Apparatus-dependent contributions to $g-2$?

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The University of Washington $g-2$ experiments have progressed to the marvelous precision of 10^{12} in their measurement of the magnetic moment of the electron. A further improvement of a factor of 10 should occur in the near future. Concomitant with this accuracy is the necessity of understanding small corrections. These experiments employ a Penning trap whose electrodes behave as conducting walls for a microwave cavity. We investigate the effect of this cavity on the spin and cyclotron frequencies. Previous work, employing calculations that are not gauge invariant, implies that the present level of accuracy cannot be exceeded because of cavity effects on the spin-precession frequency. In contrast to this work, we find no significant correction to the spin-precession frequency, but we do find a correction to the cyclotron frequency which may be important.

I. INTRODUCTION AND SUMMARY

A University of Washington experiment[1] isolates a single electron or a single positron in a Penning trap to obtain a "geonium atom" and then measures the magnetic moments of these elementary particles with unprecedented precision. The recent results[2] for the g factor of the electron,

$$\frac{g-2}{2}=1\,159\,652\,193(4)\times 10^{-12}\ , \tag{1.1}$$

display a relative accuracy of 4×10^{-9}. This extraordinary accuracy demands that a careful examination be made of possible sources of systematic error—physical mechanisms that alter the experimental determination of $(g-2)/2$. The electrostatic field in the Penning trap is produced by electrodes that also behave as conducting walls which form a microwave cavity. In this paper we shall investigate the action of these effective cavity walls on the experimental determination of $(g-2)/2$. Part of our motivation for this investigation arises from a recent paper[3] whose result implies that the presence of this cavity makes it essentially impossible to improve the present precession. This and previous related work[4] are incorrect. Thus we shall present our work in some detail starting from first principles.

The electron in the geonium experiment orbits in a strong magnetic field at the cyclotron frequency ω_c, and its spin processes at the frequency ω_s. The g factor is related to these frequencies by

$$\frac{g-2}{2}=\frac{\omega_s-\omega_c}{\omega_c}=\frac{\omega_a}{\omega_c}\ . \tag{1.2}$$

We see that corrections to the spin frequency ω_s and the cyclotron frequency ω_c are of equal importance. Moreover, since

$$\omega_a=\omega_s-\omega_c\simeq\frac{\alpha}{2\pi}\omega_c\simeq\omega_c\times 10^{-3}\ , \tag{1.3}$$

we see that to determine $(g-2)/2$ to the present relative accuracy of about one part in 10^9, the fractional corrections to both ω_c and ω_s must be smaller than one part in 10^{12} or else these corrections must be computed with appropriate accuracy and taken into account in the experimental analysis. The experiments may well improve by an order of magnitude in accuracy in the near future, in which case these corrections should be bounded by $\delta\omega_c/\omega_c\leq 10^{-13}$ and $\delta\omega_s/\omega_s\leq 10^{-13}$.

Our results apply to a cavity with arbitrary geometry. However, since our aim is to explain carefully the basic physics and the order of magnitude of the effects of cavity walls on the geonium experiment, we being in Sec. II by replacing the Penning-trap electrodes, which have complicated shapes, by a single perfectly conducting plane. We further simplify the physics by considering a classical particle with charge e moving about a cyclotron orbit in a magnetic field $\mathbf{B}$ at a normal distance $\mathbf{R}$ from the conducting wall. The effect of the wall is exactly reproduced by an image of charge $-e$ at a distance $-\mathbf{R}$ on the other side of the wall moving with the same velocity except that the sign of its component parallel to $\mathbf{R}$ is reversed. See Fig. 1. The image construction familiar for the electrostatic case generalizes to the full electromagnetic field since the retardation times to the plane from the charge and its image are identical.

The image charge produces a retarded radiation field which acts on the true charge, with the leading spatial behavior being given by $1/(2R)$ in contrast to the $1/(2R)^2$ behavior of the Coulomb force. Since the electron in the Penning trap is at a large distance from the electrodes in comparison with the wavelength $2\pi c/\omega_c$, the leading behavior of the radiation field gives the most significant contribution. We show in Sec. II that this effect produces a shift in the cyclotron frequency given to order e^2 by

$$\frac{\delta\omega_c}{\omega_c}=\frac{r_0}{4R}[1+(\hat{\mathbf{R}}\cdot\hat{\mathbf{B}})^2]\cos(2\omega_c R/c)\ . \tag{1.4}$$

Here $r_0=e^2/4\pi mc^2\simeq 2.8\times 10^{-13}$ cm is the classical electron radius and $\hat{\mathbf{R}}$ and $\hat{\mathbf{B}}$ are unit vectors along the wall normal and magnetic field, respectively. The cosine fac-

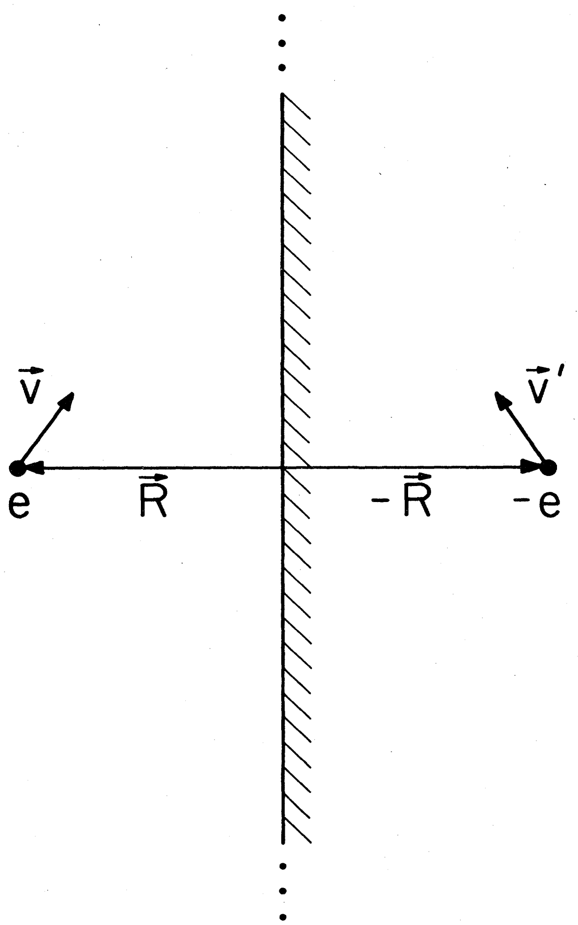

FIG. 1. Image construction of the electromagnetic field outside the conducting wall denoted by the hatched surface.

tor results from the retarded propagation of the electromagnetic field. Since for a typical trap $R \approx 1/3$ cm and $\omega_c \approx 1\times 10^{12}$ sec^{-1}, giving $2\omega_c R/c \approx 20$, the correction in Eq. (1.4) may be as large as several parts in 10^{13}. We find that a more careful examination is warranted of the cavity-induced frequency shift of the cyclotron frequency. This has recently been done[5] for the case of a cylindrical cavity whose walls have a finite conductivity. It is worth noting that the cavity also changes the decay rate of the cyclotron orbit from its free-space value.[6]

The cavity alters, in principle, the spin-precession frequency ω_s as well as the cyclotron frequency. Although the spin $\frac{1}{2}$ of the electron is intrinsically quantum mechanical, the size of this effect can be estimated semiclassically using the image method. In this case an image magnetic dipole moment of order $e\hbar/mc$ produces a retarded magnetic radiation field which contains the factor $\cos(2\omega_s R/c)$. The gradient operation associated with the moment can act either on the $1/R$ behavior of the monopole field or on the cos $(2\omega_s R/C)$ retardation factor. Since $2\omega_s R/c \approx 20$, the latter is the more important contribution. Thus the fractional shift in the spin frequency $\delta\omega_s/\omega_s$ is changed in order of magnitude from the result (1.4) for the corresponding cyclotron shift by an additional factor of $\hbar\omega_s/mc^2 \simeq 10^{-9}$. In contrast to previous

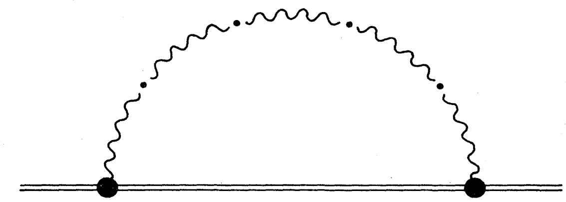

FIG. 2. Graphical illustration of the energy shift given in Eqs. (1.6) and (1.7). The double line represents the electron propagation in the external binding fields. The dotted-wavy line represents the difference of the photon propagators for the cavity and free space.

work,[3,4] we conclude that the cavity effect on the spin is completely negligible.

We verify and generalize these results to an arbitrary cavity in Sec. III using the exact apparatus of quantum electrodynamics. We avoid short-distance singularities and the attendant necessity of renormalization by considering only the alteration brought about by the cavity. We work only with the modification of the photon propagator

$$\delta D_{\mu\nu} = D_{\mu\nu}^{\text{cavity}} - D_{\mu\nu}^{\text{free space}} , \tag{1.5}$$

which is nonsingular but which forces $D_{\mu\nu}^{\text{cavity}}$ to satisfy the correct boundary conditions at the cavity walls. In the absence of the cavity, the electron is bound in a small orbit of energy E described by the Dirac wave function $\Psi_E(\mathbf{r})$. To order e^2, the presence of the cavity shifts this energy eigenvalue to $E+\delta E$, where

$$\delta E = \int d^3r\, d^3r' \Psi_E(\mathbf{r}) \delta\mathscr{M}(E;\mathbf{r},\mathbf{r}') \Psi_E(\mathbf{r}') , \tag{1.6}$$

in which

$$\delta\mathscr{M}(E;\mathbf{r},\mathbf{r}') = ie^2 \int \frac{dE'}{2\pi} \gamma^\mu S(E';\mathbf{r},\mathbf{r}')\gamma^\nu \delta D_{\mu\nu}(E-E';\mathbf{r},\mathbf{r}') \tag{1.7}$$

is the electron self-energy or mass operator. The Green's function $S(E';\mathbf{r},\mathbf{r}')$ describes the propagation of the electron in the binding field provided by the Penning trap with its strong magnetic field. This form for the energy shift, which is illustrated in Fig. 2, is the familiar form which is used, for example, in modern calculations of the Lamb shift.[7]

We use the radiation gauge for $\delta D_{\mu\nu}$ to simplify the calculation. In this gauge the leading contributions are obtained by performing the dipole approximation about the center of the well-localized electron. In the radiation gauge

$$\delta D_{00}(E;\mathbf{r},\mathbf{r}') = -\delta\mathscr{D}(\mathbf{r},\mathbf{r}') \tag{1.8}$$

is independent of energy corresponding to an instantaneous Coulomb interaction with image charges that represent the effect of the cavity walls. We find that in the leading dipole approximation

$$\delta E_{\text{Coulomb}} = -\tfrac{1}{2} e^2 \delta\mathscr{D}(\mathbf{R},\mathbf{R}) . \tag{1.9}$$

This is just the classical potential energy of the electron at its average position in the presence of the induced charges on the cavity walls. This potential energy is independent of the spin and velocity of the electron, and it does not alter the spin and cyclotron frequencies. [A more complete discussion of the Coulomb effect is presented in Ref. 5(b).]

At this point we should pause to clarify our notation. For the simple case of a single plane, $\mathbf{R}$ is the vector from the plane to the center of the electron orbit shown in Fig. 1. In general, $\mathbf{R}$ denotes the average position of an electron inside an arbitrary cavity. For the particular case of interest, the Penning trap, $\mathbf{R}$ denotes the position of the center of the trap where the electron is well localized. In this case, we measure $\mathbf{R}$ from a convenient origin located on a trap electrode so that the *magnitude* of $\mathbf{R}$ *also denotes the size of the trap.*

To evaluate the contribution of the radiation-field propagator δD_{kl}, we express the propagator in terms of a dispersion relation

$$\delta D_{kl}(\omega;\mathbf{r},\mathbf{r}')=\int_0^\infty d\lambda^2\frac{d_{kl}(\lambda;\mathbf{r},\mathbf{r}')}{\lambda^2-\omega^2-i\epsilon} . \tag{1.10}$$

This makes manifest the analytic properties of the quantum propagator with the $i\epsilon$, $\epsilon\to 0^+$, providing the positive-frequency boundary conditions of the time-ordered product. We then perform a Foldy-Wouthuysen transformation to cast the expression for $\delta E_{\rm rad}$ in a form suitable for the appropriate nonrelativistic approximation. In this way we find that to leading order

$$\delta E_{\rm rad}=(\psi_E,H_1\psi_E) , \tag{1.11}$$

where ψ_E is the nonrelativistic counterpart of the Dirac wave function Ψ_E and H_1 is an effective perturbation Hamiltonian given by

$$H_1=-\frac{e^2}{m^2}\int_0^\infty d\lambda[\Pi^k+\tfrac{1}{2}(\boldsymbol{\sigma}\times\nabla)^k]\frac{1}{H_0-E+\lambda-i\epsilon} \times[\Pi^l+\tfrac{1}{2}(\boldsymbol{\sigma}\times\nabla')^l]d_{kl}(\lambda;\mathbf{r},\mathbf{r}')]\,|_{\mathbf{r}=\mathbf{r}'=\mathbf{R}} . \tag{1.12}$$

Here $\boldsymbol{\Pi}=\mathbf{p}-e\,\mathbf{A}$ is the kinetic momentum operator, $\boldsymbol{\sigma}$ are the Pauli spin matrices, and H_0 is the nonrelativistic Hamiltonian of the electron. Not surprisingly, formula (1.12) is precisely the result one obtains by directly coupling the quantized radiation field to the nonrelativistic, quantum-mechanical description of the electron motion, including the spin contribution to the current. We give the derivation of this result from quantum electrodynamics in Sec. III since the relevant literature is in error.[3,4]

To obtain the shift in the cyclotron frequency, we note that the operators Π^k can be expressed as a linear combination of raising and lowering operators which shift the energy eigenvalue E by $\pm\hbar\omega_c$. Moreover, once this shift action is performed, the order of the raising and lowering operators is immaterial since their commutator is a c-number which does not contribute to the energy difference $\Delta\delta E$ between adjacent levels which produces a frequency shift. Accordingly, we may make the replacement

$$\Pi^k\frac{1}{H_0-E+\lambda+i\epsilon}\Pi^l \to\tfrac{1}{2}\Pi^k\Pi^l\left[\frac{1}{\omega_c+\lambda+i\epsilon}+\frac{1}{-\omega_c+\lambda+i\epsilon}\right] \tag{1.13}$$

in Eq. (1.12). The terms in the large parentheses in Eq. (1.13) combine with the dispersion integral in Eq. (1.12) to reform the photon propagator correction δD_{kl}, and we find that, effectively,

$$H_1=-\frac{e^2}{m}\delta D_{kl}(\omega_c;\mathbf{R},\mathbf{R})\frac{\Pi^k\Pi^l}{2m} . \tag{1.14}$$

Now in the quantum expectation value

$$\left\langle\frac{\Pi^k\Pi^l}{2m}\right\rangle=\tfrac{1}{2}(\delta^{kl}-\hat{B}^k\hat{B}^l)\left\langle\frac{\Pi^2}{2m}\right\rangle =\tfrac{1}{2}(\delta^{kl}-\hat{B}^k\hat{B}^l)E , \tag{1.15}$$

and since $\Delta E=\hbar\omega_c$, we see that Eq. (1.15) produces the frequency shift

$$\delta\omega_c=-\frac{e^2\omega_c}{2m}(\delta^{kl}-\hat{B}^k\hat{B}^l)\delta D_{kl}(\omega_c;\mathbf{R},\mathbf{R}) . \tag{1.16}$$

Since $\omega_c>0$, the combination $\omega_c^2-i\epsilon$ in the dispersion integral (1.10) for the photon propagator is equivalent to $(\omega_c-i\epsilon)^2$. Thus, the time-ordered function δD_{kl} can be replaced by the corresponding retarded Green's function. Therefore, the result (1.16) for the cyclotron frequency shift is purely a classical result. Since the propagator is complex, $\delta\omega_c$ contains an imaginary part which gives the alteration in the cyclotron decay rate brought about by the cavity. The real part of $\delta\omega_c$ gives the frequency shift.

The spin contribution to the energy shift can be readily determined from (1.12) to be of the form $\hbar\omega_s(r_0/R)(\hbar\omega_s/mc^2)f(2\omega_s R/c)$ and thus of order $\hbar\omega_s/mc^2$ smaller than the shift of the cyclotron frequency: The spin contribution comes from the $\boldsymbol{\sigma}\times\nabla$ terms where, by the same argument as above, the gradient acts on the retarded photon propagator δD_{kl} which is of the form $f(R\omega_s/c)/R$ where f is of order 1. The leading term in the gradient then yields a factor of order ω_s/c from the retardation; thus, the spin term is of order

$$(\hbar\omega_s/c)^2/\langle\Pi^2\rangle\approx\hbar\omega_s/2mc^2$$

times the shift in the cyclotron frequency.

These expressions are the main conclusions of our work. The leading-order corrections to the energy levels of an electron in a cavity are purely the classical results. The correction to the spin is entirely negligible regardless of the cavity geometry but the corrections to the cyclotron frequency are important. The detailed dependence on the cavity geometry is included in the expression for δD_{kl}.

Previous publications have dealt with the special cases of a single-plane conducting wall[4] and a pair of conducting walls.[3] In these works much larger effects were reported; however, the calculations as presented are not gauge invariant. In order to understand the origin of the discrepancy, we will show that a gauge-invariant calculation yields a vanishing contribution of the order reported there so that the corrected calculation is consistent with ours. Since the form and order of magnitude of our result is independent of the details of the cavity (or plates), we shall not present a detailed comparison in the two-plate case although the result may be obtained from that of the single plate by doing the (convergent) sum over the infinite set of images.

It is a simple matter to compute δD_{kl} for a single-plane conducting wall (see the Appendix) and verify Eq. (1.4). Clearly, the frequency shift for an arbitrary cavity will also be of roughly the same order of magnitude as that for the wall.

The results of previous workers[3,4] on the effects which we have considered are not gauge invariant. This is explicitly stated in the discussion of Eq. (6.1) in Ref. 3. It is easy to see that the result of Ref. 4 is also not gauge invariant. These authors make a Foldy-Wouthuysen reduction of the complete Hamiltonian for the radiation field plus an electron near a plane wall. In our notation, the corrected version of one of their contributions [their Eq. (2.5)] reads

$$H_1' = \frac{e^2}{16m^2R^2}\boldsymbol{\sigma}\cdot\hat{\mathbf{R}}\times\boldsymbol{\Pi} \ . \tag{1.17}$$

Clearly

$$\langle \boldsymbol{\Pi} \rangle = im\langle [H,\mathbf{r}]\rangle = 0 \ ,$$

and so Eq. (1.17) gives a vanishing result. Instead of this evaluation, the authors of Ref. 4 write $\boldsymbol{\Pi}=\mathbf{p}-e\mathbf{A}$ and assume that the expectation value of the gauge-noninvariant canonical momentum $\mathbf{p}$ vanishes. They then use a gauge such that $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{R}$ and so find that Eq. (1.17) yields a large contribution of order $(e^2/mR)(e/m)\boldsymbol{\sigma}\cdot\mathbf{B}$ rather than the correct result: zero. A similar (noncompensating) error is made in the authors' evaluation of the remaining term in their Eq. (2.6). This may be made manifest by observing that the single factor of R in the denominator of their final result Eq. (2.8) arises from a factor of R^{-2} canceled by the $\mathbf{R}$ from $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{R}$ using a particular choice of origin. Thus these earlier results depend upon the position of a "center" of a perfectly homogeneous magnetic field. They are altered by the translation

$$\mathbf{A}=\tfrac{1}{2}\mathbf{B}\times\mathbf{R}\rightarrow\tfrac{1}{2}\mathbf{B}\times(\mathbf{R}-\mathbf{R}_0) \ ,$$

which is equivalent to a gauge transformation. Thus there is an additional term in their expression, $e^3\boldsymbol{\sigma}\times\hat{\mathbf{R}}\cdot\mathbf{R}_0/m^2R^2$, which depends upon the origin (gauge). Since any physical quantity must be gauge invariant, these results must be in error and not even their order of magnitude can be trusted.

Before turning to give the details of our calculations, we should emphasize that the electron is very well localized in the $g-2$ experiment. Thus, an approximation in which the photon propagator is represented by a mode sum appropriate to a cavity but the electron propagator represents unbound, plane waves is not relevant to the experiment. Such a computation gives an unphysical logarithmic divergence because the effect of the image charge diverges when the electron approaches the wall, which it does with sufficient probability when it is represented by plane waves. Indeed, if the magnetic field is treated as a first-order perturbation, an essentially free electron introduced into a conducting cavity will be attracted by an image charge, hit the wall, and be lost from the trap.

II. CLASSICAL CALCULATION

To assess the size of the shift in the cyclotron frequency brought about by a microwave cavity, we replace the cavity walls by a single, perfectly conducting plane. The electron of charge e moves about its cyclotron orbit at velocity $\mathbf{v}(t)$ a distance $\mathbf{R}$ from the wall. As discussed in the preceding section, the electromagnetic properties of the conductor are reproduced by an image charge $-e$ at a distance $-\mathbf{R}$ on the other side of the plane. The image charge moves at velocity $\mathbf{v}'(t)$ which is equal to $\mathbf{v}(t)$ except that the sign of its component parallel to $\mathbf{R}$ is reversed. The wavelength c/ω_c is very small in comparison with the distance to the plane. Hence, the leading effect of the plane is the long-distance radiation field produced by the motion of the image, a field that falls off as $1/2R$. Moreover, the size of the cyclotron orbit is also very small in comparison with the distance to the plane and so the dipole approximation suffices, where $\mathbf{R}$ is taken to be the position of the center of cyclotron orbit. Thus the cyclotron equation of motion is altered by the addition of the familiar (Lienard-Wiechert) retarded radiation field:[8]

$$\dot{\mathbf{v}}(t)-\boldsymbol{\omega}_c\times\mathbf{v}(t)=-\frac{e^2/4\pi}{2Rmc^2}\hat{\mathbf{R}}\times[\hat{\mathbf{R}}\times\dot{\mathbf{v}}(t-2R/c)] \ . \tag{2.1}$$

The overdot denotes the time derivative.

The radiative correction changes the orbital period τ_0 to $\tau_0+\delta\tau$. To compute $\delta\tau$, we write the unperturbed velocity as

$$\mathbf{v}_0(t)=\boldsymbol{\omega}_c\times\boldsymbol{\rho}_0(t) \ , \tag{2.2}$$

take the scalar product of Eq. (2.1) with $\boldsymbol{\rho}_o(t)$, and integrate over the exact period τ. Since

$$\boldsymbol{\rho}_0(t)\cdot[\dot{\mathbf{v}}(t)-\boldsymbol{\omega}_c\times\mathbf{v}(t)]=\frac{d}{dt}[\boldsymbol{\rho}_0(t)\cdot\mathbf{v}(t)] \ , \tag{2.3}$$

and $\mathbf{v}(t)$ is periodic[9] with period τ while $\boldsymbol{\rho}_0(t)$ is periodic with period τ_0, we obtain, to leading order in $\delta\tau$,

$$[\boldsymbol{\rho}_0(\tau)-\boldsymbol{\rho}_0(0)]\cdot\mathbf{v}(0) \simeq \delta\tau\mathbf{v}_0{}^2=\frac{e^2/4\pi}{2Rmc^2}\int_0^\tau dt\,\mathbf{v}_0(t)\hat{\mathbf{R}}\times[\hat{\mathbf{R}}\times\mathbf{v}_0(t-2R/c] \ . \tag{2.4}$$

Here we have integrated by parts in the last term so as to place the time derivative of $\mathbf{v}(t)$ on $\boldsymbol{\rho}(t)$. Now

$$\mathbf{v}_0(t-2R/c)=\mathbf{v}_0(t)\cos(2\omega_cR/c) -\hat{B}\times\mathbf{v}_0(t)\sin(2\omega_cR/c) \ , \tag{2.5}$$

where $\hat{\mathbf{B}}=\hat{\boldsymbol{\omega}}_c$ is the unit vector along the magnetic-field direction. Moreover, averaging $v_0^k(t)v_0^l(t)$ over an orbit produces a unit dyadic in the plane of the orbit,

$$\frac{1}{\tau}\int_0^\tau dt\, v_0^k(t)v_0^l(t)=\mathbf{v}_0{}^2\tfrac{1}{2}(\delta^{kl}-\hat{B}^k\hat{B}^l) \ . \tag{2.6}$$

Using Eqs. (2.5) and (2.6) in Eq. (2.4) and remembering that $\delta\tau/\tau=-\delta\omega_c/\omega_c$, $r_0=e^2/4\pi mc^2$, we secure

$$\frac{\delta\omega_c}{\omega_c}=\frac{r_0}{4R}[1+(\hat{\mathbf{R}}\cdot\hat{\mathbf{B}})^2]\cos(2R\omega_c/c)\ . \tag{2.7}$$

This is the result quoted in Eq. (1.4) in the preceding section.

III. QUANTUM CALCULATION

As discussed in the Introduction and Summary, in quantum electrodynamics, when the photon propagator is altered by $\delta D_{\mu\nu}$, to order e^2 an energy eigenvalue E is shifted by

$$\delta E=\int d^3r\, d^3r'\overline{\Psi}_E(\mathbf{r})\delta\mathcal{M}(E;\mathbf{r},\mathbf{r}')\Psi_E(\mathbf{r}')\ , \tag{3.1}$$

where

$$\delta\mathcal{M}(E;\mathbf{r},\mathbf{r}')=ie^2\int\frac{dE'}{2\pi}\gamma^\mu S(E';\mathbf{r},\mathbf{r}')\times\gamma^\nu\delta D_{\mu\nu}(E-E';\mathbf{r},\mathbf{r}')\ . \tag{3.2}$$

The unperturbed Dirac wave functions are defined by

$$H_D\Psi_E=E\Psi_E\ , \tag{3.3}$$

where

$$H_D=\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}+V+\beta m\ . \tag{3.4}$$

We use the Dirac matrices $\gamma^0=\beta$ and $\boldsymbol{\alpha}=\beta\boldsymbol{\gamma}$. The operator $\boldsymbol{\Pi}=\mathbf{p}-e\mathbf{A}$ is the gauge-invariant, kinetic momentum. The Dirac Green's function $S(E',\mathbf{r},\mathbf{r}')$ is the coordinate matrix element of the operator

$$S(E')\beta=\frac{1}{H_D-E'}\ . \tag{3.5}$$

To obtain the correct time-ordered boundary conditions for the electron's Green's function, the energy integral over E' must run along the contour shown in Fig. 3. The calculation is facilitated with the use of the radiation gauge where

$$\delta D_{00}(E';\mathbf{r},\mathbf{r}')=-\delta D(\mathbf{r},\mathbf{r}') \tag{3.6}$$

is the correction to the static, Coulomb Green's function brought about the cavity, while δD_{kl} is the similar correction to the spatially transverse, radiation field Green's function.

We first compute the Coulomb energy correction resulting from δD_{00}. In leading order we may neglect the spread of the electron's wave function and make the dipole approximation, replacing $\delta\mathcal{D}(\mathbf{r},\mathbf{r}')$ by $\delta\mathcal{D}(\mathbf{R},\mathbf{R})$, where $\mathbf{R}$ is the position of the center of the orbit. Then, using an operator notation, we have

$$\delta E_{\text{Coulomb}}=-e^2\delta\mathcal{D}(\mathbf{R},\mathbf{R})_i\int\frac{dE'}{2\pi}\Psi_E^\dagger\frac{1}{H_d-E'}\Psi_E$$
$$=-e^2\delta\mathcal{D}(\mathbf{R},\mathbf{R})i\int\frac{dE'}{2\pi}\frac{1}{E-E'}\ . \tag{3.7}$$

To completely define the energy integration contour in Eq. (3.7), we note that the Fourier transform of the

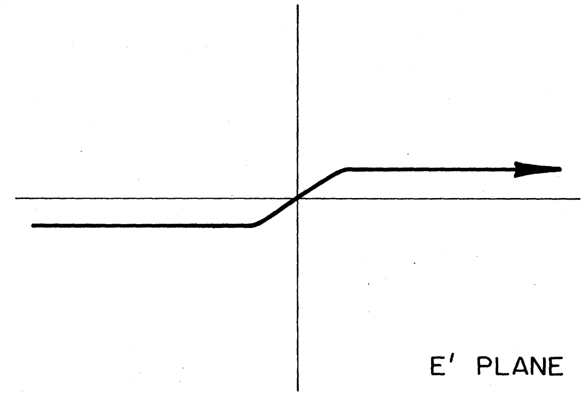

FIG. 3. Contour which defines the time-ordered electron's Green's function.

Coulomb propagator correction (3.6) gives an instantaneous propagator involving $\delta(t-t')$. Therefore, a symmetrical equal-time limit must be defined by the energy contour; it must be the average of closing the contour in both the upper and the lower half plane. Accordingly,

$$\delta E_{\text{Coulomb}}=\tfrac{1}{2}e^2\delta\mathcal{D}(\mathbf{R},\mathbf{R})\ , \tag{3.8}$$

which is the result displayed in Eq. (1.9).

To obtain the contribution of the radiation field correction, we first perform a unitary, Foldy-Wouthuysen transformation

$$\Psi_E=U\psi_E,\ \beta\psi_E=+\psi_E\ . \tag{3.9}$$

To our needed accuracy, it suffices to take

$$U=\exp(-\beta\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}/2m)\ , \tag{3.10}$$

which gives

$$U^\dagger H_D U=\beta(m+H_0)+O(1/m^2)\ , \tag{3.11}$$

where

$$H_0=\frac{\boldsymbol{\Pi}^2}{2m}+V+\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{3.12}$$

is the nonrelativistic Hamiltonian. Employing this transformation on the wave functions in Eq. (3.1) and performing appropriate integrations by parts, we encounter, to our order,

$$U^\dagger\alpha^k U=\alpha^k+\beta\frac{\Pi^k}{m}+\frac{1}{2m}\beta\boldsymbol{\alpha}\cdot\frac{1}{i}\boldsymbol{\nabla}\alpha^k\ . \tag{3.13}$$

Here the explicit gradient $\boldsymbol{\nabla}$ acts on the coordinate of the photon propagator. This gradient appears in the form

$$\boldsymbol{\alpha}\cdot\frac{1}{i}\boldsymbol{\nabla}\alpha^k=\frac{1}{i}\nabla^k+(\boldsymbol{\sigma}\times\boldsymbol{\nabla})^k\ , \tag{3.14}$$

and the ∇^k/i does not contribute since the photon propagator is transverse. Finally, using

$$U^\dagger S(E')\beta U=\frac{1}{\beta(H_0+m)-E'}\ , \tag{3.15}$$

we obtain in the dipole approximation

$$\delta E_{\rm rad}=ie^2\int\frac{dE'}{2\pi}\psi_E^\dagger\left[\frac{\Pi^k}{m}+\frac{1}{2m}(\sigma\times\nabla)^k\right]\frac{1}{H_0+m-E'}\left[\frac{\Pi^l}{m}+\frac{1}{2m}(\sigma\times\nabla')^l\right]\psi_E\delta D_{kl}(E-E';\mathbf{r},\mathbf{r}')\,|_{\mathbf{r}=\mathbf{r}'=\mathbf{R}}\ . \tag{3.16}$$

Note that since ψ_E is an eigenvector of β with eigenvalue $+1$, only "even terms," terms that commute with β, survive in the $\psi_E^\dagger\cdots\psi_E$ matrix element and β may be replaced by 1. Although the $\alpha^k\cdots\alpha^l$ contribution is overall an even term, it is not significant because it gives rise to the denominator

$$\frac{1}{-H_0-m-E'}\simeq-\frac{1}{2m}\ , \tag{3.17}$$

which commutes with α^k. Hence, since δD_{kl} is symmetrical in k,l this contribution involves $\frac{1}{2}\{\alpha^k,\alpha^l\}=\delta^{kl}$, which produces a state-independent energy shift that can be omitted. (Since this state-independent shift is already of order e^2/mR^2, we may neglect higher-order corrections to it.) In a similar way one may verify that the higher order in $1/m$ corrections to Eq. (3.13) are not significant.

The energy integral in Eq. (3.16) remains to be done. To evaluate this integral, we express the photon-propagator correction as a dispersion relation

$$\delta D_{kl}(\omega;\mathbf{r},\mathbf{r}')=\int_0^\infty d\lambda^2\frac{d_{kl}(\lambda;\mathbf{r};\mathbf{r}')}{\lambda^2-\omega^2-i\epsilon}\ . \tag{3.18}$$

The energy integration now involves the simple poles and contour shown in Fig. 4. Thus we may close the contour in the upper half plane to secure

$$\delta E_{\rm rad}=-\frac{e^2}{m^2}\int_0^\infty d\lambda\psi_E^\dagger[\Pi^k+\tfrac{1}{2}(\sigma\times\nabla)^k]\frac{1}{H_0+m-E+\lambda+i\epsilon}[\Pi^l+\tfrac{1}{2}(\sigma\times\nabla')^l]\psi_E d_{kl}(\lambda;\mathbf{r},\mathbf{r}')\,|_{\mathbf{r}=\mathbf{r}'=\mathbf{R}}\ . \tag{3.19}$$

Here E is the total energy including the rest mass. Changing to the nonrelativistic energy, $E-m\to E$, gives exactly the formula (1.12) which we have discussed in the Introduction and Summary.

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APPENDIX

Here we compute the photon-propagator correction δD_{kl} for a perfectly conducting plane. First we recall that

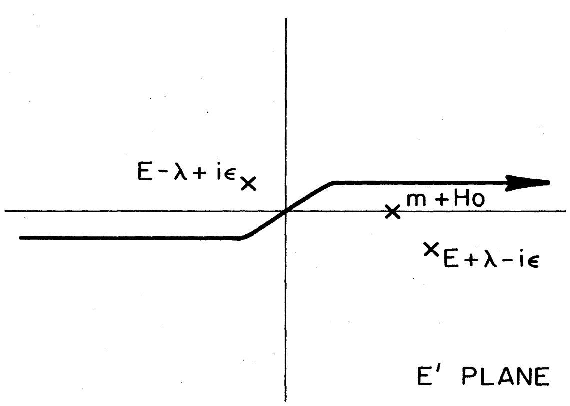

FIG. 4. Poles and contour for the energy integral giving $\delta E_{\rm rad}$ when Eq. (3.18) is introduced into Eq. (3.16).

the free-space propagator is given by

$$D_{kl}(\omega;\mathbf{r},\mathbf{r}')=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\times\left[\delta_{kl}-\frac{k_kk_l}{k^2}\right]\frac{1}{k^2-\omega^2-i\epsilon}\ . \tag{A1}$$

We establish a coordinate system such that the conductor is in the x-y plane. The boundary condition that the tangential electric field vanish at this plane is obeyed by adding the correction

$$\delta D_{kl}(\omega;\mathbf{r},\mathbf{r}')=-\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{q}\cdot\mathbf{r}'}\times\left[t_{kl}-\frac{k_kq_l}{k^2}\right]\frac{1}{k^2-\omega^2-i\epsilon}\ . \tag{A2}$$

Here

$$\mathbf{q}=(k_1,k_2,-k_3) \tag{A3}$$

and

$$t_{kl}=\delta_{kl}-2\hat{z}_k\hat{z}_l\ . \tag{A4}$$

It is easy to check that this correction is transverse,

$$\nabla_k\delta D_{kl}=0=\nabla_l'\delta D_{kl}\ . \tag{A5}$$

For the evaluation of the cyclotron frequency shift we need the long-distance limit of

$$\delta D_{kl}(\omega;\mathbf{R},\mathbf{R})=-\int\frac{d^3k}{(2\pi)^3}e^{2ik_3R}\times\left[t_{kl}-\frac{k_kq_l}{k^2}\right]\frac{1}{k^2-\omega^2-i\epsilon}\ . \tag{A6}$$

This form has cylindrical symmetry about the z axis. Hence it must be composed of scalar function times the projection into the x-y plane

$$\delta_{kl}^{(\perp)}=\delta_{kl}-\hat{z}_k\hat{z}_l\ , \tag{A7}$$

and another scalar function times $\hat{z}_k\hat{z}_l$. To compute the long-distance limit of these scalar functions, we introduce spherical coordinates with $k_3=k\cos\theta$. Since the $\exp(2ikR\cos\theta)$ in the integral (A6) varies rapidly in the $\mathbf{R}\to\infty$ limit, with adjacent regions in k canceling, the leading contribution comes from the angular end points $\cos\theta=\pm 1$. Since δD_{33} involves the factor $1-\cos^2\theta$, it is smaller asymptotically than are the $\delta D_{11},\delta D_{22}$ terms. Thus in the long-distance limit

$$\delta D_{kl}(\omega;\mathbf{R},\mathbf{R})=-\delta_{kl}^{(\perp)}\int\frac{d^3k}{(2\pi)^3}\frac{e^{2ikR\cos\theta}}{k^2-\omega^2-i\epsilon}\ . \tag{A8}$$

[The $k_1{}^2$ and $k_2{}^2$ terms in Eq. (A6) can be neglected since they contain $\sin^2\theta$.] Evaluating first the angular integrals in Eq. (A8) and then using standard contour methods for the remaining integral gives

$$\delta D_{kl}(\omega;\mathbf{R},\mathbf{R})=-(\delta_{kl}-\hat{z}_k\hat{z}_l)\frac{e^{2i\omega R}}{4\pi(2R)}\ . \tag{A9}$$

Replacing $\hat{\mathbf{z}}$ by an arbitrary direction $\hat{\mathbf{R}}$ and inserting this in Eq. (1.16) produces the result (1.4) after the real part is taken.

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[6]This has been seen experimentally by G. Gabrielse and H. G. Dehmelt [Phys. Rev. Lett. **55**, 67 (1985)].

[7]See, for example, G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) **35**, 271 (1965), Eq. (2.1).

[8]See, for example, J. D. Jackson, *Classical Electrodynamics,* 2nd edition (Wiley, New York, 1975), Sec. 14.1, Eq. (14.14).

[9]We neglect the radiative damping. It is not difficult to see that this damping, which shrinks the orbit but does not change the orientation of $\mathbf{v}(t)$ does not affect our result. The alteration of the cyclotron damping constant γ_c caused by the plane can be computed in a similar way by taking the scalar product of Eq. (2.1) with $\mathbf{v}_0(t)$ rather than with $\boldsymbol{\rho}_0(t)$.