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Study of light baryons in the three-quark-cluster model: Exact calculations

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Within the nonrelativistic model, all the baryons built with three quarks of flavors, u, d, and s ($N,\Delta,\Lambda,\Sigma,\Xi,\Omega$) are studied with the Bhaduri-Cohler-Nogami potential. It is shown that the free-parameter simplest model is able to reproduce most of the experimental properties and allows the extraction of the rare effects which need a more elaborate model. The validity of the description in terms of a harmonic-oscillator basis is also tested.

I. INTRODUCTION

At the beginning, the quark hypothesis was only a convenient mathematical tool for explaining classification properties of elementary particles. Now, after the modern developments of gauge theories and the advent of quantum chromodynamics, the quark structure of elementary particles is firmly established. Moreover, the quark theory is no longer restricted to the domain of elementary-particle physics, but reaches other fields, such as nuclear physics. The simplest systems to be studied are the mesons since they involve only one quark and one antiquark. More complicated are the baryons built with three quarks. Still more complicated is the description of hadron-hadron interaction, and finally the properties of atomic nuclei. Several theories are devoted to such studies. The most fundamental approach is probably the lattice gauge theory,[1] but there is very much progress to be made in that direction and complicated systems are widely out of scope presently. Less ambitious and more phenomenological are the descriptions in terms of bags or of nonrelativistic quark-quark potentials. The various bag approaches[2] (MIT bag, little bag, cloudy bag, soliton bag, etc.) are more satisfying from the relativistic point of view but suffer from several drawbacks, such as a bad treatment of the center-of-mass motion. On the other hand, the nonrelativistic potential model[3] is more rigorous for this aspect but obviously lacks relativistic foundations. Curiously, from all studies already done within this model, it seems to explain a lot of situations even in cases where relativistic effects are expected to be important. Some explanations for such a success have been claimed[4] but, up to now, the understanding of the problem is far from clear. Which theory to choose is essentially a matter of taste. As nuclear physicists, we were naturally led to work with the nonrelativistic model. Within that framework, the Schrödinger equation for mesons is reduced to the resolution of a differential equation which is exact in principle. The study of baryons, which is typically a three-body problem, is more involved and, to our knowledge, was tackled only through variational prescriptions or diagonalization within a restricted Hilbert space. For the nucleon-nucleon interaction, one cannot avoid an approximate treatment such as the Born-Oppenheimer approximation,[5] the generator-coordinate method,[6] or the resonating-group method.[7]

In this paper, we present a systematic study, in the framework of the nonrelativistic model, of all the baryons built with light quarks (u, d, and s valence quarks) i.e., the six systems N, Δ, Λ, Σ, Ξ, and Ω. All previous calculations, concerning both ground and excited states, were performed within some approximation schemes; when comparing the results to experimental data it was thus difficult to interpret the discrepancies as the deficiency of the used potentials or as a consequence of the truncation in the Hilbert space. Here, we solve *exactly* the three-body problem in terms of the Faddeev formalism. A first aim of this paper is thus to test the validity of previous approaches, such as the pioneering works of Isgur and Karl,[8] which relied on the expansion of the wave functions in terms of an oscillator basis; the detailed comparison between the exact and the approximate treatments will provide a definite answer to the above-mentioned ambiguity. A second aim is to know which kind of physical information could be or could not be explained by the very simple quark-quark potential, which proved to be very successful in the description of many meson properties.

The paper is organized as follows. In Sec. II, the frame of the model is described precisely. The next two sections are devoted to the resolution of the three-body problem, either exactly or by diagonalization in a harmonic-oscillator basis. In Sec. V, the results of the above two methods are compared carefully, while in Sec. VI are presented and discussed the extensive results obtained with the Faddeev formalism for the six considered baryon systems. Finally the conclusions are drawn in the last section.

II. DESCRIPTION OF THE SYSTEMS

In this paper, we consider a cluster of three valence quarks of flavor u, d, and s. The u and d quarks are supposed to be the two components of an isospin doublet. Besides its isospin degree of freedom, each ith particle is characterized by its constituent mass m_i ($m_u=m_d=m$), its color c_i, its intrinsic spin $s_i=\frac{1}{2}$, and its position $\mathbf{r}_i$ in ordinary space. Using the isospin degree of freedom, the u and d quarks are treated as identical particles and consequently each of the six systems considered here

$(N,\Delta,\Lambda,\Sigma,\Xi,\Omega)$ contains at least two identical particles. The isospin formalism, although not necessary, is very convenient especially for the description of systems with three identical particles (nonstrange baryons N,Δ). The total wave function must be antisymmetric in the exchange of any two identical particles. Concerning the color degree of freedom, there is only one possibility to form a color singlet with three quarks in three color states, namely, a Slater determinant, which is therefore totally antisymmetric. Since the color wave function is unique for all considered systems, we will ignore it once all for all and require for the rest of the wave function—which will be called simply the wave function—to be symmetric in the exchange of identical particles. Since each quark has spin $\frac{1}{2}$, one can form three spin wave functions $\chi_{\sigma S}=[\frac{1}{2}(\frac{1}{2}\frac{1}{2})_\sigma]_S$ with different symmetries: $\chi_0=\chi_{0\,1/2}$ is of mixed symmetry, antisymmetric in $2\leftrightarrow3$; $\chi_1=\chi_{1\,1/2}$ is of mixed symmetry, symmetric in $2\leftrightarrow3$; and $\chi_2=\chi_{1\,3/2}$ is totally symmetric. When all the particles have isospin $\frac{1}{2}$, one can adopt the same coupling for the isospin wave function and note

$$\eta_{tT}=[\tfrac{1}{2}(\tfrac{1}{2}\tfrac{1}{2})_t]_T \ .$$

For nonstrange baryons it will be very useful to define a wave function in the isospin-spin space with a definite symmetry. The list of the various possibilities is as follows: Totally symmetric,

$$|S\rangle=\frac{1}{\sqrt{2}}(\chi_0\eta_0+\chi_1\eta_1), \quad |s\rangle=\chi_2\eta_2 \ ;$$

mixed symmetry, symmetric in $2\leftrightarrow3$,

$$|MS\rangle=\frac{1}{\sqrt{2}}(\chi_0\eta_0-\chi_1\eta_1) \ ,$$
$$|ms\rangle=\chi_2\eta_1, \quad |ms'\rangle=\chi_1\eta_2 \ ; \qquad (2.1)$$

mixed symmetry, antisymmetric in $2\leftrightarrow3$,

$$|MA\rangle=\frac{1}{\sqrt{2}}(\chi_0\eta_1+\chi_1\eta_0) \ ,$$
$$|ma\rangle=\chi_2\eta_0, \quad |ma'\rangle=\chi_0\eta_2 \ ;$$

totally antisymmetric,

$$|A\rangle=\frac{1}{\sqrt{2}}(\chi_1\eta_0-\chi_0\eta_1) \ .$$

Concerning the space wave function, it is trivial to factorize a center-of-mass function whose role is to ensure the translational invariance (contrary to bag models) and to give a center-of-mass energy. The internal coordinates are chosen as the usual Jacobi coordinates

$$\mathbf{x}_i=\left[\frac{2m_jm_k}{m_j+m_k}\right]^{1/2}(\mathbf{r}_j-\mathbf{r}_k) \ ,$$
$$\mathbf{y}_i=\left[\frac{2m_i(m_j+m_k)}{m_i+m_j+m_k}\right]^{1/2}\left[\mathbf{r}_i-\frac{m_j\mathbf{r}_j+m_k\mathbf{r}_k}{m_j+m_k}\right] \ . \qquad (2.2)$$

Each couple $(\mathbf{x}_i,\mathbf{y}_i)$ is related to another one $(\mathbf{x}_j,\mathbf{y}_j)$ by an orthogonal transformation. Considering, for example, the $(\mathbf{x}_1,\mathbf{y}_1)$ channel, we put the pair (2,3) in a relative angular momentum l and the last particle 1 in an angular momentum $\boldsymbol{\lambda}$ relative to the center of mass of the pair. Finally the total orbital angular momentum $\mathbf{L}$ is obtained by summing l and $\boldsymbol{\lambda}$, and the total spin $\mathbf{J}$ by coupling $\mathbf{L}$ and $\mathbf{S}$. It is easy to show that the parity of the state is $(-1)^{l+\lambda}$ (H being invariant under parity, $l+\lambda$ is either odd or even); moreover, if the particles 2 and 3 are identical the symmetry of the wave function imposes the constraint $\sigma+t+l$ even.

After this geometrical description of the system, let us come to its dynamical description. A number of quark-quark potentials were proposed in the past. The central part contains usually a Coulombic term $(1/r)$ coming from one-gluon-exchange contributions[9] and a phenomenological confining term which is either quadratic[8] (r^2), linear[10] (r), logarithmic[11] $[\ln(r/r_0)]$, or power-like[12] (r^β). In addition, there is also a hyperfine (or spin-spin) term. Contrary to atomic or nuclear potential, the quark-quark hyperfine term is very strong and, in fact, it is responsible for many important features in hadron properties, such as the π-ρ and N-Δ splittings and the hard core of the nucleon-nucleon interaction.[13] In most of the potentials, this term is derived from one-gluon exchange through the Fermi-Breit approximation of the Bethe-Salpeter equation. This approximation gives a contact $\delta(\mathbf{r})$ term.[9] This Fermi-Breit expression was widely used in previous perturbative treatments, but it makes no sense in an exact actual calculation—such as the one performed here—since it leads to a collapse. In fact, in situations where the contact term is attractive and where there is no centrifugal repulsion between two particles ($l=0$), the potential energy can become more and more negative when the particles come closer and closer (with a dependence in $-1/r^3$); the kinetic energy becomes more and more positive (with a dependence in $1/r^2$) but is not able to compensate this effect and the total binding energy tends to negative infinity. Thus, in actual complete calculations, it is absolutely necessary to use potentials with finite range. On the other hand, there exist also a spin-orbit and a tensor force;[9] there are not essential for the gross features of the spectroscopy but they have to be taken into account for a finer analysis.[8] Here, our philosophy is to retain the simplest potential as long as it gives results compatible with other approximations.

For all these reasons, we believe that, among others, the potential proposed by Bhaduri, Cohler, and Nogami[14] is a good one. In their original paper, Bhaduri, Cohler, and Nogami give the form of the quark-antiquark potential. However, as long as the color scalar is of the form $\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j$ and we deal with the color singlet, it is trivial to show that the quark-quark potential is half the quark-antiquark potential. Thus the quark-quark potential used in this paper is defined by

$$V_{ij}(r_{ij})=\frac{1}{2}\left[-\frac{\kappa}{r_{ij}}+\frac{r_{ij}}{a^2}-D+\frac{\hbar^2\kappa_\sigma}{m_im_jC^2}\frac{\exp(-r_{ij}/r_0)}{r_0{}^2r_{ij}}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right] . \qquad (2.3)$$

The mass of the u and d quark $m=337$ MeV is fitted on the magnetic moment of the proton and the other parame-

ters $\kappa_\sigma=\kappa=102.67$ MeV fm, $a=0.0326$ $(\text{MeV}^{-1}\,\text{fm})^{1/2}$, $D=913.5$ MeV, $r_0=0.45\overline{45}$ fm are fitted on the charmonium spectra and on the π-ρ splitting; last, the strange-quark mass $m_s=600$ MeV is deduced from the energy value of the $\phi(1020)$. It is remarkable that this very simple potential, obtained essentially from charmonium, is still good for b-quarkonium and even for light mesons (including the pion). The ground states of the baryons, calculated in the work of Bhaduri, Cohler, and Nogami with a variational method suggested by Fesbach and Rubinow, are also nicely reproduced. It is our aim to test this simple potential, with the *same parameters* (we have no free parameters), on the excited states of light baryons. All our calculations were performed with potential (2.3) without any change. Since this potential does not include spin-orbit and tensor force, L and S are good quantum numbers and levels occur in (L,S) multiplets. It is also important to stress that there is no coupling with continuum channels (such as those induced by the pion field or the quark-antiquark sea) and hence our eigenstates are bound states with definite energy which do not exhibit a resonance feature.

III. EXACT SOLUTION OF THE THREE-QUARK PROBLEM

Among the various approaches to an exact treatment of the three-body problem, the Faddeev-Merkuriev equations are numerically the most tractable, at least for local currently used potentials. In this method, the wave function ψ is written as the superposition of three Faddeev components ϕ_i

$$\psi=\phi_1+\phi_2+\phi_3 \ . \tag{3.1}$$

Each one obeys a coupled partial derivative equation which reads

$$(E-T-V_i)\phi_i=V_i(\phi_j+\phi_k) \ , \tag{3.2}$$

where E is the total energy of the system, T the total-kinetic-energy operator, and V_i is the potential-energy operator between particles j and k. Since V_i depends on the $\mathbf{x}_i$ Jacobi distance only, it is natural to express each of the components ϕ_i with the corresponding $(\mathbf{x}_i,\mathbf{y}_i)$ coordinates. By summing the three Faddeev equations (3.2) for $i=1,2,3$, one obtains the Schrödinger equation for the wave function ψ, as defined by (3.1).

Each Faddeev component is expanded in terms of states labeled by the quantum numbers λ,l,σ,t defined in the previous section. One can write the Faddeev equation (3.2) as

$$(E-T)\phi_i=V_i\psi \ . \tag{3.3}$$

The source term $V_i\psi$ has roughly the spatial expansion R_0 of the baryon. Its contribution to the ϕ_i amplitude is negligible, due to the centrifugal barrier, as soon as angular momentum l is greater than $\sqrt{2mE}R_0/\hbar$ (in practice for $l>2$).

Simplifications arise due to identity of at least two quarks. Let us consider first the nonstrange (N and Δ) system or the Ω system. The isospin formalism makes the three particles indistinguishable: ϕ_2 and ϕ_3 are deduced from ϕ_1 by cyclic permutations and we are left with only one relevant Faddeev equation. On the other hand, for the other three systems (Λ, Σ, and Ξ), ϕ_3 is deduced from ϕ_2 by transposition of particles 2 and 3, and hence there remain two coupled Faddeev equations. More details about the formalism or the numerical solution can be found in Ref. 15. Usually, the energy reference corresponds to the case where all particles are infinitely separated. For confining potentials this makes no sense and there exist Faddeev components for each energy reference, with different asymptotic behaviors. One can take advantage of this freedom by choosing an energy reference which makes the Faddeev component rapidly decreasing. The stability of the eigenenergy versus the energy reference has been checked numerically.

IV. HARMONIC-OSCILLATOR BASIS

Unlike the Faddeev equations, which solve the three-body problem directly in the configuration space, one may be tempted to develop the wave function on basis states. An harmonic-oscillator (HO) basis is proved to be a very convenient one since most of the calculations can be done analytically. Moreover, contrary to the nuclear three-body problem where one needs several hundred basis states, the expansion of the wave function in terms of the HO basis is expected to be rapidly convergent in case of a confining potential. In the past, most baryon calculations were performed within this framework. We define the HO wave functions $\phi_{nlm}(\mathbf{r})$ with the usual phase convention (see, for instance, Ref. 16). The size parameter $b=(\hbar/m\omega)^{1/2}$ can be chosen arbitrarily *a priori*, but the numerical determination of b is a crucial problem and is discussed in detail in Sec. V. For a three-body problem, since the center-of-mass function is factorized, we are left with two internal Jacobi coordinates $\mathbf{x}_1,\mathbf{y}_1$ [or other sets $(\mathbf{x}_2,\mathbf{y}_2)$ or $(\mathbf{x}_3,\mathbf{y}_3)$] and it is natural to introduce the basis states as the coupling of $\phi_{nl}(\mathbf{x}_1)$ and $\phi_{\nu\lambda}(\mathbf{y}_1)$ to the total orbital angular momentum L. Thus we define the space wave function

$$|\,nl\nu\lambda;L\,\rangle=[\phi_{nl}(\mathbf{x}_1)\phi_{\nu\lambda}(\mathbf{y}_1)]_L \ . \tag{4.1}$$

Now the problem of permutations of particles (especially when the three particles are identical) is of crucial importance. Traditionally it is solved by group-theoretical arguments. We prefer to have a more physical and palpable representation. Since a cyclic permutation transforms $(\mathbf{x}_1,\mathbf{y}_1)$ into $(\mathbf{x}_2,\mathbf{y}_2)$ by an orthogonal operation, the problem is solved for a long time by atomic and nuclear spectroscopists. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two coordinates and $\mathbf{r},\mathbf{R}$ two rotated coordinates,

$$\begin{aligned}&\mathbf{r}=\cos\frac{\beta}{2}\mathbf{r}_1-\sin\frac{\beta}{2}\mathbf{r}_2, \quad \mathbf{r}_1=\cos\frac{\beta}{2}\mathbf{r}+\sin\frac{\beta}{2}\mathbf{R} \ ,\\ &\mathbf{R}=\sin\frac{\beta}{2}\mathbf{r}_1+\cos\frac{\beta}{2}\mathbf{r}_2, \quad \mathbf{r}_2=-\sin\frac{\beta}{2}\mathbf{r}+\cos\frac{\beta}{2}\mathbf{R} \ .\end{aligned} \tag{4.2}$$

The HO basis functions in the two sets of coordinates are related by an orthonormal matrix

$$[\phi_{n_1l_1}(\mathbf{r}_1)\phi_{n_2l_2}(\mathbf{r}_2)]_{\lambda\mu} = \sum_{nlNL} \langle nlNL;\lambda \mid n_1l_1n_2l_2;\lambda\rangle_\beta [\phi_{nl}(\mathbf{r})\phi_{NL}(\mathbf{R})]_{\lambda\mu} . \tag{4.3}$$

The $\langle nlNL;\lambda \mid n_1l_1n_2l_2;\lambda\rangle_\beta$ are called the Brody-Moshinsky (BM) coefficients with angle β (or Smirnov coefficients). It is important to note that the summation in (4.3) is restricted to a finite number of terms due to conservation of energy and angular momentum. The conventions in (4.2) and (4.3) are those of the original paper by Moshinsky.[17] The BM coefficients can be calculated by recurrence relations. For identical particles, the cyclic permutations generate BM coefficients with angles $\beta=\pm 2\pi/3$. It is then possible to construct basis wave functions completely symmetric in the exchange of particles in terms of HO functions defined with the set $(\mathbf{x}_1,\mathbf{y}_1)$ only:

$$\Phi_i(1,2,3) = \sum_{\sigma,t,n,l,\nu,\lambda} d^i_{\sigma tnl\nu\lambda}\eta_{tT}\chi_{\sigma S}[\phi_{nl}(\mathbf{x}_1)\phi_{\nu\lambda}(\mathbf{y}_1)]_L \tag{4.4}$$

with $\mathscr{P}\Phi_i(1,2,3)=\Phi_i(1,2,3)$ where $\mathscr{P}$ is any (1,2,3) permutation. The $d^i_{\sigma tnl\nu\lambda}$ numbers are universal geometrical coefficients derived with the help of BM coefficients. The correct symmetrized functions $\Phi_i(1,2,3)$ are given in the Appendix up to four-quanta excitation energies. In our applications, we calculated analytically all the Φ_i up to four quanta and diagonalized the Hamiltonian in the space spanned by these states. The resulting wave functions are given directly in terms of HO functions. The wave functions calculated with the Bhaduri-Cohler-Nogami potential are also given in the Appendix.

V. COMPARISON BETWEEN HO AND FADDEEV CALCULATIONS

We consider in this section the nucleon, which is the system with three identical particles and $T=\frac{1}{2}$. It contains much physical information and allows an interesting comparison between the exact Faddeev treatment and the diagonalization in a spaced spanned by HO functions. The first problems which arise in the approximate model are: how fast is the convergence with the number of basis states and which size parameter b to choose for actual calculations? In fact, both problems are related since at the limit of an infinite basis, the results are independent of b. However, once the basis is truncated, they acquire special numerical importance. One possible way of determining b is the variational constraint

$$\delta\langle\Psi_0(b) \mid H \mid \Psi_0(b)\rangle = 0 , \tag{5.1}$$

where $\mid\Psi_0(b)\rangle$ is the ground state in the truncated Hilbert space; this is the procedure usually employed. If N stands for the maximum number of quanta in the HO basis, the value of b_1 thus obtained is *a priori* depending on N. For $N=4$ (there are 7 states with $L=0$ and $S=\frac{1}{2}$ in that case) b_1 is found equal to 0.58 fm and the ground-state energy $E_0(b_1)=1040$ MeV to be compared to the exact one $E_0=1024$ MeV. This result seems encouraging; however the quantum energy $\hbar\omega$ resulting from this b_1 value is equal to 687 MeV, a value too high leading to a dilated spectrum. For instance, the first and second $L=0$, $S=\frac{1}{2}$ states are located at 898 and 1012 MeV, respectively, to be compared to the exact values 762 and 917 MeV. This criterion for determining b_1 is thus valid for the ground state but not for the excited states. This is a serious drawback if one is interested in the whole spectrum. Moreover, since L and S are good quantum numbers the same procedure can be used for each (L,S) band leading to a b parameter depending also on L and S. If the b_1 value determined from the ground ($L=0$, $S=\frac{1}{2}$) band is retained for other excited (L,S) bands, these will be badly reproduced (this is the case effectively). If, on the other hand, we select the b values minimizing each band head the original simplicity of the model is lost and the calculations of transitions between different bands become more complicated. One is interested in a criterion for determining a *unique* value of b which gives an overall agreement for the *whole* spectrum. This is closely related to a good convergence versus the number of quanta N in the basis. If we find a b value which decouples as much as possible the states with different number of quanta, this convergence criterion will be fulfilled. In particular, one can require that the overlap of the ground state $\mid\Psi_0(b)\rangle$ with the 0 quantum state

$$\mid S\rangle[\phi_{00}(\mathbf{x}_1,b)\phi_{00}(\mathbf{y}_1,b)]_0$$

is maximum:

$$R(b)=\langle\Psi_0(b) \mid S;0000;0(b)\rangle \text{ max} . \tag{5.2}$$

For $N=4$ the b_2 value obtained from this criterion is $b_2=0.66$ fm for which $E_0=1046$ MeV and $R(b)=0.988$. Since the ground state is practically the pure 0 quantum state it is obviously decoupled from other states and, consequently, this b value is practically independent of N. One sees that although b_2 is quite different from b_1, leading to quite different spectrum, the ground-state energies do not differ significantly indicating that E_0 is a flat minimum versus b. However, the correlations in the ground state, although small, are not negligible; in fact $E_0(N=0,\ b_2)=1087$ MeV, $E_0(N=2,\ b_2)=1065$ MeV, $E_0(N=4,\ b_2)=1046$ MeV. The quantum energy $\hbar\omega=530$ MeV obtained with this new criterion is quite realistic and the spectrum is much better in that case, even in a more restricted space. For example, the first and second excited $L=0$, $S=\frac{1}{2}$ states are now $E_0^*(N=2,\ b_2)=837$ MeV and $E_0^{**}(N=2,\ b_2)=971$ MeV. To have an idea of the convergence, we give the same quantities calculated in a space built with all states up to $N=4$ quanta $E_0^*(N=4,\ b_2)=808$ MeV, $E_0^{**}(N=4,\ b_2)=936$ MeV. These values are close to the exact ones and it appears that, with the criterion (5.2) for determining b, the spectrum for states up to $N=2$ is well reproduced with an HO basis up to $N=4$. This is clearly shown in Fig. 1 where the whole relative spectrum (up to $N=2$) of the nucleon is displayed both with an exact calculation and with an HO basis up to $N=4$ [remember that the potential does not include spin-orbit and tensor force and consequently the states J appear in multiplets (L,S)]. We see that the HO basis is a good one, especially for negative-

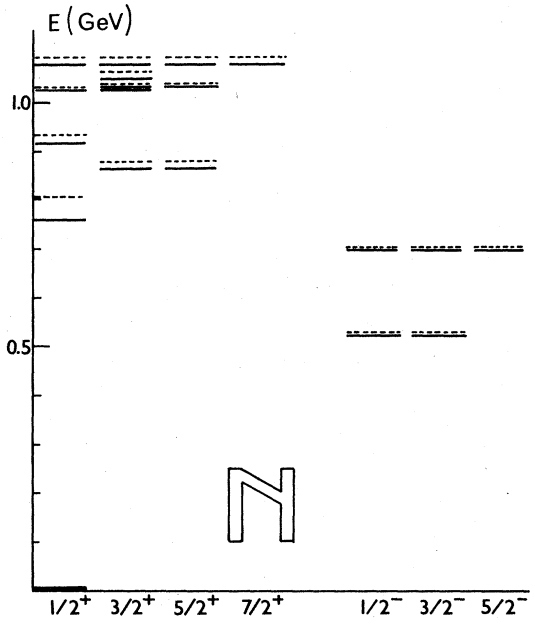

FIG. 1. Comparison between an exact treatment (solid line) and an approximate one using a harmonic-oscillator basis (dashed line) for the nucleon spectrum ($T=\frac{1}{2}$). The HO basis contains states up to four quanta and uses a size parameter $b=0.66$ fm. All levels up to two quanta are reported.

parity states; the differences with the exact calculations are more pronounced for low values of L ($L=0$). Nevertheless the overall agreement is remarkable. Concerning the energies, the HO approximation is certainly meaningful. One can now ask the question: Does this validity remain for other observables? To answer this question we select one-body and two-body observables. Among the one-body observables we calculate the mass square radius $\langle R_m{}^2\rangle$ with

$$R_m{}^2=\sum_{i=1}^{3} m_i(\mathbf{r}_i-\mathbf{R})^2 \Big/ \sum_{i=1}^{3} m_i \ , \tag{5.3}$$

$$\langle R_m{}^2\rangle=\tfrac{1}{3}\langle\Psi\,|\,\mathbf{y}_1{}^2\,|\,\Psi\rangle \ . \tag{5.4}$$

We also study the charge square radius $\langle R_c{}^2\rangle$, where

$$R_c{}^2=\sum_{i=1}^{3} e_i(\mathbf{r}_i-\mathbf{R})^2 \ , \tag{5.5}$$

then

$$\langle R_c{}^2\rangle=\frac{e}{6}\langle\Psi\,|\,(1+3\tau_z^1)\mathbf{y}_1{}^2\,|\,\Psi\rangle \tag{5.6}$$

which can be recast, through the Landé theorem, in the form $\langle R_c{}^2\rangle=A+T_zB$. Another interesting quantity is the magnetic moment in units of the quark magneton,

$$\mu/\mu_q=\tfrac{1}{6}\left\langle\Psi_{JJ}\left|\sum_{i=1}^{3}(l_z^i+\sigma_z^i)(1+3\tau_z^i)\right|\Psi_{JJ}\right\rangle \tag{5.7}$$

which, for the ground state $L=0$, $S=\frac{1}{2}$, takes the form

$$\mu/\mu_q=\frac{S}{3}+T_zC \ . \tag{5.8}$$

Among two-body observables the Coulomb energy, calculated by perturbation, is of special importance,

$$E_C=\frac{e^2}{36}\left\langle\Psi\left|\sum_{i<j}\frac{(1+3\tau_z^i)(1+3\tau_z^j)}{r_{ij}}\right|\Psi\right\rangle \ . \tag{5.9}$$

Denoting

$$A=\left\langle\Psi\left|\frac{1}{x_1}\right|\Psi\right\rangle$$

and (5.10)

$$B=\left\langle\Psi\left|\frac{\tau_z^1}{x_1}\right|\Psi\right\rangle\Big/T_z$$

one can show that

$$E_C=\frac{e^2}{12}[18(A-B)T_z{}^2+(6A-3B)T_z-\tfrac{7}{2}A] \ . \tag{5.11}$$

The numerical values of these quantities, calculated both with the HO basis ($N=4$ and b_2) and with exact Faddeev formalism, are reported in Table I for the two isospin components $T_z=\frac{1}{2}$ (proton) and $T_z=-\frac{1}{2}$ (neutron) of the nucleon. It is clear that for one-body physical quantities ($\langle R_m{}^2\rangle$, $\langle R_c{}^2\rangle$, μ/μ_q) the harmonic-oscillator approximation is a really good one, while the discrepancy on Coulomb energy is more pronounced [curiously the difference $E_C(p)-E_C(n)$ is also well reproduced].

Having in mind that the correlations between two particles in the cluster may be a crucial test, we introduce the correlation function $\rho(x_0)$ which gives the probability per volume unit to find two particles separated by the distance x_0, independently of their direction. More precisely,

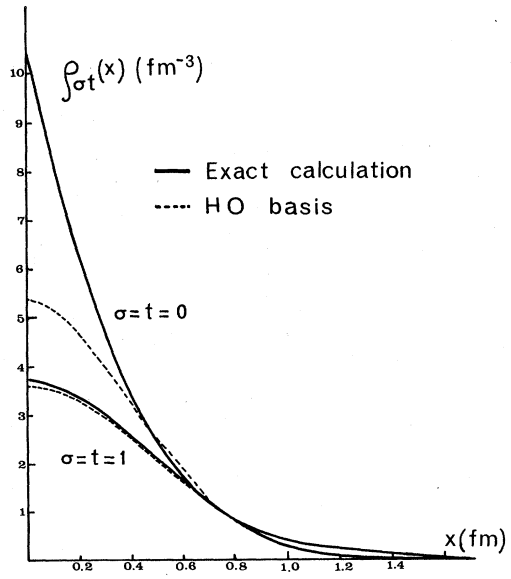

FIG. 2. Correlation function, as defined by (5.12), calculated in an exact treatment (solid curve) and in an approximate one using a harmonic-oscillator basis (dashed curve). The two most important channels $\sigma=t=0$ and $\sigma=t=1$ are plotted.

$$\rho_{\sigma t}^{LST}(x_0)=\int d\hat{\mathbf{x}}_0\langle\Psi^{LST}(\mathbf{x},\mathbf{y})\,|\,P_\sigma P_t\delta(\mathbf{x}_0-\mathbf{x})\,|\,\Psi^{LST}(\mathbf{x},\mathbf{y})\rangle\ , \tag{5.12}$$

where P_σ and P_t are the projectors on states where the particle pair is coupled to σ for the spin, to t for the isospin. In Fig. 2, the $\rho(x_0)$ correlation is plotted for the nucleon in the two important channels $\sigma=t=0$ and $\sigma=t=1$. The channel $\sigma=t=1$ (for which the spin-spin interaction is repulsive) is correctly described by the HO approximation. On the other hand, the channel $\sigma=t=0$ (for which the spin-spin interaction is strongly attractive) in the HO basis deviates more and more from the exact one as the particles come closer and closer. At origin the discrepancy reaches a factor of 2. This effect may be very important for quantities which need the correlations for small distance, such as the leptonic decay of strange baryons; in that case it is necessary to make a full Faddeev calculation. We hope that this lengthy and detailed comparison between the exact treatment and the diagonalization in a restricted space spanned by harmonic-oscillator functions sheds some light on the validity of previous works, more particularly those of Isgur and Karl,[8] dealing with the same subject.

VI. RESULTS AND DISCUSSION

In this section, the study of baryons built up with u, d, s quarks is presented. All the calculations are performed with the Bhaduri-Cohler-Nogami potential discussed in Sec. II, and with the Faddeev formalism presented in Sec. III. To our knowledge, this paper is the first one reporting such extensive and precise calculations. These represent the following six systems: nonstrange baryons with $T=\frac{1}{2}$,

$p\ (uud)$, $n\ (udd)$;

nonstrange baryons with $T=\frac{3}{2}$,

$\Delta^{++}(uuu)$, $\Delta^{+}\ (uud)$, $\Delta^{0}\ (udd)$, $\Delta^{-}\ (ddd)$;

strangeness-1 baryons with $T=0$,

$\Lambda\ (uds)$;

strangeness-1 baryons with $T=1$,

$\Sigma^{+}\ (uus)$, $\Sigma^{0}\ (uds)$, $\Sigma^{-}\ (dds)$;

strangeness-2 baryons ($T=\frac{1}{2}$),

$\Xi^{0}\ (uss)$, $\Xi^{-}\ (dss)$;

strangeness-3 baryons ($T=0$),

$\Omega^{-}\ (sss)$.

For each system, the static properties $\langle R_m^2\rangle$, $\langle R_c^2\rangle$, μ/μ_N. E_c already discussed in the previous section are given for the ground state, and the relative spectrum up to *two-quanta excitation energy* is plotted in Figs. 3–8. These systems are those for which the nonrelativistic model is expected to be worse; they are also those for which the experimental data are more abundant.

TABLE I. Comparison between an exact treatment and the approximate one using a harmonic-oscillator basis for the nucleon system. The one-body ($\langle R_m^2\rangle$, $\langle R_c^2\rangle$, μ/μ_q) and two-body (Coulomb energy E_C) observables are calculated with the Bhaduri-Cohler-Nogami potential. The HO basis contains states up to four quanta and uses a size parameter $b=0.66$ fm.

Observables \ Treatment		HO basis	Faddeev calculations
$\langle R_m^2\rangle$ (fm^2)		0.227	0.218
$\langle R_c^2\rangle$	p	0.247	0.240
(e fm^2)	n	−0.020	−0.025
μ/μ_q	p	0.990	0.987
	n	−0.657	−0.653
E_C	p	−0.098	−0.152
(MeV)	n	−0.884	−0.947

A. Absolute energies

The calculated mass for the nucleon is found to be 1024 MeV. This value is 85 MeV higher than the experimental one, but this is not significant. We employed the potential without changing any parameter and the constant term D in (2.3) can be used to fit exactly the nucleon mass. This term has no influence on the wave function and consequently on the nucleon structure. What is more important for testing the potential are the masses of the ground states of the systems relative to the nucleon mass. These are listed in Table II. The overall agreement is quite good; however the discrepancy increases with the system mass. To remedy this effect, Bhaduri, Cohler, and Nogami proposed to include in the potential a three-body constant term depending on the quark masses. On the other hand, Ono[18] preferred to use a spin-spin range r_0 in (2.3) depending on the quark masses.

B. Relative energy spectrum

The energy spectra of the six considered systems relative to their own ground state are plotted in Figs. 3–8. All states up to around 1-GeV excitation energy, corresponding roughly to two quanta, are reported. Calculated values are given by horizontal bars while experimental data are drawn through their full width. The comparison between experimental spectra (resonance energies) and our calculated results (discrete bound states) should be taken with caution; indeed, our model cannot include the cou-

TABLE II. Ground-state masses in MeV for the considered systems relative to the calculated nucleon mass (M_S-M_N); these results were obtained by an exact three-body treatment.

System S	Experiment	Theory
Δ	293	306
Λ	177	174
Σ	254	237
Ξ	379	355
Ω	733	660

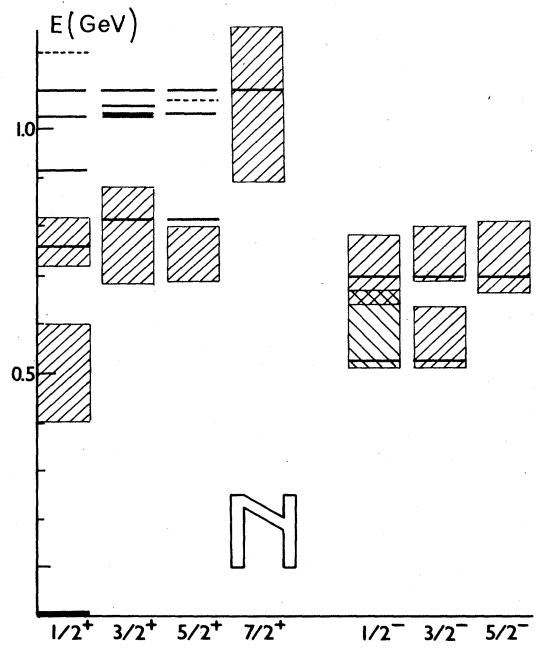

FIG. 3. Relative spectrum for the nucleon system (*uud* and *udd*) calculated with the Bhaduri-Cohler-Nogami potential and within an exact three-body treatment. Positive-parity levels are displayed in the left-hand side, while the negative-parity levels are on the right-hand side. The calculated values are indicated by a solid line; the experimental data are shown through their full width, while uncertain experimental levels are denoted by dashed lines.

pling to the continuum channels. Thus we have no prediction for the energy shift and the full width, which may be of the same order of magnitude. Sometimes, uncertain experimental levels are indicated with dashed lines. As already stated, the calculated states appear in (L,S) multiplets. These multiplets can be split by the introduction of spin-orbit or tensor forces. Nevertheless, from an overall

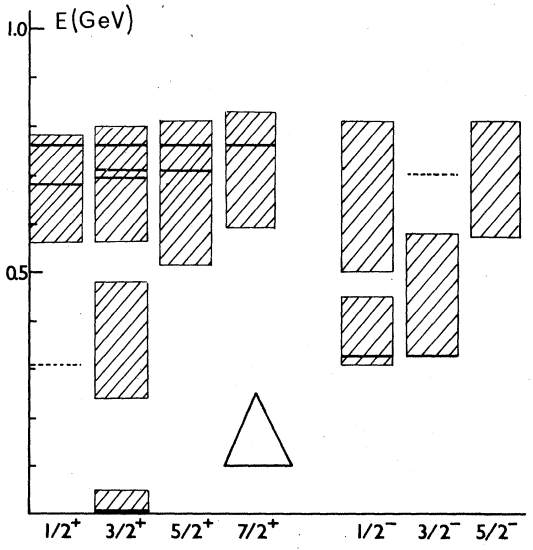

FIG. 4. Same as Fig. 3 for the Δ system (*uuu*, *uud*, *udd*, *ddd*).

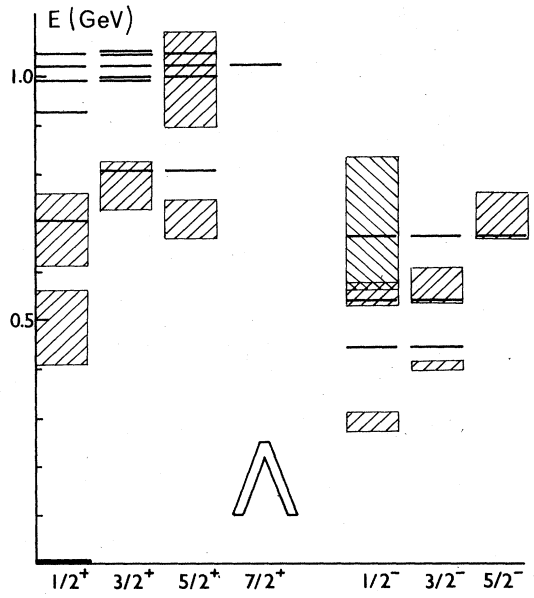

FIG. 5. Same as Fig. 3 for the Λ system (*uds*).

view of Figs. 3–8, the simple potential of Bhaduri, Cohler, and Nogami looks quite able to reproduce the main features of the spectra for all six considered systems. In particular, the negative parity levels are fairly well described. However, one cannot interpret, within this model, the challenging discrepancy of the $\Delta(1900)\frac{1}{2}^-$ and $\Delta(1930)\frac{5}{2}^-$ states which lie experimentally far too low as compared to our three-quanta excitation levels (which appear at around 1.1-GeV excitation energy). Moreover another discrepancy concerns the Λ(1405) which is theoretically too high. However, our feeling is that taking into account the spin-orbit or tensor force in the potential would improve the situation in this peculiar case. On the other hand, most of positive parity states are correctly located. Nevertheless, in each system there is a dramatic

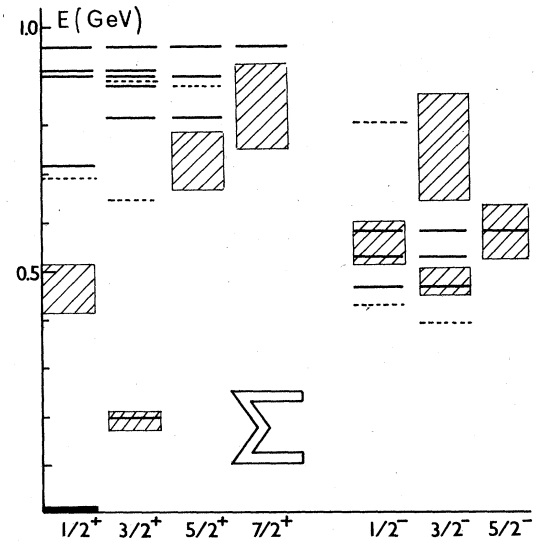

FIG. 6. Same as Fig. 3 for the Σ system (*uus*, *uds*, *dds*).

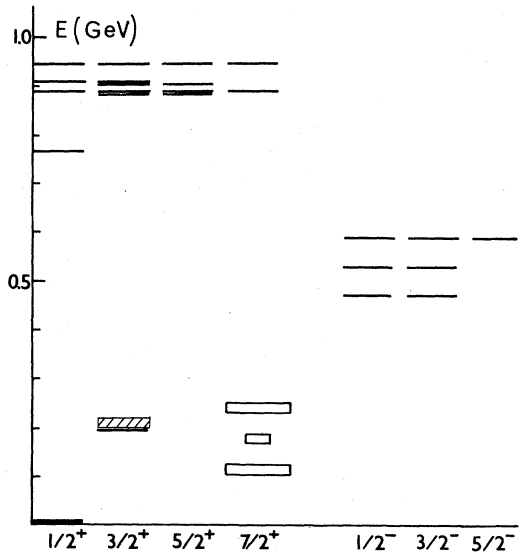

FIG. 7. Same as Fig. 3 for the Ξ system (*uss*, *dss*).

exception concerning the first excited state with the same quantum numbers as the ground state, namely, the Roper resonance $N(1440)$ and the analogous levels $\Delta(1600)$, $\Lambda(1600)$, and $\Sigma(1660)$. The puzzle comes from the fact that they are lying experimentally in the first negative parity energy range. This cannot be understood in terms of HO states. Our complete calculations clearly show that this phenomenon is not a consequence of truncating the Hilbert space. Our deep impression is that this puzzle cannot be solved by changing the potential and/or introducing spin-orbit and tensor forces. If the potential is determined in order to fit these peculiar levels, the description of the rest of the spectrum (especially the negative parity states) would fail in counterpart.[19] It was argued that these levels could be constructed on a deformed basis.[20] From our study, we can conclude that, if it is true, they cannot be understood as the deformation of the mean field coming from the quark-quark potential. Personally, we think that the structure of these puzzling levels is more complicated than the simple three-quark structure studied in this paper. Maybe the polarizability due to the quark-antiquark sea is especially important in these special cases. In any event, the mechanism responsible for the lowering of Roper and analogous resonances is still not understood and no doubt that extensive works will be done on this subject in the near future. Since the experimental spectrum of the Ξ and Ω systems is practically unknown, we hope that our results could serve as a theoretical reference for future experimental determinations.

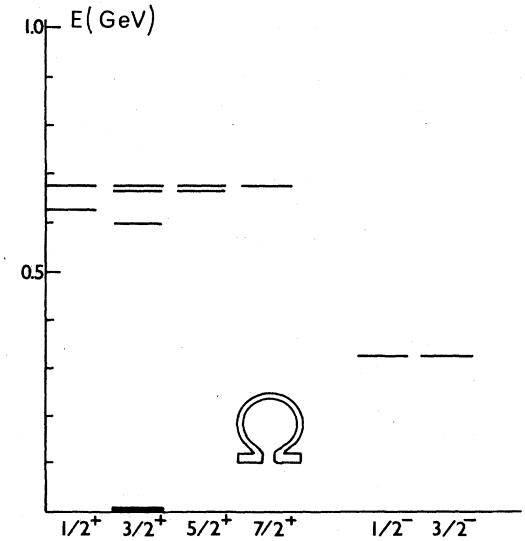

FIG. 8. Same as Fig. 3 for the Ω system (*sss*).

C. Physical observables

In this subsection we consider the one-body and two-body observables mentioned in the previous section; they are calculated for the ground state of each system and displayed in Table III. Experimental values, when known, are indicated between parentheses. The mass square radius is a decreasing function of the system mass (comparing systems with the same spin-space symmetry); this is intiuitively understood when one realizes that for heavier systems the kinetic energy is reduced allowing quarks to be closer to each other. These values give particle size around 0.4 to 0.5 fm. Charge square radius varies significantly from one system to the other due to different electric charges of the quarks; if the spatial wave function were completely symmetric, one would have the exact identity $\langle R_c^2\rangle = Z\langle R_m^2\rangle$. This is practically the case for all systems, even when the masses are different. The calculated values are smaller than the measured ones. It is traditionally claimed that the difference is due to the pion cloud, or equivalently to the coupling of the valence quarks to the quark-antiquark sea; quantitative calculations in that direction would be very instructive. Since, for all the systems, the ground-state wave function has practically pure $\lambda = l = 0$ components, the orbital contribution to the magnetic moment is quite negligible and the spin contribution gives the values derived by "the naive quark model." The general trend is well reproduced except the anomaly of the Ξ^-. The calculated Coulomb energies are of order of one MeV, that is small as compared to the total mass. However, they are rather different from one system to the other (ranging from 0.8 MeV for $n-p$ to 2.5 MeV for $\Xi^- - \Xi^0$) showing the importance of an exact calculation. From a glance at Table III, it is clear that the Coulomb energy alone is not able to explain the mass splitting of the isospin multiplets (the neutron is more bound than the proton); the only way to recover the correct masses is to break the fundamental SU(2) isospin symmetry for u and d quarks and to assume a small difference $m_d - m_u = \Delta m$. Apart from this constituent mass correction, there exist also correcting terms coming from kinetic energy and flavor dependence of the color magnetic force.[21,22] These terms are, in first order of perturbation, proportional to Δm but with destructive interference in the coefficients.[22] Thus they need a careful and proper treatment which is out of the scope of this pa-

TABLE III. One-body and two-body observables calculated by an exact treatment with the Bhaduri-Cohler-Nogami potential for the ground states of the six considered systems. Experimental values, when known, are written within parentheses.

System \ Observables	$\langle R_m^2 \rangle$ (fm^2)	$\langle R_c^2 \rangle$ (e fm^2)	μ/μ_N	E_C (MeV)
p		0.240	2.752	−0.152
	0.218	(0.66)	(2.793)	
n		−0.025	−1.822	−0.947
		(−0.108)	(−1.913)	
Δ^{++}		0.578	5.579	2.894
Δ^{+}		0.289	2.789	0
	0.289			
Δ^{0}		0	0	−0.723
Δ^{-}		−0.289	−2.789	0.723
Λ	0.179	0.040	−0.522	−0.866
			(−0.613)	
Σ^{+}		0.334	2.653	0.029
			(2.379)	
Σ^{0}	0.193	0.059	0.794	−0.756
Σ^{-}		−0.217	−1.414	0.749
			(−1.10)	
Ξ^{0}		0.091	−1.316	−1.412
	0.161		(−1.25)	
Ξ^{-}		−0.177	−0.386	1.218
			(−1.85)	
Ω^{-}	0.175	−0.175	−1.567	0.945

per and which is postponed for a future publication. However for the Σ system the combination $\frac{1}{2}(\Sigma^+ + \Sigma^-) - \Sigma^0$ cancels exactly these dangerous terms and we are left with the Coulomb correction only. The calculated value for this peculiar case is 1.145 MeV to be compared to the experimental one 0.89 MeV; the agreement is satisfactory.

VII. CONCLUSIONS

In this paper we solved exactly the three-body problem for baryons composed of u, d, and s quarks. The quark-quark potential employed in our calculations is due to Bhaduri, Cohler, and Nogami; we took it without modification and thus our results are obtained without any free parameter (except the mass difference $m_d - m_u$ necessary to fit the members of isospin multiplets). On the other hand, the nucleon properties were also investigated in terms of a harmonic-oscillator basis. This way of truncating the total Hilbert space was abundantly used in the past (see, for instance, the extensive pioneering works of Isgur and Karl[8]); thus a serious comparison with an exact reference is highly instructive. From our study it appears that the current prescription for determining the oscillator size b by minimizing $\langle \Psi | H | \Psi \rangle$ gives good results only for the ground state but leads to a dilated spectrum. A better choice for b is to impose that the ground state is as pure as possible on the zero quantum state. In that case the different levels are rather decoupled from each other and convergence is faster. For example, a space restricted to states up to four quanta is large enough to reproduce nicely the spectrum of states up to two quanta. With these prescriptions to restrict the Hilbert space, the energy spectrum as well as one-body properties are very well described. However two-body properties such as Coulomb energies or quark correlations in the cluster can be seriously different from their exact values. This effect is important for the leptonic decay of strange baryons for which one needs to rely on an exact Faddeev calculation.

The Bhaduri-Cohler-Nogami potential has a very simple form; in particular it does not contain spin-orbit or tensor forces. Nevertheless the spectrum obtained with it is in rather good agreement with the experimental data concerning the studied baryons. This shows that the most important part of the physical content is already included in it. However, the Roper and analogous resonances cannot fit in this framework. Our feeling is that this is not a matter of potential but rather a matter of structure. Changing the potential would not modify qualitatively the general features of the spectra; we think that the structure of the Roper resonance is more complicated than the three-quark structure studied in this paper. Probably there is an important coupling of the three valence quarks to the quark-antiquark sea. In the same spirit the

$\Delta(1900)\frac{1}{2}^-$ and $\Delta(1930)\frac{5}{2}^-$ are calculated much too high compared to their experimental values. Even a complete exact treatment maintains them in the three-quanta region, while experimentally they lie below the two-quanta region. This phenomenon is not understood up to now. Finally the nonrelativistic model applied to systems for which $\langle v/c \rangle$ is around unity works quite well as if the relativistic effects were masked. This is also an interesting puzzle.

APPENDIX

In this appendix, we want to report some technical aspects concerning the nucleon wave function in terms of an harmonic-oscillator basis. The correctly symmetrized basis functions Φ_i up to $N=4$ quanta and the eigenvectors Ψ_α (up to two quanta) obtained with the Bhaduri-Cohler-Nogami potential are presented for $L=0,1,2$ with $S=\frac{1}{2}$ and $\frac{3}{2}$ in Tables IV–X. As explained in Sec. IV, the Φ_i were deduced from the use of Brody-Moshinsky coefficients with angles $\beta=\pm 2\pi/3$. In order to simplify the notations let us put, as in (4.1),

$$|\, nl\nu\lambda; L\,\rangle = [\phi_{nl}(\mathbf{x}_1)\phi_{\nu\lambda}(\mathbf{y}_1)]_L \ . \tag{A1}$$

Since separate tables are presented for each L value, the L index in (A1) will be omitted in the tables. The color wave function is antisymmetric and was purely forgotten

TABLE IV. Correctly symmetrized basis functions Φ_i up to $N=4$ quanta and the eigenvectors Ψ_α (up to two quanta) obtained with the Bhaduri-Cohler-Nogami potential, for $L=0$, $S=\frac{1}{2}$, positive parity.

N	Φ_i	Ψ_1	Ψ_2	Ψ_3
0	$\lvert S\rangle \lvert 0000\rangle$	0.988	−0.062	0.087
2	$\frac{1}{\sqrt{2}}\lvert S\rangle(\lvert 1000\rangle+\lvert 0010\rangle)$	−0.019	−0.932	−0.293
	$\frac{1}{2}[\lvert MS\rangle(\lvert 1000\rangle-\lvert 0010\rangle)-\sqrt{2}\lvert MA\rangle\lvert 0101\rangle]$	0.112	0.263	−0.920
4	$\frac{1}{4}\lvert S\rangle[\sqrt{5}(\lvert 2000\rangle+\lvert 0020\rangle)+\sqrt{6}\lvert 1010\rangle]$	0.089	0.162	0.084
	$\frac{1}{\sqrt{96}}\lvert S\rangle[\sqrt{6}(\lvert 2000\rangle+\lvert 0020\rangle)-2\sqrt{5}\lvert 1010\rangle+8\lvert 0202\rangle]$	0.042	0.084	−0.055
	$\frac{1}{2}[\lvert MS\rangle(\lvert 2000\rangle-\lvert 0020\rangle)-\lvert MA\rangle(\lvert 0111\rangle+\lvert 1101\rangle)]$	0.036	−0.148	0.215
	$\frac{1}{\sqrt{96}}\{\lvert MS\rangle[\sqrt{6}(\lvert 2000\rangle+\lvert 0020\rangle)-2\sqrt{5}\lvert 1010\rangle-4\lvert 0202\rangle]$ $-2\sqrt{6}\lvert MA\rangle[\lvert 0111\rangle-\lvert 1101\rangle]\}$	0.030	0.054	−0.054

TABLE V. As in Table IV, for $L=1$, $S=\frac{1}{2}$, negative parity.

N	Φ_i	Ψ_1
1	$\frac{1}{\sqrt{2}}(\lvert MS\rangle\lvert 0001\rangle-\lvert MA\rangle\lvert 0100\rangle)$	0.978
3	$\frac{1}{\sqrt{12}}\lvert S\rangle(-\sqrt{3}\lvert 0011\rangle+\sqrt{5}\lvert 1001\rangle+2\lvert 0201\rangle)$	0.128
	$\frac{1}{\sqrt{12}}\lvert A\rangle(-\sqrt{3}\lvert 1100\rangle+\sqrt{5}\lvert 0110\rangle+2\lvert 0102\rangle)$	−0.031
	$\frac{1}{4}[\lvert MS\rangle(\sqrt{5}\lvert 0011\rangle+\sqrt{3}\lvert 1001\rangle)-\lvert MA\rangle(\sqrt{5}\lvert 1100\rangle+\sqrt{3}\lvert 0110\rangle)]$	−0.145
	$\frac{1}{\sqrt{48}}[\lvert MS\rangle(\sqrt{3}\lvert 0011\rangle-\sqrt{5}\lvert 1001\rangle+4\lvert 0201\rangle)$ $-\lvert MA\rangle(\sqrt{3}\lvert 1100\rangle-\sqrt{5}\lvert 0110\rangle+4\lvert 0102\rangle)]$	−0.067

TABLE VI. As in Table IV, for $L=1$, $S=\frac{1}{2}$, positive parity.

N	Φ_i	Ψ_1
2	$\lvert A\rangle\lvert 0101\rangle$	−0.966
4	$\frac{1}{\sqrt{2}}\lvert A\rangle(\lvert 1101\rangle+\lvert 0111\rangle)$	0.253
	$\frac{1}{2}[\lvert MS\rangle\lvert 0202\rangle+\lvert MA\rangle(\lvert 1101\rangle-\lvert 0111\rangle)]$	0.047

TABLE VII. As in Table IV, for $L=2$, $S=\frac{1}{2}$, positive parity.

N	Φ_i	Ψ_1	Ψ_2
2	$\frac{1}{\sqrt{2}}\|S\rangle(\|0200\rangle+\|0002\rangle)$	0.780	−0.564
	$\frac{1}{2}[\|MS\rangle(\|0200\rangle-\|0002\rangle)-\sqrt{2}\|MA\rangle\|0101\rangle]$	−0.570	−0.777
	$\frac{1}{\sqrt{20}}\|A\rangle[\sqrt{3}(\|1101\rangle-\|0111\rangle)-\sqrt{7}(\|0301\rangle-\|0103\rangle)]$	−0.010	−0.020
	$\frac{1}{\sqrt{5}}\|S\rangle(\|1200\rangle+\|0012\rangle+\sqrt{3}\|0202\rangle)$	−0.148	0.100
	$\frac{1}{\sqrt{120}}\|S\rangle[\sqrt{21}(\|1200\rangle+\|0012\rangle)-2\sqrt{7}\|0202\rangle+5(\|1002\rangle+\|0210\rangle)]$	−0.114	0.141
	$\frac{1}{\sqrt{40}}\{\|MS\rangle[\sqrt{3}(\|1200\rangle-\|0012\rangle)+\sqrt{7}(\|1002\rangle-\|0210\rangle)]$ $-\sqrt{10}\|MA\rangle(\|0301\rangle+\|0103\rangle)\}$	0.107	0.015
4	$\frac{1}{\sqrt{40}}\{\|MS\rangle[-\sqrt{7}(\|1200\rangle-\|0012\rangle)+\sqrt{3}(\|1002\rangle-\|0210\rangle)]$ $+\sqrt{10}\|MA\rangle(\|1101\rangle+\|0111\rangle)\}$	−0.121	−0.219
	$\frac{1}{\sqrt{240}}\{\|MS\rangle[\sqrt{15}(\|1200\rangle+\|0012\rangle)-\sqrt{35}(\|1002\rangle+\|0210\rangle)-2\sqrt{5}\|0202\rangle]$ $+\|MA\rangle[\sqrt{42}(\|1101\rangle-\|0111\rangle)+3\sqrt{2}(\|0301\rangle-\|0103\rangle)]\}$	−0.069	0.010

TABLE VIII. As in Table IV, for $L=0$, $S=\frac{3}{2}$, positive parity.

N	Φ_i	Ψ_1
2	$\frac{1}{2}[\|ms\rangle(\|1000\rangle-\|0010\rangle)-\sqrt{2}\|ma\rangle\|0101\rangle]$	0.947
4	$\frac{1}{2}[\|ms\rangle(\|2000\rangle-\|0020\rangle)-\|ma\rangle(\|0111\rangle+\|1101\rangle)]$	−0.322
	$\frac{1}{\sqrt{96}}\{\|ms\rangle[\sqrt{6}(\|2000\rangle+\|0020\rangle)-2\sqrt{5}\|1010\rangle-4\|0202\rangle]$ $-2\sqrt{6}\|ma\rangle[\|0111\rangle-\|1101\rangle]\}$	0.013

TABLE IX. As in Table IV, for $L=1$, $S=\frac{3}{2}$, negative parity.

N	Φ_i	Ψ_1
1	$\frac{1}{\sqrt{2}}(\|ms\rangle\|0001\rangle-\|ma\rangle\|0100\rangle)$	0.966
3	$\frac{1}{4}[\|ms\rangle(\sqrt{5}\|0011\rangle+\sqrt{3}\|1001\rangle)-\|ma\rangle(\sqrt{5}\|1100\rangle+\sqrt{3}\|0110\rangle)]$	−0.257
	$\frac{1}{\sqrt{48}}[\|ms\rangle(\sqrt{3}\|0011\rangle-\sqrt{5}\|1001\rangle+4\|0201\rangle)$ $-\|ma\rangle(\sqrt{3}\|1100\rangle-\sqrt{5}\|0110\rangle+4\|0102\rangle)]$	−0.022

TABLE X. As in Table IV, for $L=2$, $S=\frac{3}{2}$, positive parity.

N	Φ_i	Ψ_1
2	$\frac{1}{2}[\|ms\rangle(\|0200\rangle-\|0002\rangle)-\sqrt{2}\|ma\rangle\|0101\rangle]$	0.949
4	$\frac{1}{\sqrt{40}}\{\|ms\rangle[\sqrt{3}(\|1200\rangle-\|0012\rangle)+\sqrt{7}(\|1002\rangle-\|0210\rangle)]$ $-\sqrt{10}\|ma\rangle[\|0301\rangle+\|0103\rangle]\}$	−0.034
	$\frac{1}{\sqrt{40}}\{\|ms\rangle[-\sqrt{7}(\|1200\rangle-\|0012\rangle)+\sqrt{3}(\|1002\rangle-\|0210\rangle)]$ $+\sqrt{10}\|ma\rangle[\|1101\rangle+\|0111\rangle]\}$	0.313
	$\frac{1}{\sqrt{240}}\{\|ms\rangle[\sqrt{15}(\|1200\rangle+\|0012\rangle)-\sqrt{35}(\|1002\rangle+\|0210\rangle)-2\sqrt{5}\|0202\rangle]$ $+\|ma\rangle[\sqrt{42}(\|1101\rangle-\|0111\rangle)+3\sqrt{2}(\|0301\rangle-\|0103\rangle)]\}$	0.006

in the tables since it is the same for each function. The rest of the basis functions Φ_i is thus completely symmetric. The notations for the spin-isospin functions, which appear in front of the spatial functions (A1), are those defined by Eqs. (2.1). The eigenvectors Ψ_α are expanded in the Φ_i basis and were calculated with $N=4$ and $b=0.66$ fm, as was discussed in detail in Sec. V. The components of Ψ_α are also listed in the tables. The wave functions are intended for the readers who want to make calculations more sophisticated than the usual ones, with a harmonic-oscillator basis.

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