient. All of our calculations of $E^{(4)}$ are variational, but we will present one comparison with an exact numerical evaluation of the binding energy of the four-body system that shows the variational calculations to be very accurate.

A. $Q\bar{Q}$

The potential energy from the bag model depends upon two parameters, the bag constant, B, and the coupling constant, α_s, and the latter runs with the size of the system. In addition, there are the masses of the quarks, m_i. In previous calculations of $Q\bar{Q}$ and Q^3 spectra,[3,4] the bag constant was kept fixed at the value obtained in the original fit[9] to the light hadron spectrum, namely, $B^{1/4}=0.145$ GeV. The present calculations began with this same value, but we also vary B to see if the binding energy of the four-body system is sensitive to it. For systems composed of c or b quarks, which are the main ones in the present study, our procedure for choosing the constants is as follows. For each assumed value of B, α_s and m_i are adjusted so that the first *two* S states of a given $Q\bar{Q}$ system have their energies at the experimental $^3S_1$ values.[10] The correlations amongst these parameters are shown in Fig. 4. Note that α_s is required to be smaller in the heavier system if B is kept fixed.

Some idea of the reliability of the potential energy $V^{(2)}$ can be obtained by calculating the energies of the higher S states in the $c\bar{c}$ and $b\bar{b}$ systems and comparing with the experimental values. The largest discrepancy occurs for $\Upsilon(10570)$ where the calculated energy is 109 MeV too large. The fact that the potential energy is too large at larger distances is probably not serious because the dimensions of the $Q^2\bar{Q}^2$ states that we discuss below are much closer to those of the $1S$ state than the $4S$ state of the $Q\bar{Q}$ system.

We have calculated the ground-state energies of the $Q\bar{Q}$ systems variationally as well as by numerical integration of the Schrödinger equation. Simple exponential wave functions, $\psi(r)=N\exp(-\alpha r)$, yield extremely accurate energies, within 1 MeV of the correct values. Gaussian trial wave functions are much poorer.

We have also studied the $s\bar{s}$ and $t\bar{t}$ systems using the very same potential energy. As mentioned in the Introduction, the s quarks are somewhat relativistic, and in addition, spin effects are much more important for them. We have, nevertheless, simply repeated our calculations for this quark. The bag constant is kept at the value $B^{1/4}=0.145$ GeV, and the coupling constant held at its value in the $c\bar{c}$ system, $\alpha_s=0.747$; with these choices, $m_s=0.319$ GeV puts the $\phi(1020)$ at its correct energy.[10]

The parameters are even more arbitrary for the t quark, and we have chosen $B^{1/4}=0.145$ GeV, $\alpha_s=0.2$, and $m_t=40$ GeV. This value for α_s is roughly what is expected for a system of this size based on the logarithmic varia-

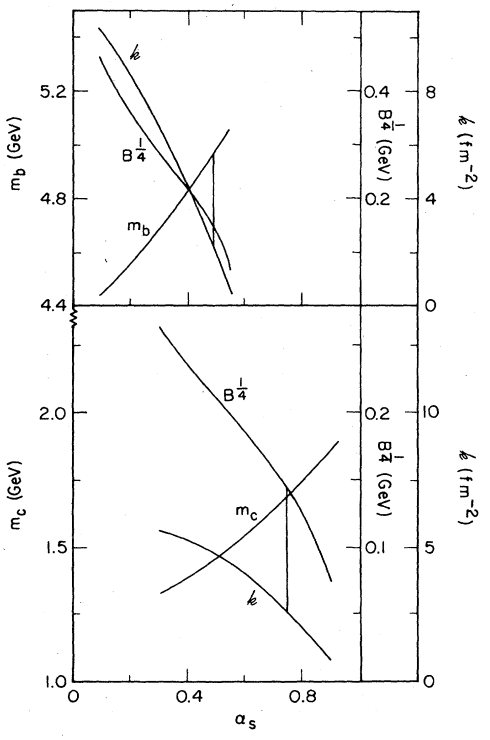

FIG. 4. Relationship of coupling constant, bag constant, and quark mass needed to fit the first two s states of quarkonium, using the potential energy from Eq. (19). The lower portion refers to $c\bar{c}$ and the upper portion to $b\bar{b}$. The vertical lines correspond to $B^{1/4}=0.145$ GeV. k is the slope parameter appearing in the confining term of the potential energy.

tion with distance. With these values, the $1s$ state of $t\bar{t}$ has an energy that is 0.699 GeV below the sum of the quark masses.

Table I summarizes the results for the $Q\bar{Q}$ systems with the fixed value $B^{1/4}=0.145$ GeV. Comparison of $s\bar{s}$ and $c\bar{c}$ shows the effect of varying the quark mass with all other parameters fixed. The small dependence of $M(2S)-M(1S)$ on quark mass has been noted before. It appears to arise from two causes: the changing importance of Coulomb and confining terms, and the variation of coupling constant with the size of the system.

It is seen from Table I that the masses of the $1s$ states are *less* than the sum of the quark masses [$E(1s)<0$] for all mesons heavier than $s\bar{s}$. This is a confirmation of the Born-Oppenheimer approximation described in the Introduction, provided the quarks are sufficiently heavy, for the following reason. In the fixed-cavity approximation[9] to the MIT bag model, where the quarks are put into the *lowest* mode, the sum of quark energies and volume energy is *greater* than the sum of the masses. By localizing the quarks in space the small increase in kinetic energy is more than compensated for by the saving in color Coulomb potential energy.[5] The s quark is a borderline case for which neither approximation is very accurate.[5]

B. $Q^2\bar{Q}^2$

Since we deal only with the lower eigenvalue of the potential matrix in the present paper, Eq. (29), only one component of the wave function, ϕ_1, need be specified. For the ground state of the four-body system we choose a cluster trial wave function consisting of a product of internal wave functions for the mesons and a wave function in the intercluster coordinate, the latter also chosen to be an exponential,

$$\phi(\mathbf{r}_i)=\exp[-\alpha(r_{12}+r_{34})-\beta R_{12,34}] \tag{35}$$

with α and β variational parameters. For systems with identical quarks such as $c^2\bar{c}^2$, the Pauli principle must be imposed. As spin interactions are being neglected, the spin wave function can be chosen as required to make the overall space-color-spin wave function antisymmetric under the interchange of any pair of identical particles. This means that we are at liberty to choose the spatial symmetry that gives the lowest eigenvalue of Eq. (29). (The color coordinates occur only in the χ_α, which are themselves symmetric under the interchange of identical particles.) Expecting this to occur for the symmetric case, we take

$$\psi(\mathbf{r}_i)=N(1+P_{13})\phi_1(\mathbf{r}_i)\ , \tag{36}$$

and verify that the antisymmetric choice (or the unsymmetrized one) gives a higher variational energy. The spin wave function going with Eq. (36) has both pairs of identical particles in $S=0$ states.

The six-dimensional integrations required to evaluate

$$E^{(4)}=\langle\psi\,|\,H^{(4)}\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle$$

are performed by first making a change of variable for each of the three coordinates having infinite range, r_{12},

TABLE I. Results for the $Q\bar{Q}$ mesons with the fixed value $B^{1/4}=0.145$ GeV. All energies and masses are in GeV. The first column specifies the meson by quark flavor; α_s is the strong coupling constant; m_Q is the quark mass; $E(1S)$ is the eigenvalue of the $1S$ state; $M(1S)=2m_Q+E(1S)$ is the mass of the $1S$ state; $M(2S)$ is the mass of the $2S$ state; and the final column is the variational parameter α in the exponential trial wave function for the ground state. The numbers for the t quark are merely illustrative.

Meson	α_s	m_Q	$E(1S)$	$M(1S)$	$M(2S)-M(1S)$	α (fm^{-1})
ϕ $(s\bar{s})$	0.747	0.319	0.382	1.020	0.598	1.5
ψ $(c\bar{c})$	0.747	1.685	−0.275	3.095	0.590	4.9
Υ $(b\bar{b})$	0.495	4.970	−0.479	9.461	0.551	8.8
$(t\bar{t})$	0.2	40	−0.699	79.301	0.569	29

r_{34}, and $R_{12,34}$, to variables defined on the unit interval, via

$$r = c \ln \frac{1+x}{1-x} ,$$

and then using a Gauss integration procedure. Typical values are $c=3$, and eight integration points per variable, and the numerical results are stable under changes about these values.

For all four systems, $Q^2\bar{Q}^2$, with $Q=s$, c, b, and t, and for all the parameter values considered in Sec. III A above, we find that the dimeson is stable (bound) against breakup into two mesons. The results are shown in Fig. 5 where it is seen that the binding energies are in the range 0.16–0.22 GeV if the unmodified spherical bag potential is used [$\gamma=\infty$ in Eq. (32)]. The binding energies are only slightly smaller from the modified potential with $\gamma=1$, with the largest difference occurring, as expected, for the lightest quark. These energies are also shown in Fig. 5. As the quark mass increases and the size of the system decreases, the bag surface moves *relatively* farther away from the quarks, and the spherical approximation becomes better. R/R_1 in Eq. (32b) decreases and the difference between the modified and unmodified potentials becomes smaller. Assuming the validity of the approximation that only the smaller eigenvalue of V need be considered, the difference between the larger and smaller values in Fig. 5 is a measure of the uncertainty in the dimeson binding energy.

For the unmodified potential, the results are also presented in Table II, including the values of the variational parameters α and β. Note that the values of α are the same as in Table I, or nearly so, suggesting that there is very little distortion of the individual mesons in the dimeson, in spite of the fact that $\beta \sim \frac{2}{3}\alpha$. The variational wave function, on the other hand, is probably not nearly so accurate as the variational energy.

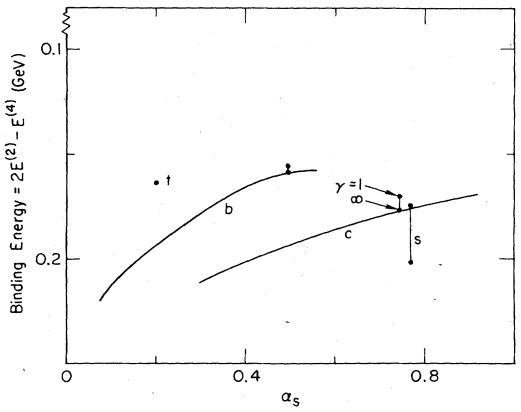

FIG. 5. Binding energy of the dimesons $Q^2\bar{Q}^2$ with respect to the sum of the energies of two separated mesons. All results are based on the approximation of using only the lower eigenvalue of the potential energy matrix. For each quark flavor, the more deeply bound result arises from the spherical bag potential without any modification, i.e., $\gamma=\infty$ in Eq. (32); the less deeply bound value results from setting the cutoff parameter $\gamma=1$. For the c and b quarks, the binding energies are shown as a function of the coupling constant α_s, with the bag constant and quark mass adjusted as in Fig. 4. The vertical lines correspond to $B^{1/4}=0.145$ GeV.

TABLE II. Results for some dimesons $Q^2\bar{Q}^2$, using the unmodified spherical bag potential v_1. For each flavor, the parameters are the same as in Table I. Energies are in GeV. $E^{(4)}$ is the ground-state energy of the four-body system; $E^{(2)}$ is the ground-state energy of the two-body system [the same as $E(1S)$ in Table I.]; α and β are the variational parameters in the four-body wave function, Eqs. (35) and (36).

Flavor	$E^{(4)}$	$2E^{(2)}-E^{(4)}$	α (fm^{-1})	β (fm^{-1})
s	+0.563	0.201	1.5	1.4
c	−0.729	0.178	4.9	3.3
b	−1.119	0.161	9.1	6.2
t	−1.561	0.164	29	18

The question arises whether a better wave function may not yield significantly more binding energy, and we have tried a couple of other forms for the trial function but with no appreciable gain in energy. In addition, Schmidt[11] has obtained an exact numerical solution for the binding energy that shows the variational results are very accurate. Using a Monte Carlo technique on the $c^2\bar{c}^2$ system, both variational (with the parameters in Tables I and II) and Green's function (exact), he finds that the exact binding energy is only 7 ± 2 MeV greater than the variational energy. The uncertainty is purely statistical.

IV. DISCUSSION

In the Introduction a calculation was described that we wish we knew the answer to, namely, what are the eigenstates and eigenvalues of QCD in the presence of a collection of static, localized quarks and antiquarks. This would provide the basis for a coupled-channel Schrödinger equation treatment of slowly moving quarks. One question of immediate interest concerns the *number* of low-lying states that are important for the quark dynamics. From lattice gauge theory for two quarks and two antiquarks, one might have guessed that the answer is three, since there are three topologically distinct kinds of gauge-invariant connections that can be formed. (This includes contraction of three links with the antisymmetric tensor ϵ_{abc}.) The bag model as we have used it, however, or indeed any potential model would say that there are two states, since that is the number of independent color singlets that can be constructed from the color generators of (just) the four quarks.

The derivation of the potential energy from the bag model according to Eq. (4), $\partial W/\partial R_\alpha=0$, is the standard procedure for a system with constraints ($P_\alpha=0$), as described, for example, by Dirac.[12] It represents the physical statement that since there is no inertia associated with the surface, it adjusts *instantaneously* to the values of the other dynamical variables.[13] This procedure was used for light quarks by Rebbi and De Grand,[14] and for heavy quarks in Refs. 2–6 and 15. The spectra obtained in this way are different, in general, from those obtained from $\partial\langle H\rangle/\partial R_\alpha=0$, where $\langle H\rangle$ is the energy for a fixed sur-

face and the surface is varied as the *last* step rather than the first.[9,16] We now compare our potential energy with some others that have been used.

Ader, Richard, and Taxil[17] also start with a spherical approximation to the bag, using the full Green's function, but then follow a procedure that is different from both of those just mentioned. At one point they find the minimum eigenvalue of $W(F_i,\mathbf{r}_i,R)$ with R treated as a number, and then find its minimum with respect to R. Although the eigenvalues and eigenvectors obtained by this procedure are different, in general, from ours (even if the dipole approximation is made in both treatments), it turns out, at least in the 2×2 matrix case, that they are only slightly different.[18] The fact, therefore, that $c^2\bar{c}^2$ was found in Ref. 17 to be bound by only 35 MeV is either the result of their having used the full spherical Green's function rather than the dipole approximation, or else of an inadequate wave function.

With regard to the latter point, we find that a trial wave function that is symmetrized in space and color, as in Eq. (36), produces considerably more binding energy than an unsymmetrized wave function. For $c^2\bar{c}^2$, for example, an unsymmetrized wave function gives a binding energy that is only $\sim\frac{1}{2}$ that shown in Table II. The explanation for the effectiveness of symmetrization for equal-mass particles is just that the potential energy is symmetric under interchange of space and color. For identical particles the Pauli principle merely fixes the spin wave function for the lowest-energy state.

In spite of theoretical arguments against using two-body potentials to describe the confining part of the interaction,[5] and the additional defect that they lead to van der Waals forces between separated hadrons, they have been used for multiquark systems.[19] Following the standard procedure, one writes

$$V^{2B}=-\tfrac{1}{2}\sum_{i\neq j}F_i\cdot F_j w^{2B}(r_{ij}) \tag{37}$$

and chooses w^{2B} to correspond to the $Q\bar{Q}$ system where $F_i\cdot F_j=-\frac{4}{3}$,

$$w^{2B}(r)=\tfrac{3}{4}v^{Q\bar{Q}}(r)\ . \tag{38}$$

By examining v_1^{2B}, the lower eigenvalue of V^{2B}, it is straightforward to show that the four-body system is at least as attractive as two $Q\bar{Q}$ pairs that are not interacting with each other. By this we mean that for any configuration of the four particles, and for any choice of the function $v^{Q\bar{Q}}$,

$$v_1^{2B}\leq\min(v^{Q\bar{Q}}(r_{12})+v^{Q\bar{Q}}(r_{34}),v^{Q\bar{Q}}(r_{14})+v^{Q\bar{Q}}(r_{23}))\ . \tag{39}$$

The proof follows by finding the maximum of $e_1(y_{ij})$ [the lower eigenvalue from Eq. (24)] with respect to y_{13}, which occurs at $y_{13}=\max(y_{12},y_{14})$ and yields the inequality

$$e_1(y_{ij})\leq\tfrac{4}{3}\min(y_{12},y_{14})\ . \tag{40}$$

Setting $y_{ij}=w^{2B}(r_{ij})+w^{2B}(r_{kl})$ with i, j, k, and l, all different, the left side of Eq. (40) is just v_1^{2B}, thereby proving Eq. (39).

We now show that the additional attraction is sufficient to produce binding, by including the kinetic-energy terms and solving the Schrödinger equation (still using the one-channel approximation). In order to make the comparison of the two-body potential approach with the bag model as meaningful as possible, we set $v^{Q\bar{Q}}(r)$ in Eq. (38) equal to $V^{(2)}(r)$ from Eq. (19), since that is the $Q\bar{Q}$ potential coming from the bag model. With this choice the matrix elements and eigenvalues of V^{2B} are qualitatively very similar to those of $V^{(4)}$, with numerical differences $\lesssim0.1$ GeV. This similarity is partly due to the importance of the two-body Coulomb terms. In addition, the lower eigenvalue v_1^{2B} is guaranteed to go to the correct limit as the $Q\bar{Q}$ pairs separate. This can be seen from Eq. (23) by choosing the clusters to be (12)(34), since the matrix element V_{11}^{2B} only involves interaction *within* the two clusters, and V_{18}^{2B} vanishes at large separation. Taking v_1^{2B} to be the single-channel potential energy, and the same choices for α_s and B as in Tables I and II, the $c^2\bar{c}^2$ system is bound by 0.115 GeV, which is *considerably less* than the value obtained from the bag-model potential.

We take note finally of a phenomenological many-body potential that was inspired by the string model. van der Waals forces are avoided by the simple expedient of making the off-diagonal matrix element vanish discontinuously as one $Q\bar{Q}$ pair separates from the other.[20] As no Coulomb potential was included it is not meaningful to compare their numerical results with those of the present paper.

V. SUMMARY

We have used the potential energy coming from the MIT bag model to describe a system of two heavy quarks and two heavy antiquarks; for the reasons presented in Sec. IV we believe this is the best static potential that is available. The separations of the particles in the bound state turn out to be sufficiently small that a spherical approximation to the bag shape is adequate. Even at large distances the dipole approximation to the homogeneous part of the Green's function gives a value for the lower eigenvalue of the potential matrix that is fairly accurate.

In this paper, we have taken the lower eigenvalue of the 2×2 potential matrix (in color space) and have solved the four-body Schrödinger equation variationally. Using a wide range of bag-model parameters and quark masses chosen to fit the $c\bar{c}$ and $b\bar{b}$ spectra, we find that the dimesons $c^2\bar{c}^2$ and $b^2\bar{b}^2$ are bound. The binding energy is in the range 0.16—0.22 GeV and depends on the bag-model parameters. Similar results are obtained for $t^2\bar{t}^2$ and $s^2\bar{s}^2$, although the approximations are less reliable for lighter quarks.

We have made a rough estimate of the hyperfine interaction using our $c^2\bar{c}^2$ wave function, and find that it lowers the energy of the ground state by ~0.02 GeV. Since the ground state of $c\bar{c}$ is lowered by ~0.09 GeV, the binding-energy estimate made in this paper must be lowered by $2(0.09)-0.02=0.16$ GeV. This could unbind the system, but the answer must await the solution of the

two-channel problem.

The range of the variational wave function in the $Q\bar{Q}$ separation is very similar to that in a single meson; and is approximately 50% larger in the separation of one $Q\bar{Q}$ pair from the other. The color wave function in the single-channel problem is already very interesting since χ_1 varies considerably with the positions of the particles; some of this variation is shown in Figs. 2 and 3 where the mixing angle from Eq. (26) is presented for two geometries. In the coupled-channel problem, the color degree of freedom becomes truly dynamical, no longer factorizing in the wave function.

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