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Amplitudes of the two-nucleon interaction at 579 MeV

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A complete amplitude analysis is presented of the 579-MeV data set for *p-p* elastic scattering. Although all continua of ambiguities are eliminated by the extensive nature of the data, discrete ambiguities remain, as expected from general criteria for such ambiguities. In particular, four solutions are found. Further experiments are specified which can resolve this remaining ambiguity also. A comparison is also made with previous amplitude results at this energy.

I. INTRODUCTION

The two-nucleon interaction up to about 6 GeV is the best explored strong interaction by a long shot, and thus it constitutes a firm data base for our concrete knowledge of strong interactions. In the lowest energy range a program of partial-wave analysis carried out over several decades[1] has provided the best phenomenological description of this process. Beyond 500 MeV, however, the phase-shift description is not necessarily the most convenient formalism, since it has some difficulties in dealing with inelastic processes, and because it requires an increasingly more complete angular distribution of observables in order to function, and since the number of angular momentum states keeps growing.

Thus at these higher energies an analysis into complex reaction amplitudes (which are functions of energy and angle) is preferable. To perform this in the simplest possible way, with the smallest possible uncertainties, we have to find a polarization formalism in which the relationship between observables and the bilinear products of amplitudes ("bicoms") is as simple as possible. Such a class of formalism is the optimal formalism,[2] in which this relationship can be described by a matrix which is as diagonal as Hermiticity allows: The matrix consists only of a string of small submatrices along the main diagonal.

The optimal formalism has been used to analyze data on reactions of various kinds,[3–5] and to draw various conclusions from these data concerning the dynamics underlying the reaction, such as one-particle-exchange mechanisms,[6] Regge-pole mechanisms,[7] QCD models,[8] as well as possible novel mechanisms.[4]

The aim of the present paper is to use the optimal formalism to provide a phenomenological description of the amplitudes for *p-p* elastic scattering at 579 MeV. At this energy a very impressive set of experimental data[9,10] is now available which is extensive enough to allow a determination of the amplitudes at various angles without a *continuum* of ambiguities. In fact, a solution has been given by the experimental group[9] which carried out the measurements. Our aim therefore was to attain the same solution via the use of the optimal formalism and also to explore the extent to which there remain *discrete* ambiguities in the amplitude solution at this energy with the present set of data. This is of importance since if one is to deduce dynamical conclusions from such measurements, such discrete ambiguities can seriously diminish the conclusiveness and unambiguity of such conclusions. Indeed, this is the case for the results already obtained from *p-p* scattering at other energies.[7] Such an analysis of ambiguities also contributes to a more systematic design of future experimental programs which are expected to unfold on many of the intermediate- and high-energy accelerators.

II. THE ANALYSIS

Experience with other reactions and with *p-p* scattering at other energies has shown[3–5,11] us that for any parity-conserving reaction the optimal formalism most suitable for the phenomenological determination of the amplitudes is the transversity formalism, since it "naturally" blends with the constraints imposed on the reaction amplitudes by parity conservation. We will therefore use this formalism to determine the amplitudes from the measurements.

The observables that have been measured at this energy and their expressions in terms of transversity amplitudes are given in Table I. The results of this table were adopted from Table VI of Ref. 12. We used all these data in arriving at the amplitudes: σ, P, C_{NN}, D_{NN}, and K_{NN} gave the magnitude of the transversity amplitudes; then C_{LL}, C_{SS}, and C_{LS} provided $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)-\phi(\epsilon)$ up to some ambiguities; and finally the rest of the observables were used to determine $\phi(\gamma)$ and $\phi(\epsilon)$ and to eliminate some ambiguities. This is described in detail below.

A. Determination of magnitudes

We see that five observables already measured determine the five magnitudes of the five amplitudes. We will ignore the angular dependence of the differential cross section and normalize the latter to be always unity, so that we have

$$\sigma \equiv |\alpha|^2+|\beta|^2+2(|\gamma|^2+|\delta|^2+|\epsilon|^2)=1. \quad (2.1)$$

Thus we need only four observables, namely P, C_{NN},

TABLE I. Observables for pp elastic scattering in various notations and their relationships to transversity amplitudes. Legend: (1) The Argonne name for the observable in the c.m. system (see Ref. 12, Table VI). (2) The Argonne c.m. observable notation (see Ref. 12, Table VI). (3) Our observable notation in the c.m. system (see Ref. 12, Table VI). (4) Expression for the observable in terms of bilinear products of transversity amplitudes (see Ref. 12, Table VI). (5) CEN Saclay and SIN Geneva laboratory observable notation. (6) Laboratory observable in terms of c.m. observables.

1	2	3	4	5	6
σ	(0,0;0,0)	$(A,A;A,A)$	$\lvert\alpha\rvert^2+\lvert\beta\rvert^2+2(\lvert\gamma\rvert^2+\lvert\delta\rvert^2+\lvert\epsilon\rvert^2)$	σ	$\sigma^{\text{c.m.}}$
P	$-(0,y;0,0)$	$(A,\Delta;A,A)$	$\lvert\alpha\rvert^2-\lvert\beta\rvert^2$	P	$P^{\text{c.m.}}$
C_{NN}	$-(y,y;0,0)$	$(\Delta,\Delta;A,A)$	$\lvert\alpha\rvert^2+\lvert\beta\rvert^2+2[-\lvert\gamma\rvert^2+\lvert\delta\rvert^2-\lvert\epsilon\rvert^2]$	C_{NNOO}	$C_{NNOO}^{\text{c.m.}}=A_{OONN}^{\text{c.m.}}$
C_{LL}	$-(z,z;0,0)$	$(R,R;A,A)$	$2\,\text{Re}[(\alpha+\beta)\delta^*+2\gamma\epsilon^*]$	A_{OOKK}	$A_{OOKK}^{\text{c.m.}}=C_{KKOO}^{\text{c.m.}}$
C_{SS}	$(x,x;0,0)$	$(I,I;A,A)$	$-2\,\text{Re}[(\alpha+\beta)\delta^*-2\gamma\epsilon^*]$	A_{OOSS}	$A_{OOSS}^{\text{c.m.}}=C_{SSOO}^{\text{c.m.}}$
C_{SL}	$-(x,z;0,0)$	$(I,R;A,A)$	$2\,\text{Im}[(\alpha-\beta)\delta^*]$	A_{OOSK}	$A_{OOSK}^{\text{c.m.}}=C_{SKOO}^{\text{c.m.}}$
D_{NN}	$(0,y;0,y)$	$(A,\Delta;A,\Delta)$	$\lvert\alpha\rvert^2+\lvert\beta\rvert^2+2[\lvert\gamma\rvert^2-\lvert\delta\rvert^2-\lvert\epsilon\rvert^2]$	D_{NONO}	$D_{NONO}^{\text{c.m.}}$
$-D_{SS}$	$(0,x;0,x)$	$(A,I;A,I)$	$2\,\text{Re}[(\alpha+\beta)\gamma^*-2\delta\epsilon^*]$	$D_{S'OSO}$	$-D_{SS}^{\text{c.m.}}\cos\theta_s+D_{LS}^{\text{c.m.}}\sin\theta_s$
D_{LS}	$(0,z;0,x)$	$-(A,R;A,I)$	$2\,\text{Im}[(\alpha-\beta)\gamma^*]$	$D_{S'OKO}$	$-D_{LS}^{\text{c.m.}}\cos\theta_s-D_{LL}^{\text{c.m.}}\sin\theta_s$
K_{NN}	$-(y,0;0,y)$	$(\Delta,A;A,\Delta)$	$\lvert\alpha\rvert^2+\lvert\beta\rvert^2+2[-\lvert\gamma\rvert^2-\lvert\delta\rvert^2+\lvert\epsilon\rvert^2]$	K_{ONNO}	$K_{ONNO}^{\text{c.m.}}$
K_{SS}	$-(x,0;0,x)$	$-(I,A;A,I)$	$-2\,\text{Re}[(\alpha+\beta)\epsilon^*-2\gamma\delta^*]$	$K_{OS''SO}$	$K_{SS}^{\text{c.m.}}\cos\theta_s-K_{SL}^{\text{c.m.}}\sin\theta_s$
$-K_{SL}$	$-(x,0;0,z)$	$(I,A;A,R)$	$2\,\text{Im}[(\alpha-\beta)\epsilon^*]$	$-K_{OS''KO}$	$K_{SL}^{\text{c.m.}}\cos\theta_s-K_{LL}^{\text{c.m.}}\sin\theta_s$
H_{NSS}	$-(y,x;0,x)$	$-(\Delta,I;A,I)$	$-2\,\text{Re}[(\alpha-\beta)\gamma^*]$	$M_{S'OSN}$	$-H_{NSS}^{\text{c.m.}}\cos\theta_s-H_{NSL}^{\text{c.m.}}\sin\theta_s$
H_{NLS}	$(y,z;0,x)$	$-(\Delta,R;A,I)$	$2\,\text{Im}[(\alpha+\beta)\gamma^*-2\delta\epsilon^*]$	$-M_{S'OKN}$	$H_{NLS}^{\text{c.m.}}\cos\theta_s+H_{NSS}^{\text{c.m.}}\sin\theta_s$
H_{SNS}	$(x,y;0,x)$	$-(I,\Delta;A,I)$	$-2\,\text{Re}[(\alpha-\beta)\epsilon^*]$	$N_{OS''SN}$	$H_{SNS}^{\text{c.m.}}\cos\theta_R-H_{SNL}^{\text{c.m.}}\sin\theta_R$
H_{LNS}	$(z,y;0,x)$	$-(R,\Delta;A,I)$	$2\,\text{Im}[(\alpha+\beta)\epsilon^*+2\gamma\delta^*]$	$N_{OS''KN}$	$H_{LNS}^{\text{c.m.}}\cos\theta_R-H_{SNS}^{\text{c.m.}}\sin\theta_R$

D_{NN}, and K_{NN} to determine the magnitudes of the amplitudes. Note that this determination of the *magnitudes* (as opposed to the *phases*) can always be done without any discrete ambiguities, since such magnitudes are of course always positive, and hence determining the magnitude squared from the equations which are linear in them gives also a unique value for the magnitude itself.

The values of the magnitudes thus determined are given in Table II.

B. Determination of the phases

We will now describe the algebraic procedure that can be used to determine the relative phases between the amplitudes. It is at this point that all discrete ambiguities enter, since the measurements provide the sine and cosine of these relative phases, and the angles are multivalued functions of sine and cosine.

Since we can determine only the relative phases between amplitudes, we can set one amplitude real without infringing on the generality of the description. We will choose δ to be real. Then, from Table I we see that $C_{LL}-C_{SS}$ and C_{LS} give information on $\phi(\alpha)$ and $\phi(\beta)$. In particular, we have for these two quantities

$$U\equiv\frac{C_{LL}-C_{SS}}{4\lvert\delta\rvert}=\lvert\alpha\rvert\cos\phi(\alpha)+\lvert\beta\rvert\cos\phi(\beta) \tag{2.2}$$

and

$$V\equiv\frac{C_{LS}}{2\delta}=\lvert\alpha\rvert\sin\phi(\alpha)-\lvert\beta\rvert\sin\phi(\beta)\ , \tag{2.3}$$

and from these we get

$$\cos[\phi(\alpha)+\phi(\beta)]=\frac{U^2+V^2-\lvert\alpha\rvert^2-\lvert\beta\rvert^2}{2\lvert\alpha\rvert\,\lvert\beta\rvert}\ . \tag{2.4}$$

We note that Eq. (2.4) indicates a twofold discrete ambiguity for the left-hand side. Equation (2.4) also provides a constraint (and hence a consistency check) on experiments since the left-hand side has an absolute value equal to or smaller than unity.

The above set of equations yields [using $\eta\equiv\phi(\alpha)+\phi(\beta)$]

$$\cos\phi(\alpha)=\frac{U(\lvert\alpha\rvert+\lvert\beta\rvert\cos\eta)-V\lvert\beta\rvert\sin\eta}{U^2+V^2}\ ,$$
$$\sin\phi(\alpha)=\frac{V(\lvert\alpha\rvert+\lvert\beta\rvert\cos\eta)+U\lvert\beta\rvert\sin\eta}{U^2+V^2} \tag{2.5}$$

TABLE II. The magnitudes of the transversity amplitudes as obtained from our analysis.

$\theta_{\text{c.m.}}$ (deg)	$\lvert\alpha\rvert$	$\Delta\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\Delta\lvert\beta\rvert$	$\lvert\gamma\rvert$	$\Delta\lvert\gamma\rvert$	δ	$\Delta\lvert\delta\rvert$	$\lvert\epsilon\rvert$	$\Delta\lvert\epsilon\rvert$
66	0.720	0.006	0.392	0.010	0.287	0.011	0.211	0.015	0.194	0.016
70	0.717	0.005	0.422	0.009	0.285	0.011	0.188	0.016	0.194	0.016
74	0.703	0.005	0.482	0.008	0.280	0.013	0.185	0.020	0.155	0.024
78	0.674	0.006	0.506	0.007	0.291	0.012	0.173	0.021	0.175	0.021
82	0.649	0.006	0.527	0.007	0.265	0.013	0.174	0.020	0.224	0.016
86	0.636	0.006	0.569	0.006	0.253	0.014	0.167	0.021	0.208	0.017
90	0.600	0.006	0.605	0.006	0.241	0.015	0.161	0.022	0.229	0.015

and

$$\cos\phi(\beta)=\frac{U(|\beta|+|\alpha|\cos\eta)+V|\alpha|\sin\eta}{U^2+V^2}\ ,$$
$$\sin\phi(\beta)=\frac{-V(|\beta|+|\alpha|\cos\eta)+U|\alpha|\sin\eta}{U^2+V^2}\ . \tag{2.6}$$

The available data reflect the constraint of Eq. (2.4) rather poorly. This is shown in Table III.

The above set of equations gives two solutions for $\phi(\alpha)$ and two solutions for $\phi(\beta)$, thus leaving us altogether with four solutions for the set $\{\phi(\alpha),\phi(\beta)\}$.

If we had measurements also on H_{SSN} (see Table I), then we could define

$$D\equiv\frac{H_{SSN}}{2|\delta|}=|\alpha|\cos\phi(\alpha)-|\beta|\cos\phi(\beta)\ , \tag{2.7}$$

and from it and Eq. (2.3) we would get

$$\cos[\phi(\alpha)-\phi(\beta)]=\frac{D^2+V^2-|\alpha|^2-|\beta|^2}{2|\alpha||\beta|}\ , \tag{2.8}$$

which, however, also has a twofold ambiguity in it, and hence adding it to the previous information would only reduce the previous quadruple ambiguity to a double one.

In our quantitative determination of $\phi(\alpha)$ and $\phi(\beta)$ whenever the right-hand side of Eq. (2.4) was larger than unity, we replaced that number by unity but retained the error limits. Thus, a number of 1.65 ± 0.95 was replaced by 1.00 ± 0.30, thus, in fact, probably underestimating the uncertainty. This causes some problems, such as our not getting exactly zero for the single spin-flip amplitude at $\theta_{\text{c.m.}}=90°$.

Having determined $\phi(\alpha)$ and $\phi(\beta)$, at least up to some discrete ambiguities, we now turn to the determination of $\phi(\gamma)$ and $\phi(\epsilon)$. From our previously considered observables we also have

$$C_{LL}+C_{SS}=8\,\text{Re}\gamma\epsilon^* \tag{2.9}$$

which determines the phase between $\phi(\gamma)$ and $\phi(\epsilon)$ up to a twofold discrete ambiguity. This equation also gives a constant and hence a consistency check on the measurements.

So far we have determined the phases of α and β (relative to δ) and the relative phase between γ and ϵ. Thus we need at least one more measurement pertaining to the relative phase between the group α, β, and δ on the one hand and the group γ and ϵ on the other. That can be supplied by

$$D_{SS}=4\,\text{Re}\gamma\epsilon^*\cos\theta_R-2\,\text{Re}(\alpha\gamma^*e^{i\theta_R}+\beta\gamma^*e^{-i\theta_R})$$
$$=X\cos\phi(\gamma)+Y\sin\phi(\gamma)\ , \tag{2.10}$$

where X and Y are given by

$$X=4|\delta||\epsilon|\cos[\phi(\gamma)-\phi(\epsilon)]\cos\theta_R$$
$$-2|\gamma|\{|\alpha|\cos[\phi(\alpha)+\theta_R]+|\beta|\cos[\phi(\beta)-\theta_R]\}\ , \tag{2.11}$$

$$Y=4|\delta||\epsilon|\sin[\phi(\gamma)-\phi(\epsilon)]\cos\theta_R$$
$$-2|\gamma|\{|\alpha|\sin[\phi(\alpha)+\theta_R]+|\beta|\sin[\phi(\beta)-\theta_R]\}\ , \tag{2.12}$$

where θ_R is the recoil angle. Using

$$\cos\xi=\frac{X}{(X^2+Y^2)^{1/2}}\quad\text{and}\quad\sin\xi=\frac{Y}{(X^2+Y^2)^{1/2}} \tag{2.13}$$

we can write Eq. (2.10) as

$$\frac{D_{SS}}{(X^2+Y^2)^{1/2}}=\cos[\phi(\gamma)-\xi]\ , \tag{2.14}$$

where ξ is uniquely determined.

In terms of the same X and Y we can obtain a similar relationship for $\sin[\phi(\gamma)-\Xi]$. Thus *for a fixed* $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)-\phi(\epsilon)$, we can determine $\phi(\gamma)$ without any ambiguities, but we recall that we "inherited" ambiguities in the first three of these quantities. Using observables other than D_{SS} (for example, D_{LS} or H_{NSS}) does not help to resolve these discrete ambiguities.

A similar procedure, with similar results, could be carried through using the corresponding K's instead of D's to determine $\phi(\epsilon)$. We have

$$K_{SS}=T\cos\phi(\epsilon)+Z\sin\phi(\epsilon) \tag{2.15}$$

with T and Z given by

$$T=4|\delta||\gamma|\cos[\phi(\gamma)-\phi(\epsilon)]\cos\theta_R$$
$$-2|\epsilon|\{|\alpha|\cos[\phi(\alpha)+\theta_R]+|\beta|\sin[\phi(\beta)-\theta_R]\} \tag{2.16}$$

and

$$Z=-4|\delta||\gamma|\sin[\phi(\gamma)-\phi(\epsilon)]\cos\theta_R$$
$$-2|\epsilon|\{|\alpha|\sin[\phi(\alpha)+\theta_R]+|\beta|\sin[\phi(\beta)-\theta_R]\} \tag{2.17}$$

from which we get

$$\cos[\phi(\epsilon)-\mu]=\frac{K_{SS}}{(T^2+Z^2)^{1/2}}\ , \tag{2.18}$$

where

TABLE III. The demonstration of the constraint given by Eq. (2.4) in our analysis. For details, see the text.

$\theta_{\text{c.m.}}$ (deg)	50	54	58	62	66	70	74	78	82	86	90
$\cos\eta$	1.875	1.776	1.746	3.170	0.480	0.431	0.472	1.092	0.826	1.092	1.210
$\Delta(\cos\eta)$	1.171	1.088	1.121	2.013	0.089	0.516	0.400	0.581	0.513	0.603	0.666

$$\cos\mu = \frac{T}{(T^2+Z^2)^{1/2}} \quad \text{and} \quad \sin\mu = \frac{Z}{(Z^2+T^2)^{1/2}} . \tag{2.19}$$

We used, however, a different procedure at this point. Since there were a number of additional observables at our disposal, we used D_{SS}^{lab}, D_{LS}, H_{NSS}, H_{NLS} (corresponding to the notation in Ref. 13 to D_{WOSO}, D_{WOKO}, M_{WOSN}, M_{WOKN}, respectively), which are given by

$$-\tfrac{1}{4}(M_{WOKN}^{\text{lab}}+D_{WOKO}^{\text{lab}}) = \text{Im}[(\,|\alpha|\,|\gamma|\,e^{i[\phi(\alpha)-(\theta_s-\omega)]})e^{i\phi(\gamma)} - (\,|\delta|\,|\epsilon|\,e^{i\{[\phi(\gamma)-\phi(\epsilon)]+(\theta_s-\omega)\}})e^{-i\phi(\gamma)}] , \tag{2.20}$$

$$-\tfrac{1}{4}(M_{WOKN}^{\text{lab}}-D_{WOKO}^{\text{lab}}) = \text{Im}[(\,|\beta|\,|\gamma|\,e^{i[\phi(\beta)+(\theta_s-\omega)]})e^{-i\phi(\gamma)} - (\,|\delta|\,|\epsilon|\,e^{i\{[\phi(\gamma)-\phi(\epsilon)]-(\theta_s-\omega)\}})e^{-i\phi(\gamma)}] , \tag{2.21}$$

$$-\tfrac{1}{4}(M_{WOSN}^{\text{lab}}+D_{WOSO}^{\text{lab}}) = \text{Re}[(\,|\alpha|\,|\gamma|\,e^{i[\phi(\alpha)-(\theta_s-\omega)]})e^{i\phi(\gamma)} - (\,|\delta|\,|\epsilon|\,e^{i\{[(\phi(\gamma)-\phi(\epsilon)]+(\theta_s-\omega)\}})e^{-i\phi(\gamma)}] , \tag{2.22}$$

$$-\tfrac{1}{4}(M_{WOSN}^{\text{lab}}-D_{WOSO}^{\text{lab}}) = \text{Re}[(\,|\beta|\,|\gamma|\,e^{i[\phi(\beta)+(\theta_s-\omega)]})e^{-i\phi(\gamma)} - (\,|\delta|\,|\epsilon|\,e^{i\{[\phi(\gamma)-\phi(\epsilon)]-(\theta_s-\omega)\}})e^{-i\phi(\gamma)}] , \tag{2.23}$$

where θ_s is the scattering angle in the laboratory system, and ω is the final transversal direction as analyzed in the polarimeter. These four equations contain only one unknown, namely, $\phi(\gamma)$, and so a χ^2 fit can be made to these four measurements to obtain the best value of $\phi(\gamma)$.

A similar procedure, using the observables K_{OWSO}, N_{OWSN}, K_{OWKO}, and N_{OWKN}, can be carried through to determine $\phi(\epsilon)$. None of that, however, relieves the inherited ambiguities originating with $\phi(\alpha)$ and $\phi(\beta)$.

To conclude this section, we remind the reader that once the transversity amplitudes have been determined, we can immediately obtain any other amplitude set (and, specifically, the helicity amplitudes) by a simple linear transformation involving only mathematics but not physics. In these transformations both the magnitudes and the phases of the other set of amplitudes depend on both the magnitudes and the phases of the transversity amplitudes. For the transversity amplitudes the magnitudes can be determined, by themselves, from an easy subset of experiments and hence with very small uncertainties, but the phases almost always involve much larger errors (even apart from the question of discrete ambiguities). Once one transforms, however, to a different set of amplitudes, both the magnitudes and the phases will exhibit larger uncertainties. Specifically, the transformations from transversity to helicity amplitudes are[12]

$$\begin{aligned} a &= \tfrac{1}{4}[\alpha+\beta+2(\gamma-\delta-\epsilon)] , \\ b &= \frac{i}{4}(\alpha-\beta) , \\ c &= \tfrac{1}{4}[\alpha+\beta+2(\gamma+\delta+\epsilon)] , \\ d &= \tfrac{1}{4}[\alpha+\beta+2(-\gamma-\delta+\epsilon)] , \\ e &= \tfrac{1}{4}[-\alpha-\beta+2(\gamma-\delta+\epsilon)] . \end{aligned} \tag{2.24}$$

III. DISCRETE AMBIGUITIES

Discrete ambiguities in amplitude determinations can occur for at least three different reasons.

First, let us consider the relationship of bicoms and observables, with exactly $2n-1$ observables available (n being the number of amplitudes). Criteria for when discrete ambiguities arise in such situations have been provided recently.[14] These criteria can be tested extremely easily through the geometrical diagrammatic analog technique described in the same paper.

Second, in a situation when we have more measurements than $2n-1$, and a least-square fitting is made, additional discrete ambiguities can arise if the uncertainties on the experimental observables are sufficiently large so that a unique solution cannot be pinpointed on the basis of χ^2 criteria, and several different solutions with fairly comparable χ^2's present themselves as candidates.

Third, as we have seen in the previous section, additional discrete ambiguities can arise when the measured observables do not yield individual bicoms but some linear combinations of these.

The second and third causes for discrete ambiguities depend on the particular situation at hand and hence it is difficult to offer an *a priori* discussion of them. The first reason, however, can be thus discussed.

We recall that the criteria in terms of the analog diagrams are that each amplitude point must be included in at least one loop diagram in which there are an odd number of solid and an odd number of broken lines. With five amplitudes, therefore, there is no way that four measurements (in addition to the five determining the magnitudes) will give freedom from discrete ambiguities. In fact, five measurements cannot do it either. With six measurements, however, we can eliminate all discrete ambiguities, for example, by measuring the cosines of $\phi(\epsilon)$, $\phi(\gamma)$, $\phi(\beta)-\phi(\alpha)$, $\phi(\beta)-\phi(\gamma)$, and $\phi(\beta)-\phi(\epsilon)$, and the sine of $\phi(\alpha)$. Thus the minimum number of measurements in this case that can bring about a completely unique determination of the amplitudes (apart from the one overall phase factor which is experimentally undeterminable) is 11. A few examples of fully complete sets are

$$\sigma, P, C_{NN}, D_{NN}, K_{NN};\; C_{LL}, C_{SS}, C_{LS};\; D_{LL}, D_{SS}, D_{LS}$$

or

$$\sigma, P, C_{NN}, D_{NN}, K_{NN};\; C_{LL}, C_{SS}, C_{LS};\; K_{LL}, K_{SS}, K_{LS} \tag{3.1}$$

or

$$\sigma, P, C_{NN}, D_{NN}, K_{NN};\; D_{LL}, D_{SS}, D_{LS};\; K_{LL}, K_{SS}, K_{LS} .$$

The existing data, however, at this energy are not numerous enough to constitute any of the above groups entirely. It might be noted that the third of the sets in Eq. (3.1) was in fact used[15] for a determination of the amplitudes of this reaction at 800 MeV.

IV. DISCUSSION OF THE RESULTS

In order to compare the amplitudes obtained by us with those obtained by previous work, we have to collect the amplitudes from all sources in the same frame. In Ref. 9 the amplitudes were given by Eq. (2.24) in the helicity frame. Reference 1 (and the SAID program described in it) tabulated the amplitudes in the frame used in Ref. 13, and displayed them in Table I of Ref. 1.

We chose the helicity frame for the intercomparison. The results are shown in Tables IV and V, which give, for each angle, our four solutions together with their uncertainties, the solution from Ref. 9, and the solution from Ref. 1. We should recall that the last of these was obtained with the help of an energy-dependent phase-shift analysis in which the unique solution at low energies and some assumptions about the smoothness of the energy dependence were used to arrive at a unique solution at higher energies.

We can observe the following features.

(1) One of our solutions (solution I) agrees very well (i.e., within experimental errors) with that of Ref. 9, which in turn agrees well with that of Ref. 1. It should be noted that the uncertainties of the amplitudes of Ref. 1 (which are much smaller than ours) depend on the assumptions built into the smooth energy dependence of the energy-dependent phase-shift analyses, and hence are not as firm a function as our uncertainties which do not depend on such assumptions.

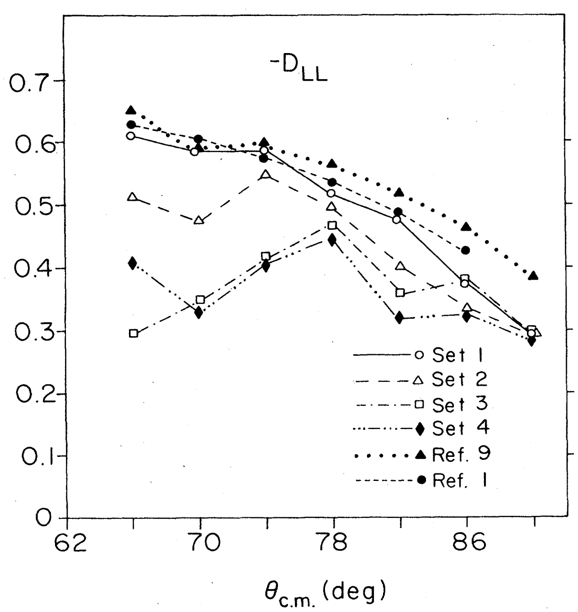

FIG. 1. Predictions of $-D_{LL}$ from the six sets of amplitudes in the laboratory system. The mixing angle ω of Ref. 9 has been taken into account.

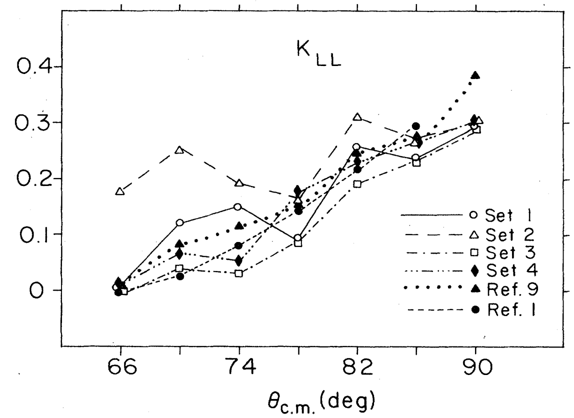

FIG. 2. Predictions of K_{LL} from the six sets of amplitudes in the laboratory system. The mixing angle ω of Ref. 9 has been taken into account.

(2) If we recalculate the already measured observables from Tables IV and V, we see that these observables can be divided into two groups. The first group, which includes the observables P, C_{NN}, D_{NN}, K_{NN}, C_{LL}, C_{LS}, D_{SS}, K_{SS}, and H_{NSS} agrees quite well for all six amplitude sets and for all five amplitudes. The second group, which includes D_{LS}, K_{SL}, H_{SNS}, and H_{NLS}, is in good agreement with some of the six sets for all amplitudes.

(3) In order to distinguish among our four solutions, some already measured observables could be measured more precisely and/or some new observables could be measured. In the former category, we have the four observables mentioned under (2) above where there appears to be some difference among the six solutions. In the second category of new observables to be measured, our investigation suggests that of the experimentally simpler measurements, D_{LL} and K_{LL} would be most suitable because such measurements would complete the sets indicated in Eq. (3.1) and thus provide a fully complete set for the determination of the amplitudes. Figures 1 and 2 show the predictions for these observables. Two other experimentally somewhat more difficult observables which

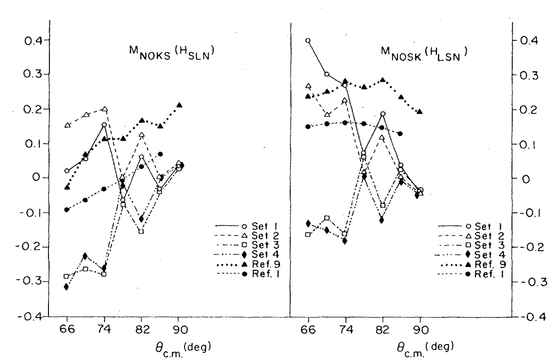

FIG. 3. Predictions of M_{NOSK} (H_{LSN}) and M_{NOKS} (H_{SLN}) from the six sets of amplitudes in the laboratory system. The mixing angle ω of Ref. 9 has been taken into account.

TABLE IV. The magnitudes of the five helicity amplitudes as given by our four solutions and by Refs. 9 and 1.

$\theta_{c.m.}$ (deg)	Solution	$\|a\|$	$\Delta\|a\|$	$\|b\|$	$\Delta\|b\|$	$\|c\|$	$\Delta\|c\|$	$\|d\|$	$\Delta\|d\|$	$\|e\|$	$\Delta\|e\|$
66	Set I	0.394	0.031	0.100	0.005	0.477	0.017	0.034	0.041	0.277	0.027
	Set II	0.421	0.040	0.100	0.005	0.425	0.024	0.063	0.051	0.313	0.024
	Set III	0.353	0.016	0.089	0.005	0.428	0.019	0.247	0.034	0.316	0.020
	Set IV	0.379	0.016	0.089	0.005	0.439	0.016	0.198	0.017	0.306	0.016
	Ref. 9	0.408		0.092		0.479		0.062		0.259	
	Ref. 1	0.398	0.002	0.086	0.001	0.483	0.003	0.079	0.004	0.269	0.004
70	Set I	0.425	0.036	0.085	0.005	0.448	0.019	0.070	0.044	0.292	0.024
	Set II	0.435	0.041	0.085	0.005	0.406	0.027	0.079	0.051	0.333	0.022
	Set III	0.388	0.016	0.080	0.005	0.431	0.018	0.208	0.026	0.309	0.018
	Set IV	0.397	0.016	0.080	0.005	0.419	0.017	0.196	0.025	0.320	0.018
	Ref. 9	0.413		0.086		0.460		0.061		0.289	
	Ref. 1	0.407	0.002	0.072	0.001	0.475	0.003	0.089	0.005	0.283	0.003
74	Set I	0.430	0.039	0.071	0.005	0.448	0.025	0.040	0.035	0.304	0.025
	Set II	0.432	0.039	0.071	0.005	0.436	0.026	0.039	0.038	0.318	0.022
	Set III	0.395	0.017	0.059	0.005	0.450	0.017	0.183	0.012	0.306	0.017
	Set IV	0.399	0.017	0.059	0.005	0.446	0.017	0.177	0.017	0.311	0.017
	Ref. 9	0.422		0.069		0.459		0.051		0.299	
	Ref. 1	0.417	0.002	0.058	0.001	0.463	0.003	0.098	0.005	0.299	0.003
78	Set I	0.415	0.020	0.042	0.005	0.451	0.016	0.142	0.040	0.312	0.016
	Set II	0.424	0.020	0.042	0.005	0.437	0.017	0.123	0.041	0.328	0.017
	Set III	0.415	0.020	0.042	0.005	0.451	0.016	0.142	0.040	0.312	0.016
	Set IV	0.424	0.020	0.042	0.005	0.437	0.017	0.123	0.041	0.328	0.017
	Ref. 9	0.429		0.043		0.450		0.081		0.315	
	Ref. 1	0.425	0.002	0.043	0.001	0.446	0.002	0.104	0.005	0.318	0.002
82	Set I	0.442	0.035	0.031	0.006	0.415	0.022	0.112	0.047	0.342	0.023
	Set II	0.441	0.038	0.031	0.006	0.393	0.023	0.125	0.049	0.363	0.021
	Set III	0.416	0.015	0.039	0.006	0.406	0.017	0.188	0.035	0.348	0.016
	Set IV	0.415	0.015	0.039	0.006	0.393	0.016	0.191	0.038	0.361	0.018
	Ref. 9	0.442		0.034		0.426		0.068		0.337	
	Ref. 1	0.433	0.002	0.029	0.000	0.426	0.002	0.109	0.006	0.339	0.002
86	Set I	0.427	0.020	0.018	0.006	0.415	0.015	0.134	0.042	0.355	0.015
	Set II	0.427	0.021	0.018	0.006	0.403	0.016	0.136	0.046	0.367	0.016
	Set III	0.427	0.020	0.018	0.006	0.415	0.015	0.134	0.042	0.355	0.015
	Set IV	0.427	0.021	0.018	0.006	0.403	0.016	0.136	0.046	0.367	0.016
	Ref. 9	0.436		0.020		0.419		0.071		0.358	
	Ref. 1	0.437	0.002	0.014	0.000	0.406	0.001	0.112	0.006	0.362	0.001
90	Set I	0.426	0.023	0.034	0.008	0.372	0.015	0.156	0.051	0.389	0.015
	Set II	0.426	0.023	0.034	0.008	0.372	0.015	0.156	0.051	0.389	0.015
	Set III	0.426	0.023	0.034	0.008	0.372	0.015	0.156	0.051	0.389	0.015
	Set IV	0.426	0.023	0.034	0.008	0.372	0.015	0.156	0.051	0.389	0.015
	Ref. 9	0.448		0.000		0.381		0.094		0.381	
	Ref. 1	0.444	0.003	0.000	0.000	0.379	0.001	0.124	0.006	0.380	0.001

are also suitable for distinguishing among the present four solutions are shown in Fig. 3.

V. CONCLUSION AND SUMMARY

We have seen that the now available data for *p-p* scattering at 579 MeV can eliminate any *continuum* of ambiguities in the determination of the reaction amplitudes but still contain a set of four discrete solutions. One of these solutions agrees with the one found previously in Ref. 9 and also with the one that can be obtained from an energy-dependent phase-shift analysis.[1] In order to choose among the four solutions, the observables D_{LS}

TABLE V. The values of the relative phases of the helicity amplitudes as given by our four solutions and by Refs. 9 and 1, at angles where data are available. $\phi(e)=0$.

$\theta_{c.m.}$ (deg)	Solution	$\phi(a)$	$\Delta\phi(a)$	$\phi(b)$	$\Delta\phi(b)$	$\phi(c)$	$\Delta\phi(c)$	$\phi(d)$	$\Delta\phi(d)$
66	Set I	190	10	228	7	155	11	65	10
	Set II	207	13	243	9	179	15	57	16
	Set III	198	10	285	7	184	11	86	10
	Set IV	183	6	280	3	174	6	79	7
	Ref. 9	184		234		162		79	
	Ref. 1	192	1	236	1.4	173	1	89	3
70	Set I	201	12	242	8	171	14	87	16
	Set II	212	13	252	9	188	15	92	24
	Set III	187	9	285	5	179	8	91	9
	Set IV	191	8	281	5	182	8	90	9
	Ref. 9	189		223		160		78	
	Ref. 1	191	1	235	1.4	172	1	88	2.6
74	Set I	204	13	239	9	180	14	118	10
	Set II	208	13	243	9	187	14	117	9
	Set III	185	4	283	3	181	4	98	6
	Set IV	185	6	284	4	182	5	97	8
	Ref. 9	187		225		161		92	
	Ref. 1	197	1	238	1.4	175	1	94	2.5
78	Set I	200	10	269	6	182	10	97	26
	Set II	200	6	272	1	187	6	94	4
	Set III	200	10	269	6	182	10	97	26
	Set IV	200	6	272	1	187	6	94	4
	Ref. 9	221		272		192		116	
	Ref. 1	198	1	238	1.5	176	1	97	2.3
82	Set I	203	13	281	13	176	14	93	36
	Set II	210	13	285	14	184	15	104	32
	Set III	195	11	303	9	179	11	94	9
	Set IV	198	11	306	9	184	12	97	9
	Ref. 9	194		229		164		97	
	Ref. 1	202	1	240	1.5	178	1	100	2.2
86	Set I	197	12	246	12	179	11	101	25
	Set II	200	9	249	8	184	10	104	24
	Set III	197	12	246	12	179	11	101	25
	Set IV	200	9	249	8	184	10	104	24
	Ref. 9	195		224		168		100	
	Ref. 1	203	1	240	1.5	179	1	101	2.2
90	Set I	204	8	5	90	186	10	107	15
	Set II	204	8	5	90	186	10	107	15
	Set III	204	8	5	90	186	10	107	15
	Set IV	204	8	5	90	186	10	107	15
	Ref. 9	192		83		167		100	
	Ref. 1	203	1	242	1.5	180	1	101	2.2

(and consequently K_{LS}), K_{SS}, and H_{NLS} (and consequently H_{SNS}) would have to be measured very accurately, and/or the observables D_{LL} and K_{LL}, now unmeasured, would have to be measured. Our conclusion, therefore, is that the first of these sets of observables deserves remeasuring, and that the second set of observables should be measured, in order to make the amplitude determination for this reaction at this energy fully unambiguous.

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