Hadronic contributions to the $g-2$ of the muon

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We compute the hadronic vacuum-polarization contributions a_v to the muon anomalous magnetic moment a_μ. To improve on previous calculations, we use QCD at large energies, and exploit the analyticity properties of the pion form factor. We find $a_v=(7100\pm105\pm49)\times10^{-11}$, with which we have $a_\mu(\text{theory})=(116\,592\,051\pm114\pm49)\times10^{-11}$. Purely QCD bounds for a_v are also presented.

I. INTRODUCTION

The largest uncertainties in the existing theoretical calculations of the gyromagnetic ratio of the muon, $\frac{1}{2}(g-2)=a_\mu$, come from the order-α^2 hadronic contributions to it, particularly from two sources: the medium-energy (1 to 2 GeV) vacuum-polarization contribution, and the low-energy pion-form-factor contribution. The uncertainties due to higher-order hadronic contributions, higher-order QED contributions, and weak-interaction contributions have been drastically diminished in the last years.[1(a),2]

In the present paper we consider the vacuum-polarization contributions to a_μ (Fig. 1), to be denoted by a_v. The reasons why we think it worthwhile to make a new calculation are the following. First of all, there are now available precise data in the region $1\lesssim t\lesssim 9$ GeV^2 for the (imaginary part of) the vacuum-polarization function $\Pi(t)$. Second, we also have new good data on the pion form factor $F(t)$, which allows a precise determination of $\text{Im}\Pi(t)$ at low t. Finally we have the fact that due to our understanding of strong interactions via QCD, the calculations present a theoretical solidity lacking in previous, more phenomenological approaches.[3]

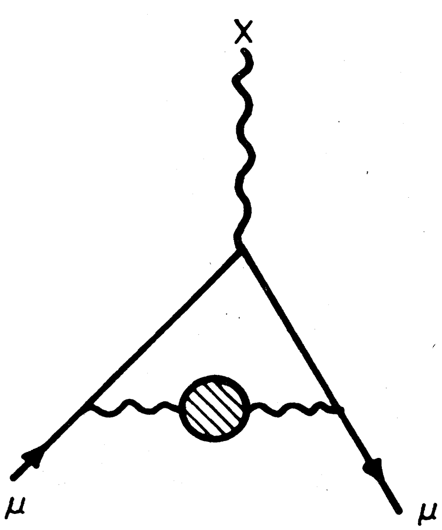

FIG. 1. Vacuum-polarization contribution to a_μ.

With respect to the last point, it is interesting to remark that pure QCD calculations of a_v are possible using techniques of QCD sum rules.[4] This is presented in Sec. II of this paper. The results are somewhat academic ($46\pm11\le10^9a_v\le1100\pm300$) in that the upper bound is way above experiment, but they have the merit to exist. In Sec. III we present the full calculation, using all available information and experimental data. The article is finished in Sec. IV, where we add all contributions to find

$$a_\mu^{\text{theor}}=[116\,592\,051\pm114(\text{stat})]\times10^{-11} \tag{1.1}$$

an error nine times smaller than the experimental one:[1(b)]

$$a_{\mu^+}^{\text{expt}}=(116\,591\,000\pm1200)\times10^{-11}\,,$$
$$a_{\mu^-}^{\text{expt}}=(116\,593\,600\pm1200)\times10^{-11}\,, \tag{1.2a}$$

or, if combining both

$$a_\mu^{\text{expt}}=(116\,592\,200\pm900)\times10^{-11}\,. \tag{1.2b}$$

Equation (1.1) is our best result, obtained with the method of analysis we consider more reliable.

If, however, we assume the P-wave isospin-1 $\pi\pi$ phase shift to be given $\pi\pi$-scattering analysis, we obtain a slightly different value:

$$a_\mu^{\text{theor}}=[116\,591\,997\pm110.5(\text{stat})]\times10^{-11}\,. \tag{1.3}$$

The discussion of why we think (1.1) is more reliable may be found in the text, Sec. III B.

We may discard (1.3) or, alternatively, consider the difference between it and (1.1) as a measure of possible systematic uncertainties in our analysis and write

$$a_\mu^{\text{theor}}=[116\,592\,024\pm112(\text{stat})\pm27(\text{syst})]\times10^{-11}\,. \tag{1.4}$$

II. PURE QCD BOUNDS

a_v may be written in terms of Π as[2]

$$a_v=\int_{4m_\pi^2}^{\infty}dt\,K(t)R(t)\,, \tag{2.1a}$$

where

$$R(t)=12\pi\,\text{Im}\Pi(t)=\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \tag{2.1b}$$

and

$$K(t)=\frac{\alpha^2}{3\pi^2 t}\int_0^1 dx\frac{x^2(1-x)}{x^2+(1-x)t/m_\mu{}^2} . \tag{2.1c}$$

By using a dispersion relation (differentiated to avoid subtractions) we may relate R to $\Pi(t)$ for $t \ll 0$, where it is calculable from QCD. We write

$$\Pi^{(N)}(t)=\frac{N!}{12\pi^2}\int_{4m_\pi{}^2}^{\infty} ds\frac{R(s)}{(s-t)^{N+1}} , \tag{2.2}$$

and from QCD we have[4,5]

$$\Pi'(t)=\frac{R^0}{-12\pi^2 t}\left[1+\frac{\alpha_s(-t)}{\pi}+r_2\left[\frac{\alpha_s(-t)}{\pi}\right]^2+\cdots\right. \\ \left. -\left[16\pi^2 A-\frac{2\pi}{3}B\right]\frac{1}{t^2}-\frac{C}{t^3}+\cdots\right] . \tag{2.3}$$

Here

$$R^0=3\sum^{n_f} Q_f{}^2$$

with Q_f the charge of the quark with flavor f, $r_2 \simeq 2.0-0.12n_f$, n_f being the (effective) number of flavors, and[4]

$$A=\frac{2}{R^0}\left[\frac{1}{g}f_\pi{}^2 m_\pi{}^2+\frac{1}{12}f_K{}^2 m_K{}^2\right]$$

$$\simeq(2.7\pm0.5)\times10^{-4}\ \text{GeV}^4 ,$$

$$B=\alpha_s\langle :G^2:\rangle_{\text{vac}}=0.044\pm0.015\ \text{GeV}^4 .$$

Finally, C is a combination of vacuum expectation values of the quark operators $\bar{q}t^a q\bar{q}'t^a q'$, estimated[4] to be

$$C\simeq-0.2\ \text{GeV}^6 .$$

The errors in (2.3) may conservatively be taken as follows: $\frac{1}{2}$ of the last known perturbative contribution; 60% the contribution of B, larger than the error given above (which was taken from Ref. 4), 20% that of A, and 100% that of C. Finally, we will take consistently the value

$$\Lambda=130^{+70}_{-40}\ \text{MeV}$$

for the QCD parameter. With this it follows that, to an overall 25% error, we may use (2.3) down to $|t| \gtrsim 1$ GeV^2.

The problem we now have is one of extremals: given $\Pi^{(N)}(t)$ for $t<-1$ GeV^2 from (2.3), find the allowed maximum and minimum values of a_v as given by (2.1). The existence of nontrivial extrema is guaranteed by the *positivity* of R which follows from (2.1b). The solution may be found with standard methods of Lagrange multipliers[6] and in this way we obtain the result quoted in the Introduction:

$$46\pm11 \le 10^9 a_v \le 1100\pm300 . \tag{2.4}$$

The errors come from the assumed errors in Eq. (2.3). The net result is fairly insensitive to the value $t_0 \sim -1$ GeV^2 at which we assume the QCD expression starts being valid, provided we do not go much above $t_0 \sim -2$ GeV^2: when we increase t_0 the lower bound decreases (the upper bound increases), but the errors also diminish and the effective result is not much altered.

More details about this part of the work will be published elsewhere;[7] here we finish this section with two comments. First of all, we could have reversed the argument, i.e., use the *experimental* value of a_μ to put bounds on C. We find

$$-0.4 \le C \le 0.10\ \text{GeV}^6 , \tag{2.5}$$

very close to the vacuum-saturation estimate[4] $C \simeq -0.20$ GeV^6. Second, it will be noted that the lower bound in (2.4) is close to the experimental value, $a_v \sim 70\times10^{-9}$; but the upper bound is hopeless. This is because $K(t)$ in (2.1c) is strongly peaked at small t. Hence we can add a narrow resonance at low energy with practically no effect at large $|t|$ where perturbative QCD is valid, but which would greatly increase the value of a_v. This emphasizes the importance of a correct treatment of the low-energy region, to which we now turn.

III. DETAILED CALCULATION OF a_v

A. The high-energy contributions

We will split the contributions to a_v into several pieces. First of all, we have the regions where perturbative QCD is expected to be valid. With the quoted value for Λ, $\Lambda=130^{+70}_{-40}$ MeV, this certainly should occur when $t \gtrsim 2$ GeV^2. Therefore, we use the QCD expression

$$R(t)=3\sum^{n_f} Q_f{}^2\left[1+\frac{\alpha_s(t)}{\pi}+r_2\left[\frac{\alpha_s(t)}{\pi}\right]^2+\cdots\right] \tag{3.1}$$

for all $t>2$ GeV^2 *except* near quark thresholds. Equation (3.1) fits perfectly the experimental data; use of it, instead of the last, allows us to get *smaller* errors. These have been estimated as (a) due to the error in Λ, and (b) the error committed by truncating the expansion (3.1), which we assume to be at most $\frac{1}{2}$ of the last term taken into account. Note that, for $t=2$ GeV^2, one already has $r_2[\alpha_s(t)/\pi]^2 \simeq 1.4\%$. Also, the error due to that of Λ is of approximately 1%.

A controversial point is whether one should use $\alpha_s(t)$ or $|\alpha_s(-t)|$ as expansion parameter in the timelike region.[8] Using the last would decrease R by some 1.5% at the lower end; the change in the value (and error) of a_v would be to

$$a_v(t>2\ \text{GeV}^2)=(840\pm10)\times10^{-11} \tag{3.2}$$

to be compared with our estimate (3.3) (see below). It will be seen that the difference is so minute that we may neglect this point. Near the $\bar{c}c$ and $\bar{b}b$ thresholds where perturbative QCD is not valid we have used experimental data.[9] The ψ and Υ resonances were evaluated in the narrow-resonance approximation. We have also included the contribution of a t quark whose mass we have allowed to vary between 20 GeV (its experimental lower bound) and 400 GeV (its theoretical upper bound). Its contribu-

TABLE I. Contributions from the $t > 2\ \mathrm{GeV}^2$ region.

Resonance or energy range [$t^{1/2}$ (GeV)]	Contribution to $10^{11}a_v$	Reference	Comments
1.4–3.1	560 ±10		QCD
3.1–3.6	44 ±5	9(b)	Exptl. data
3.6–4.9	81 ±3	9(c)	Exptl. data
4.9–9	67.5±1		QCD
9–14	19 ±1.2	9(d)	Exptl. data
14–∞	13 ±2		QCD
ψ,ψ'; $\Upsilon,\Upsilon',\Upsilon'',\Upsilon'''$	63 ±4.5	9(a)	Exptl. data

tion is negligible. The several contributions are listed in Table I; the overall result is

$$a_v(t > 2\ \mathrm{GeV}^2) = (847.5 \pm 12.6) \times 10^{-11} . \tag{3.3}$$

The region between $t = 0.8\ \mathrm{GeV}^2$ and $t = 2\ \mathrm{GeV}^2$ has been treated by using the existing experimental data[10,11] on the various channels that are open (including resonances). The lower bound, $0.8\ \mathrm{GeV}^2$ is forced on us: it is essentially the $\omega\pi$ threshold, so it is only below it that we can assume that R is dominated by the $\pi^+\pi^-$ channel. The results are insensitive to reasonable variations of the value $t = 2\ \mathrm{GeV}^2$ where we match the QCD calculation and the use of experimental data. The results for the $0.8 \le t \le 2\ \mathrm{GeV}^2$ region are summarized in Table II.

The ω resonance merits a few words on its own. To avoid problems with ω-ρ interference, we have separated off a region around $t = m_\omega^2$, $0.595 \le t \le 0.629\ \mathrm{GeV}^2$ where this interference is non-negligible. Thus the contribution of the 2π state is to be understood to be minus the ω-ρ interference, which has been evaluated by itself. The ω-ρ contribution is then

$$a_v(\omega\text{-}\rho) = (25 \pm 3) \times 10^{-11} \tag{3.4}$$

and the overall result, with obvious notation is

$$a_v(0.8 \le t \le 2\ \mathrm{GeV}^2, \omega) = (1404 \pm 100) \times 10^{-11} . \tag{3.5}$$

TABLE II. Contributions from the $0.8 \le t \le 2\ \mathrm{GeV}^2$ region.

Final state	Contribution to $10^{11}a_v$	Reference
$\pi^+\pi^-$	294 ±30	10(a)
K^+K^-	$224.5^{+34.4}_{-27.3}$	11(a)
$K_S^0K_L^0$	$196^{+33.2}_{-25.7}$	11(b)
3π (including ω)	492^{+90}_{-80}	11(c)
4π	166 ±21	11(d)
$5\pi, 6\pi$	6.3±3.4	11(c)

It will be noted that most of the error here comes from the low-energy ($0.8 \le t \le 2\ \mathrm{GeV}^2$) region. One could consider using QCD for *smaller* values of t; but this does not work. Up to $t \simeq 1.6$ (and considering only the u, d contributions) one still has agreement between the QCD contribution ($130 \pm 2.5 \times 10^{-11}$) and experiment ($125 \pm 11$); but below this they deviate. For example, the contribution between 1.2 and 2 is $366 \pm 7 \times 10^{-11}$ from QCD while using experimental data we get $278 \pm 25 \times 10^{-11}$. This discrepancy is likely due to nonperturbative contributions, not taken into account in (3.1). Therefore, we believe (3.5) to be the best one can do at present.

B. The low-energy contribution

For low energies the only states that contribute to $\sigma(e^+e^- \to \text{hadrons})$ are the 3π states (taken into account already with the ω contribution) and the $\pi^+\pi^-$ one. The 4π channel gives a negligible contribution until the opening of the $\omega\pi$ channel at $t \sim 0.8\ \mathrm{GeV}^2$. Therefore, we can write

$$a_v(t < 0.8\ \mathrm{GeV}^2, \text{ minus } \omega) \equiv a_\pi(0.8) , \tag{3.6}$$

where we define

$$\begin{aligned} a_\pi(0.8) &= \int_{4m_\pi^2}^{t_1} dt\, \rho(t)\, |F(t)|^2 , \\ \rho(t) &= \frac{1}{4}\left[1 - \frac{4m_\pi^2}{t}\right]^{3/2} K(t), \quad t_1 = 0.8\ \mathrm{GeV}^2 , \end{aligned} \tag{3.7}$$

$K(t)$ as in (2.1c). It is to be understood that the ω-ρ interference contribution has been subtracted from (3.7).

The information we have available on the pion form factor $F(t)$ is the following:

(i) $F(t)$ is a real analytic function with a cut for $4m_\pi^2 \le t < \infty$.

(ii) $F(0) = 1$.

(iii) From QCD, F decreases as $1/|t|$ (up to logarithms) when $|t| \to \infty$.

(iv) From Watson's theorem we know that the phase of F on the cut equals that of the P wave in $\pi\pi$ scattering:

$$\text{phase}\, F = \delta_1^1(t) . \tag{3.8}$$

Strict equality is valid only up to $t = 9m_\pi^2$, but it is known that, to a very good approximation (3.8) holds up to the $\omega\pi$ threshold, $t_1 \simeq 0.8\ \mathrm{GeV}^2$. As will be seen, however, we have problems with this information, to be discussed in Secs. II B 2 and III B 3 below.

(v) We have experimental data[10] on $F(t)$ not only for $t > 0$, but also in the spacelike region $t < 0$. It is the indirect inclusion of these data (via dispersion relations) that allows us to greatly decrease the errors in the contribution of F.

The value of $F'(0)$ (charge radius of the pion) is also known, but this does not add new information as it is extracted from the experimental data already described.

1. Calculation without use of data from ππ scattering

The information contained in (iv) is incorporated with the Omnés-Muskhelishvili method. We define

$$J(t)=\exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty}ds\frac{\delta_1^1(s)}{s(s-t)}\right]. \tag{3.9a}$$

We parametrize δ_1^1 as

$$2m_\pi t\left[\frac{1}{4}-\frac{m_\pi^2}{t}\right]^{3/2}\cot\delta_1^1(t)$$

$$=(m_\rho^2-t)\left[A_1+A_2\left[\frac{t}{4}-m_\pi^2\right]+A_3\left[\frac{t}{4}-m_\pi^2\right]^2\right]; \tag{3.9b}$$

we take m_ρ from the world average of the Particle Data Group tables,[9(a)] $m_\rho=0.769\pm3$ GeV, and a_1^1 from $\pi\pi$ scattering, $a_1^1=(0.04\pm0.003)m_\pi^{-3}$. A_1 is given in terms of the remaining parameters,

$$A_1=1/a_1^1(m_\rho^2-4m_\pi^2);$$

the A_2, A_3 are free parameters to be fitted; we find

$$A_2=-0.3518\ \text{GeV}^{-1},\quad A_3=2.624\ \text{GeV}^{-3},$$

which corresponds to a width of $\Gamma_\rho=158$ MeV [to be compared with the world-average value of Ref. 9(a), $\Gamma_\rho=154\pm5$ MeV].

As is known[12] there is essentially one fit for δ_1^1 consistent with m_ρ, Γ_ρ, for every value of a_1^1; we will for the moment fix a_1^1 to

$$a_1^1=0.038m_\pi^{-3}; \tag{3.10}$$

later on we will allow it to vary within its bounds. Although fit to the experimental $\pi\pi$ phase shifts was not required, we get a very good one *except* that the position of the ρ is shifted. As stated we will comment on this later on.

Defining the function

$$G(t)=F(t)/J(t), \tag{3.11}$$

we can translate the remainder of the information (i)—(v) as conditions on G as follows: since now G is analytic up to $t=t_1$, we can write a dispersion relation for it as

$$G(t)=1+\frac{t}{\pi}\int_{t_1}^{\infty}ds\frac{\text{Im}G(s)}{s(s-t)}; \tag{3.12}$$

moreover, G is known at points where we had experimental data on $F(t)$. Points to the right of t_1 have very little influence and we drop them. For points to the left of t_1, since G is real there the information on $|F|$ translates directly into values for G:

$$G(t_\alpha)=G_\alpha^{\text{expt}}\pm\epsilon_\alpha,\quad \alpha=1,\ldots,18 \tag{3.13}$$

with ϵ_α the errors induced by the experimental errors on $|F|$. Of the 18 points, nine are in the timelike and nine in the spacelike regions. They have been selected from Refs. 10.

In principle we could work directly with Eqs. (3.13), (3.12), (3.11), and (3.7), i.e., find the $\text{Im}G(t)$ that gave maximum and minimum values of $a_\pi(0.8\ \text{GeV}^2)$ compatible with the constraints (3.13). In practice it is better to find a central value for G, G_0, taking the difference $G-G_0$ as the unknown function. We thus write

$$G(t)=G_0(t)+\Delta(t),\quad \text{Im}\Delta(t)=[\text{Im}G_0(t)]\varphi(t) \tag{3.14}$$

with $\varphi(t)$ an unknown function. If we are careful and choose G_0 satisfying the requirements of Eq. (3.12), so will $\Delta(t)$.

For $G_0(t)$ we write

$$G_0(t)=f(t)+d+(b_1+b_2t+b_3t^2)\left[\frac{\sqrt{0.8}-\sqrt{0.8-t}}{t}\right]^4$$

$$+h_\omega(t)+a_1J_1(t)+a_2J_2(t). \tag{3.15}$$

The meaning, and values for the various terms (except h_ω) are given in the Appendix. The term h_ω takes into account the ω-ρ interference and, as explained in Sec. III A, is extracted from the analysis here. We took

$$h_\omega(t)=\frac{r_\omega}{\pi}\int_{p_\omega}^{q_\omega}ds\frac{(s-p_\omega)(s-q_\omega)}{s-t}$$

with

$$r_\omega=-6.38\times10^2\ \text{GeV}^{-4},\quad p_\omega=0.598\ \text{GeV}^2,$$

$$q_\omega=0.628\ \text{GeV}^2$$

which fits very well the experimental data. $G_0(t)$ itself fits the data with a $\chi^2/\text{DF}=0.95$ in the timelike region and $\chi^2/\text{DF}=2.2$ in the spacelike region.

If φ (hence Δ) were zero, this G_0 would provide us with a calculation of $a_\pi(0.8)$. We would then get our starting value,

$$a_\pi^0(0.8\ \text{GeV}^2)=4825\times10^{-11}. \tag{3.16}$$

Now, Δ is nonzero and is the function that will provide us with the corrections to (3.16), hence the final value and errors. To get them we have to make an extra assumption: otherwise we would, as in Ref. 13 get only *lower* bounds. A convenient one is to demand

$$|\varphi(t)|\le S(t). \tag{3.17a}$$

The function $S(t)$ is obtained as follows. In the region where experimental data for $|F|$ exist, we assume that the errors of $\text{Im}G$ are not (relatively) bigger than those on $|G|$ itself. We then let $S(t)$ interpolate smoothly between these errors, multiplied by a factor c. To be precise, we assume $(\delta\,\text{Im}G)/\text{Im}G\simeq c(\delta|G|^2)/|G|^2$. Then the errors in $\text{Im}G_0$ are given as

$$(\text{Im}G_0)(\varphi\pm S)=(\text{Im}G_0)\varphi\pm\delta\,\text{Im}G_0.$$

For large values of t where data do not exist, we take the errors of the phenomenological fit F_{EF} given below, again multiplied by c. This then corresponds to a χ^2 of

$$\chi^2/\text{DF}=c^2. \tag{3.17b}$$

For large values of t where experimental data are not available we have taken the errors of F as those obtained extrapolating the errors of a fit to $F(t)$ using the empirical formula

$$F_{\mathrm{EF}}(t)=\frac{A}{-t}\left[\ln\frac{-t}{\Lambda^2}\right]^{\nu},$$

$$A=(6.25\pm1.3)\times10^{-4}\ \mathrm{GeV}^2,\quad \nu=4\ ,$$

for data with $|t|\gtrsim 2\ \mathrm{GeV}^2$.

The best method to get the desired results is again that of Lagrange multipliers. We have to find the maximum and minimum of $a_\pi(0.8)$ as given by (3.7) subject to the constraints (3.13) and (3.17). We form the Lagrange functional[6]

$$\Lambda[\varphi]=2\int_{4m_\pi^2}^{t_1}dt\,Q(t)\Delta_\varphi(t)\,\mathrm{Re}G_0(t)+\int_{4m_\pi^2}^{t_1}dt\,Q(t)\Delta^2(t)$$
$$+\lambda\left[\sum_\alpha\left[\frac{G_\alpha^{\mathrm{expt}}-[G_0(t_\alpha)+\Delta_\varphi(t_\alpha)]}{\epsilon_\alpha}\right]^2-\chi_0^2\right]$$
$$+\int_{t_1}^{\infty}dt\{\mu_1(t)[S(t)-\varphi(t)]+\mu_2(t)[\varphi(t)+S(t)]\}\ . \tag{3.18}$$

Here $Q(t)=\rho(t)J(t)$ and Δ_φ is a functional of φ:

$$\Delta_\varphi(t)=\frac{t}{\pi}\int_{t_1}^{\infty}dt'\frac{\mathrm{Im}G_0(t')}{t'(t'-t)}\varphi(t')\ .$$

λ, μ_1, μ_2 are Lagrange multipliers. χ_0^2 is the overall error we admit for the experimental points, $G(t_\alpha)$ with t_α to the left of t_1.

The extremum condition is $\delta\Lambda/\delta\varphi=0$, which we write somewhat sketchily as

$$\Psi[\lambda,\varphi;t]-\mu_1(t)+\mu_2(t)=0\ . \tag{3.19}$$

The explicit expression for Ψ is very cumbersome, but may be easily obtained from (3.18) by the interested reader.

The solutions to the extremum equations (3.19) are given by functions φ_{ext} which run along the edges of (3.17a)

$$\varphi_{\mathrm{ext}}(t)=\pm S(t)$$

jumping from (+) to (−) at given points t_j. Some of these points are fixed; because Ψ is proportional to $\mathrm{Im}G_0(t)$, the points at which $\mathrm{Im}G_0(t)$ vanishes must necessarily correspond to jumps of $\varphi_{\mathrm{ext}}(t)$. Besides this, and to be able to verify the Lagrange equations, we must have an extra jump at $t=t_s$. The value of t_s is to be found by requiring

$$\Psi[\lambda,\varphi_{\mathrm{ext}};t_s]=0\ ,$$

so (3.19) can be satisfied with $\mu_{1,2}(t_s)=0$. One jump at a point t_s (besides the compulsory jumps at t_j) is sufficient to satisfy the Lagrange equations; but solutions exist with more than one jump. We have, however, checked numerically that solutions with two and three jumps are only relative extrema and thus the φ_{ext} we found with only one jump gives very likely the absolute maximum and minimum.

Before presenting the results, two last comments are in order. The problem posed [to find F verifying conditions (i)—(iv)] is *overdetermined.* Because of statistical fluctuations, it is to be expected that no F will exist compatible with [(i)—(iv)] if the χ^2 is taken too small. This indeed happens: if we decrease c below $c^2\sim1.5$, and χ_0^2/DF below unity, no solution exists. To be on the safe side, we have given generous error allowances, restrict ourselves to values $c^2>2$, $\chi_0^2/\mathrm{DF}>1.4$. Then, we have up till now kept δ_1^1 fixed, by fixing a_1^1 to $a_1^1=0.038m_\pi^{-3}$. If we allow a_1^1 to vary within the bounds (3.10), the extremal values of $a_\pi(0.8)$ do not change appreciably; but the starting value $a_\pi^0(0.8)$ does. What we can do is, for every choice of c^2, χ_0^2, select a value of a_1^1 so that $a_\pi^0(0.8)$ is centered between the extrema of $a_\pi(0.8)$. (This is the reason why $a_\pi^0(0.8)$ varies from one entry to another in Table III.) Generally speaking, this requires a value of a_1^1 on the small side, $a_1^1\simeq0.038m_\pi^{-3}$.

TABLE III. Central values and bounds for $a_\pi(0.8\ \mathrm{GeV}^2)$ not using $\pi\pi$ phase shifts. (a) Global χ^2. (b) Independent χ^2 in spacelike and timelike regions.

c^2	χ_0^2/DF	$10^{11}a_\pi(0.8\ \mathrm{GeV}^2)$
	(a)	
4	3	4879±410
4	2	4865±257
4	1.4	4856±70
2	2	4883±257
2	1.4	4848±31
	(b)	
4	2	4847±163
2	2	4834±130
2	1.4	4861±8

The results of our analysis are presented in Table III. In Table III(a) we give the results of the straightforward application of the Lagrange equations. In Table III(b) we require *separate* χ_0^2 for points in the timelike and spacelike regions. This is because the φ_{ext} we find with a global χ_0^2 manage to get it by going through the spacelike points, but missing widely the timelike ones or vice versa: so such φ_{ext} are not very believable physically. This is why we attach more credit to the calculation with separate χ^2, even if it is slightly amplified, than that with a global one. The results are quite stable against variations of c^2 and χ_0^2. We believe a reasonable value to be that given in the last entry of Table III(a):

$$a_\pi(0.8)=(4848\pm31)\times10^{-11}\ . \tag{3.20}$$

2. *Calculation using $\pi\pi$ phase shifts*

We could have found J in Eqs. (3.9) and (3.11) by fitting δ_1^1 to experimental $\pi\pi$ phase shifts.[14]

Then we write

$$2tm_\pi\left[\frac{1}{4}-\frac{m_\pi^2}{t}\right]^{3/2}\cot\delta_1^1(t)$$

$$=\frac{1}{a_1^1}+\left[\frac{t}{4}-m_\pi^2\right]\sum_{j=0}^{2}B_j\left[\frac{t}{4}-m_\pi^2\right]^j . \quad (3.21)$$

The scattering length is taken as before and we find

$$B_0=0.46\text{ GeV},\ B_1=-0.3\text{ GeV}^{-1},$$

$$B_2=-0.915\text{ GeV}^{-3}.$$

This gives a value[12,14]

$$m_\rho^{\pi\pi}=778\pm2\text{ MeV} \quad (3.22)$$

quite incompatible with what we found before, and the world average[9(a)] of 769±3 MeV.

In F both pions are real; in $\pi\pi$ scattering one of them is off-shell. This may shift the effective ρ mass, which is the reason why we trust more the previous calculation. Indeed, if we now parametrize again G_0, even adding more parameters in the form of extra contributions a_ih_i

$$h_i(t)=\frac{1}{\pi}\int_{p_i}^{q_i}ds\frac{(s-p_i)(s-q_i)}{s-t},$$

we only get a χ^2/DF of 1.1 in the timelike region and of 3.37 in the spacelike one. This discrepancy will be discussed in more detail by one of us (J.A.C.) elsewhere.[15]

The rest of the analysis is exactly like in Sec. III B 1 above and we do not repeat it. The results are reported in Tables IV(a) and IV(b). The best value for a_π with this method of analysis is, we believe, the last entry of Table IV(b), viz.,

$$a_\pi(0.8\text{ GeV}^2)=(4794\pm14)\times10^{-11} . \quad (3.23)$$

That the difference between (3.20) and (3.23) is not too large, in spite of the wide difference of methods, is very encouraging. Although for the reasons already discussed we believe (3.20) to be the best value, we may combine both and take the variation as a measure of the systematic uncertainties of our analysis. In this way we arrive at the value quoted in the Introduction, Eq. (1.4).

TABLE IV. Central value and bounds for $a_\pi(0.8\text{ GeV}^2)$ using $\pi\pi$ phase shifts. (a) Global χ^2. (b) Independent χ^2 in spacelike and timelike regions.

c^2	χ_0^2/DF	$10^{11}a_\pi(0.8\text{ GeV}^2)$
	(a)	
4	2	4791±112
4	1.8	4794±46
	(b)	
4	2	4788±46
4	1.8	4794±14

3. Discussion

To finish with this section we will give a few comments on our results, as well as on possible ways to improve them.

The method of Lagrange multipliers gives, of course, the extrema of a_π; but the functions which realize these extrema are clearly unphysical. Indeed they contain jumps of ImG, therefore ReG presents logarithmic singularities. A possible way to avoid this would be to incorporate in the formalism experimental points *on the cut,* $t_1\leq t<\infty$. We have not done this for two reasons: first this would complicate enormously the analysis; and second, the improvement would not be worth the added effort. In fact, as follows from our analysis it is practically impossible to get a χ^2 of unity, if using only points to the left of the cut. The reason for this is very likely, that systematic uncertainties exist for data on the spacelike region (where one has to extrapolate one unphysical pion mass to get F). Moreover, an improvement on the "large" t ($t\gtrsim1$) region for F would have a negligible influence on a_π. Thus although our results might improve, they would do so very slightly and at the loss of some credibility. Substantial improvement would only come with more precise *experimental* data in the ρ region and its vicinity.

IV. CONCLUSIONS

In the previous section we have discussed the various contributions to a_v. Adding also other contributions,[1(a)] we may collect here the different pieces:

$$a_\mu(\text{QED})=(116\,584\,802\pm30)\times10^{-11} , \quad (4.1a)$$

$$a_\mu(\text{weak})=(195\pm1)\times10^{-11} , \quad (4.1b)$$

$$a_\mu(\text{higher hadronic})=(-46\pm32)\times10^{-11} , \quad (4.1c)$$

$$a_v(t>2\text{ GeV}^2)=(847\pm13)\times10^{-11} , \quad (4.1d)$$

$$a_v(0.8\leq t\leq2;\omega)=(1404\pm100)\times10^{-11} , \quad (4.1e)$$

$$a_\pi(0.8\text{ GeV}^2)=(4848\pm31)\times10^{-11} . \quad (4.1f)$$

We can compare the most recent evaluation[1(a)] of a_v, using only experimental data on R:

$$a_v^{(1)}=(7070\pm60\pm170)\times10^{-11} \quad (4.2)$$

which corresponds to our result [obtained by adding (4.1d)—(4.1f)],

$$a_v=(7100\pm105\pm49)\times10^{-11} , \quad (4.3)$$

where the first error is statistical and the second is systematic. The improvement is not dramatic but is substantial.

Let us comment briefly on the difference between (4.2) and (4.3): Our statistical error is larger than that quoted in Ref. 1(a), due essentially to the three-pion contribution in the ϕ region, where we have been more pessimistic. We have already explained in the text how we have estimated the systematic error coming from the low-energy two-pion contribution. We have compared two calculations based on independent experimental data and we have assumed that the difference provides a rough estimation of the systematic error. This procedure has led us to a better result than that depicted in (4.2). All the other contributions to the systematic error, which are less dominant, go as in Ref. 1(a) except for the pieces which have been evaluated using QCD expressions and do not give any contribution to the systematic error.

Our final result is obtained adding all the contributions (4.1). Note that, since they are independent, we can add the errors statistically. In this way we get the figure quoted in the Introduction,

$$a_\mu(\text{theory})=(116\,592\,051\pm114\pm49)\times10^{-11}\ . \qquad (4.4)$$

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APPENDIX

Here we discuss the parametrization (3.15) (except h_ω). We have

$$f(t)=C\frac{0.86-t}{t}\left[\ln\frac{0.86-t}{\Lambda^2}-\ln\frac{0.86}{\Lambda^2}\right]^4 ,$$

$$C=1.02\times10^{-3}\ .$$

J_1, J_2 correspond to the ρ', ρ'' resonances, taken to be $\omega\pi$ and $\rho^+\rho^-$ resonances:

$$J_1(t)=\frac{m_1{}^2+m_1\gamma_1\sqrt{0.842}}{m_1{}^2-t+m_1\gamma_1\sqrt{0.842-t}}\ ,$$

$$J_2(t)=\frac{m_2{}^2+m_2\gamma_2\sqrt{2.372}}{m_2{}^2-t+m_2\gamma_2\sqrt{2.372-t}}\ ,$$

$$m_1=1.277\ \text{GeV},\quad m_2=1.624\ \text{GeV}\ ,$$

$$\gamma_1=0.1883,\quad \gamma_2=0.6\ .$$

Moreover,

$$a_1=0.050\,81,\quad a_2=-0.077\,34\ .$$

$f(t)$ provides the asymptotic behavior of F. The background parameters of Eq. (3.15) are

$$b_1=0.097\,63\ \text{GeV}^4,\quad b_2=0.5654\ \text{GeV}^2\ ,$$

$$b_3=-0.8461\ .$$

Finally d is given in terms of the other parameters by the condition $G_0(0)=1$.

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[15]A full discussion of the $\pi\pi$ and F incompatibility shall be given elsewhere by one of us (J.A.C.).