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Thermalization of quarks and gluons in heavy-ion collisions

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The thermalization problem in relativistic heavy-ion collisions is considered. The initial momentum distribution of the quarks and gluons in a nucleus-nucleus collision before thermalization takes place is first determined. It is then argued that the relativistic Fokker-Planck equation is the appropriate transport equation to describe the time evolution of the quark-gluon system. In terms of a scaled time variable, the thermalization time is determined without fully solving the Fokker-Planck equation. The actual proper time of local thermalization is then estimated on the basis of some reasonable inputs on the soft interactions among quarks and gluons; the resultant value is of order 0.1 fm/c. The time for global (in rapidity) thermalization is longer.

I. INTRODUCTION

A central unanswered question in ultrarelativistic heavy-ion collisions concerns the time that it would take for the quark-qluon system to achieve thermal equilibrium. This is important because the lattice gauge calculations on phase transitions[1] and the hydrodynamical calculations on the time evolution of the quark-gluon plasma[2] all assume that the system under study is in local thermal equilibrium. But in a laboratory where the quark-gluon plasma is envisaged to be created in a high-energy nucleus-nucleus collision, it is not *a priori* impossible that the thermalization time is longer than the duration of effective containment, after which the plasma has either too low a density or too low a temperature to be of much interest. In that case, many of the properties of the phase transition and their signatures that have been investigated would be irrelevant to the laboratory situation. For this reason it is crucial that the thermalization problem is fully understood.

To study the problem at the level of QCD is difficult because the perturbative method is not applicable to soft interactions. The conventional wisdom is that the time scale for hadronic interaction is of order 1 fm/c. Even on the basis of that time scale, it is still an open question as to how long a quark-gluon system achieves local thermal equilibrium. For, if it takes ten units in that time scale to thermalize, then it may be too long to justify ignoring nonequilibrium thermodynamics in the study of macroscopic behaviors of the system. The only way to gain a detailed understanding of the system while it undergoes thermalization is to find the appropriate transport equation that describes the time evolution of the momentum distribution function and to solve it.

In this paper we give arguments in favor of treating the kinetic theory of the problem by the relativistic Fokker-Planck equation. The complete solution of the problem is out of the scope of this work, although a formal solution in terms of an eigenvalue problem is feasible. We shall, however, determine the thermalization time in terms of a scaled time variable without a complete solution of the transport equation.

There are two parts to this work. The first is the momentum distribution of the quarks (referring generically to both quarks and gluons) in a nucleus-nucleus collision assuming no quark interactions. The result of this part of the work is then used as the initial condition for the transport equation to be studied in the second part. We adopt an approximation method for treating the second part so that the nature of the solution is essentially independent of the details of the initial distribution. Consequently, the two parts can be read independently. A reader interested only in the kinetic theory of thermalization can proceed directly to Sec. III.

II. QUARK DISTRIBUTION IN A COLLISIONLESS PLASMA

In a collisionless plasma in which the quarks do not interact, the momentum distribution in the fragmentation region of an $A+A$ collision has already been considered in Ref. 3. Here we treat the subject in both the central and fragmentation regions, using kinematic variables defined in the c.m. system.

Let the rapidity of each nucleon in a nucleus A colliding with another nucleus A be Y so that the velocity is $V=\tanh Y$. Define $z_\pm = t \pm z/V$. Then the trajectories of the quarks, after the nuclear collision, are confined to the forward cone bounded by the z_+ and z_- axes. Consider, in particular, the right-moving nucleus. Before collision, each nucleon in it moves along a constant-z_- line until its nucleon bag is broken by the first quark from the left-moving nucleus. That occurs at $z_+=0$ on the z_- axis. Let ζ be the value of z_- along that axis where a nucleon breaks up into a spray of partons, as shown in Fig. 1. If y is the rapidity of a quark starting from that point, then its free-particle trajectory has the time-space coordinates

$$t=\tau\cosh y+\zeta/2 , \tag{2.1}$$

$$z=\tau\sinh y-\frac{\zeta}{2}\tanh Y , \tag{2.2}$$

where τ is the proper time measured from the point of quark liberation $(z_+,z_-)=(0,\zeta)$. We shall in the follow-

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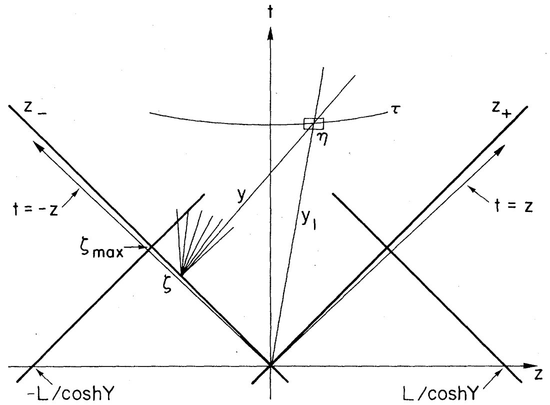

FIG. 1. Kinematical configurations of nucleus-nucleus collision. The heavy lines denote the boundaries of the two nuclei in space-time. The cell at (τ,η) is shown to have two representative quarks with rapidities y_1 and y, the former originating from $z_\pm=0$, the latter from $z_+=0$, $z_-=\zeta$.

ing ignore the transverse coordinates, regarding them as unchanging for the duration of time evolution of interest to us. Their effects are taken into account on the average by assigning a transverse mass m_T to all quarks. The momentum of a quark as a fraction of the host nucleon's momentum is

$$x=(m_T/M)e^{y-Y}\,, \tag{2.3}$$

where M is the nucleon mass.

Let the quark distribution relevant for low-p_T interaction[4,5] be $F(x)$. It can be expressed as a sum of terms corresponding to quarks and gluons separately, or for convenience combined into one generic parton distribution (in which gluons are converted into quark-antiquark pairs). The precise form of $F(x)$ will not concern us in this paper; it is only necessary to accept in the context of the parton model that in the infinite-momentum frame such a quark distribution exists even before an actual collision occurs. It is a representation of the nucleon which is given physical realization at $z_+=0$ (and at $z_-=0$ for a left-moving nucleon with corresponding changes in signs of y). For economy in notation, we denote $F(y)=F(x(y))$.

The energy-momentum tensor at (t,z) is[3]

$$T^{\mu\nu}=\sum_n \int d\tau\, k_n^\mu \frac{d}{d\tau} Z_n^\nu \delta^2(Z-Z_n(\tau))\,, \tag{2.4}$$

where Z^ν is the two-vector (t,z) and

$$k^\mu=m_T v^\mu,\quad v^\mu=(\cosh y,\sinh y)\,. \tag{2.5}$$

The contribution to the summation over the quarks, coming from the right-moving nucleus, but neglecting cooperative nuclear effects, is

$$\sum_n=\frac{N}{\zeta_{\max}}\int_0^{\zeta_{\max}} d\zeta \int_{y_1}^{y_2} dy\, F(y)\,, \tag{2.6}$$

where $\zeta_{\max}=L/\sinh Y$ and N is the number of nucleons in a longitudinal tube of length L in that nucleus in its own rest frame. There is a similar contribution from the left-moving nucleus. The limits of integration y_1 and y_2 will be specified below. Combining (2.4) and (2.6) yields

$$T^{\mu\nu}=m_T n_N \int_{y_1}^{y_2} dy\, F(y) B_Y(y) v^\mu(y) v^\nu(y)\,, \tag{2.7}$$

where $n_N=N/L$ and

$$B_Y(y)=\sinh 2Y/\sinh(Y+y)\,. \tag{2.8}$$

In a similar way the current density can be obtained

$$J^\mu=n_N \int_{y_1}^{y_2} dy\, F(y) B_Y(y) v^\mu(y)\,. \tag{2.9}$$

If (t,z) is in the overlap region of the trajectories of the colliding nuclei, $\zeta_{\max}$ would be less than $L/\sinh Y$ in order that $\tanh Y<1$. Nevertheless, (2.7) and (2.9) remain valid. The constraint $x\le 1$ is contained in the properties of $F(y)$; nuclear effects which allow x to exceed 1 are ignored in this treatment.

The dependence of J^μ and $T^{\mu\nu}$ on (t,z) is through the limits y_1 and y_2. The quarks in a space-time cell at (t,z) that have the lowest rapidity y_1 must originate from the nucleon at $\zeta=0$, while the ones that have the highest rapidity y_2 must come from $\zeta=\zeta_{\max}$. Let (t,z) be expressed as

$$t=\tau\cosh\eta,\quad z=\tau\sinh\eta\,, \tag{2.10}$$

where τ is now the proper time measured from $(t,z)=(0,0)$; it is a notation which we adhere to hereafter, superseding a different usage earlier in (2.1) and (2.2). Then we have $y_1=\eta$ and y_2 is a function of τ, η, and Y determined by the relation

$$\tanh y_2=\frac{\tau\sinh\eta+L(2\cosh Y)^{-1}}{\tau\cosh\eta-L(2\sinh Y)^{-1}}\,. \tag{2.11}$$

Note that (τ,η) specifies the coordinates of the space-time cell while y_1 and y_2 are the boundary values of the range of rapidity that the quarks in it can have, coming from the right-moving nucleus. For the quarks originating from the left-moving nucleus, the rapidity range is from y_2' to y_1, where y_1 is still η but y_2' satisfies

$$\tanh y_2'=\frac{\tau\sinh\eta-L(2\cosh Y)^{-1}}{\tau\cosh\eta-L(2\sinh Y)^{-1}}\,. \tag{2.12}$$

Their contributions to J^μ and $T^{\mu\nu}$ have the same forms as (2.7) and (2.9), except that the integrals are from y_2' to y_1, that $F(y)$ is obtained from $F(x)$ through the substitution $x=(m_T/M)\exp(-y-Y)$, i.e., $F(y)\to F(-y)$, and that $B_Y(y)\to B_Y(-y)$. We summarize by writing

$$J^\mu_{(\tau,\eta)}=\int_{-\infty}^{\infty} dy\, P(\tau,\eta,y) v^\mu(y) \tag{2.13}$$

and

$$T^{\mu\nu}(\tau,\eta)=m_T\int_{-\infty}^{\infty} dy\, P(\tau,\eta,y) v^\mu(y) v^\nu(y)\,, \tag{2.14}$$

where

$$P(\tau,\eta,y)=n_N[F(y)B_Y(y)\theta(y-\eta)\theta(y_2-y) + F(-y)B_Y(-y)\theta(y-y_2')\theta(\eta-y)]\,. \tag{2.15}$$

The quark density J^0 is frame dependent, but the rapidity distribution dN/dy of the quarks is not. They are related by

$$\int J^{\mu} d\sigma_{\mu} = \int_{-\infty}^{\infty} \frac{dN}{dy} dy \;, \qquad (2.16)$$

where $d\sigma_{\mu}$ is the spacelike surface $Z_{\mu} d\eta$. Thus we have

$$\begin{aligned} \frac{dN}{dy} &= \tau \int_{-Y}^{Y} d\eta \, P(\tau,\eta,y) \cosh(y-\eta) \\ &= n_N \tau [F(y) B_Y(y) \sinh(y-\eta_+) \\ &\quad + F(-y) B_Y(-y) \sinh(\eta_- - y)] \;, \end{aligned} \qquad (2.17)$$

where $\eta_{\pm}$ are the solutions of the equation

$$y = y_2^{\pm}(\tau,\eta_{\pm}) \;. \qquad (2.18)$$

The function $y_2^+(\tau,\eta)$ denotes the solution of (2.11), i.e., $y_2 = y_2^+(\tau,\eta)$, and the function $y_2^-(\tau,\eta)$ is the solution of (2.12), i.e., $y_2' = y_2^-(\tau,\eta)$. By virtue of an identity that can be shown, viz.,

$$\tau \sinh(y-\eta_{\pm}) = \pm L B_Y{}^{-1}(\pm y) \;, \qquad (2.19)$$

we obtain

$$\frac{dN}{dy} = N[F(y) + F(-y)] \;, \qquad (2.20)$$

which is the intuitively correct result in the absence of quark interaction. This serves as a consistency check on our result in (2.15) for the rapidity distribution of quarks in the space-time cell at (t,z) or (τ,η) in a collisionless plasma.

The determination of the quark distribution given in (2.15) is exact, subject to the condition that $F(x)$ is the quark distribution in each nucleon along the z_+ and z_- axes. While the momentum distribution of quarks in a nucleon is a well-accepted notion in the parton model, no statement is made regarding the spatial localization of the partons. To require all partons in a nucleon to be localized in a section of extent $\zeta_{\max}/N$ [cf. (2.6)] along the $z_{\pm}$ axis is actually in violation of the uncertainty principle. That localization may be sensible for partons with high rapidity by virtue of Lorentz contraction, but wee partons with low rapidity are expected to have a spatial uncertainty comparable to the hadronic size. In other words, even in the limit of infinite Y (for which a hadron is usually viewed as being an infinitely thin disk) the wee partons should be spread out over a spatial extent of order 1 fm. To take this quantum-mechanical effect into account will necessitate the introduction of a ζ-dependent quark distribution $F(y,\zeta)$ for each nucleon with appropriate spatial smearing. It would considerably complicate (2.15) for the parton distribution in the cell at (τ,η). Since the effect of spatial smearing will not influence our considerations below until a point in Sec. IV where numerical estimates are made, we shall, for the sake of clarity in our formal development, ignore the complication and use $P(\tau,\eta,y)$ in (2.15) as a specific expression of the quark distribution in a collisionless plasma. The mathematical formalism will not, of course, depend on (2.15) in detail for its validity.

III. THE THERMALIZATION PROBLEM

We now consider the thermalization process due to the interactions among the quarks. Let $\mathscr{F}(\tau,\eta,y)$ be the distribution function at the space-time cell located at (τ,η), when the quarks in the plasma interact in the normal way, as specified by QCD. $P(\tau,\eta,y)$, being the distribution in a collisionless plasma, should therefore be the initial distribution of $\mathscr{F}(\tau,\eta,y)$. Our problem is to determine how $P(\tau,\eta,y)$ evolves into $\mathscr{F}(\tau,\eta,y)$ at large τ.

Let us briefly focus on $P(\tau,\eta,y)$ still. If it is the distribution function for noninteracting quarks, then it should satisfy the relativistic transport equation for the collisionless case

$$v^{\mu} \partial_{\mu} P(\tau,\eta,y) = 0 \;. \qquad (3.1)$$

From (2.5) and (2.10) we have

$$v^{\mu} \partial_{\mu} = \cosh(y-\eta) \frac{\partial}{\partial \tau} + \sinh(y-\eta) \frac{\partial}{\tau \partial \eta} \;. \qquad (3.2)$$

In the expression (2.15) for $P(\tau,\eta,y)$ we note that y_2 and y_2' depend implicitly on τ and η. In fact, applying the operator (3.2) to (2.15), one finds that (3.1) is satisfied if

$$\cosh(y_2-\eta) \frac{\partial y_2}{\partial t} + \sinh(y_2-\eta) \frac{\partial y_2}{\tau \partial \eta} = 0 \qquad (3.3)$$

and similarly with y_2 replaced by y_2'. After some straightforward algebra, these equations can be shown to be identically satisfied with the help of (2.11) and (2.12). Thus $P(\tau,\eta,y)$ is indeed the distribution function for a collisionless plasma.

We turn our attention now to $\mathscr{F}(\tau,\eta,y)$ which satisfies the transport equation

$$v^{\mu} \partial_{\mu} \mathscr{F}(\tau,\eta,y) = C[\mathscr{F}] \;, \qquad (3.4)$$

where C is the collision operator. A complete description of this operator in the framework of QCD would be difficult and is not the objective of our endeavor here. As (3.4) stands, we have already ignored the phase space of the color variables and the kinetics in the color space.[6] Since our aim in the following is to obtain a realistic estimate of the thermalization time, we shall approach the problem in a way that incorporates as much as possible the nature of quark interaction that is characterized by QCD, yet simple enough to be amenable to a solution. An attempt to solve (3.4) has been initiated by Baym[2] in the relaxation-time approximation.

In QCD which is asymptotically free, hard collisions between quarks that involve large momentum transfers Q^2 are suppressed compared to soft interactions, because the coupling constant decreases with $\ln Q^2$. Thus in a quark-gluon plasma thermalization takes place primarily through many soft collisions. A test quark in the plasma therefore experiences many small momentum transfers, and the corresponding transport equation is the Fokker-Planck equation,[7] which describes a diffusion process in velocity space. That is the kinetic theory which we adopt for the thermalization of our system of quarks and gluons.

In our case the diffusion is in the rapidity space. The collision operator in the Fokker-Plack equation for our problem is then

$$C[\mathcal{F}]=-\frac{\partial}{\partial y}[A(y)\mathcal{F}(\tau,\eta,y)]+\frac{\partial^2}{\partial y^2}[B(y)\mathcal{F}(\tau,\eta,y)]\ , \qquad (3.5)$$

where

$$A(y)=\int_{-\infty}^{\infty}dy'\,y'w(y,y')\ , \qquad (3.6)$$

$$B(y)=\tfrac{1}{2}\int_{-\infty}^{\infty}dy'\,y'^2w(y,y')\ . \qquad (3.7)$$

Here, $w(y,y')$ is the transition rate for a quark at rapidity y to gain y' from its neighboring quarks and gluons to attain the rapidity $y+y'$. Sufficient knowledge in how to treat soft interactions in QCD should, in principle, enable one to determine $w(y,y')$, and thereby to calculate $A(y)$ and $B(y)$. In a nucleus-nucleus collision in which the plasma is expanding, $w(y,y')$ can, in general, also depend on τ and η. In the absence of a reliable procedure to account for all these complications, an approximate method is to be adopted that will retain the essence of $w(y,y')$, while averaging out its details.

Our first step is to express $\mathcal{F}(\tau,\eta,y)$ in the integral form

$$\mathcal{F}(\tau,\eta,y)=\int_{-\infty}^{\infty}d\bar{y}\,P(\tau,\eta,\bar{y})f(\tau,\eta,y-\bar{y})\ . \qquad (3.8)$$

Since $P(\tau,\eta,y)$ is the initial distribution of $\mathcal{F}(\tau,\eta,y)$ before interaction, the boundary condition for $f(\tau,\eta,y-\bar{y})$ is

$$f(\tau,\eta,y-\bar{y})\,|_{z_\pm=0}=\delta(y-\bar{y})\ , \qquad (3.9)$$

where the $z_\pm=0$ lines are the boundaries along which the quarks are liberated from their host nucleons. Since our result will not depend explicitly on Y, we shall simplify (3.9) by considering its $Y\to\infty$ limit, i.e.,

$$f(0,\eta,y-\bar{y})=\delta(y-\bar{y})\ . \qquad (3.10)$$

As τ increases, the broadening of $f(\tau,\eta,y-\bar{y})$ in $y-\bar{y}$ determines the thermalization effect that changes $P(\tau,\eta,y)$ into $\mathcal{F}(\tau,\eta,y)$.

Substituting (3.8) into (3.4) and (3.5), we have

$$\int_{-\infty}^{\infty}d\bar{y}\{v^\mu\partial_\mu[P(\tau,\eta,\bar{y})f(\tau,\eta,y-\bar{y})]-P(\tau,\eta,\bar{y})C[f(\tau,\eta,y-\bar{y})]\}=0\ . \qquad (3.11)$$

If we confine our attention to small τ, the only region where numerical estimates will be made in the following, we may regard v^μ to be very nearly $\bar{v}^\mu\equiv(\cosh\bar{y},\sinh\bar{y})$ on account of (3.10). Then we can make the approximation

$$v^\mu\partial_\mu[P(\tau,\eta,\bar{y})f(\tau,\eta,y-\bar{y})]\simeq[\bar{v}^\mu\partial_\mu P(\tau,\eta,\bar{y})]f(\tau,\eta,y-\bar{y})+P(\tau,\eta,\bar{y})\bar{v}^\mu\partial_\mu f(\tau,\eta,y-\bar{y})\ . \qquad (3.12)$$

The first term on the right-hand side vanishes because of (3.1). Since $P(\tau,\eta,y)$ is arbitrary as far as the integral in (3.11) is concerned, we have

$$\bar{v}^\mu\partial_\mu f(\tau,\eta,y-\bar{y})=C[f(\tau,\eta,y-\bar{y})]\ . \qquad (3.13)$$

Thus our problem of solving the Fokker-Planck equation (3.4) for $\mathcal{F}(\tau,\eta,y)$ is reduced to solving the same for $f(\tau,\eta,y-\bar{y})$. The advantage is that the initial condition for $f(\tau,\eta,y-\bar{y})$ is far simpler than that for $\mathcal{F}(\tau,\eta,y)$. Our focus will be on how fast $f(\tau,\eta,y-\bar{y})$ deviates from (3.10) at small τ. That will form the basis of our definition of the thermalization time. There is a complication concerning the question of whether the thermalization is local or global in rapidity. A discussion of the subject will be given at the end of this section.

To simplify (3.13) further, we define the variables σ and ρ:

$$\sigma=\tau\cosh(\eta-\bar{y}),\quad \rho=\tau\sinh(\eta-\bar{y}) \qquad (3.14)$$

which measure the time and space, respectively, of the cell at (τ,η) in the proper frame of a quark at rapidity $\bar{y}$. Since $\bar{v}^\mu\partial_\mu\sigma=1$, and $\bar{v}^\mu\partial_\mu\rho=0$, (3.13) is automatically satisfied if $f(\tau,\eta,y-\bar{y})$ is a function of σ and $y-\bar{y}$, satisfying, along constant ρ,

$$\frac{\partial}{\partial\sigma}f(\sigma,y-\bar{y})=-\frac{\partial}{\partial y}[A(y)f(\sigma,y-\bar{y})]+\frac{\partial^2}{\partial y^2}[B(y)f(\sigma,y-\bar{y})]\ . \qquad (3.15)$$

This is essentially the Fokker-Planck equation in its standard nonrelativistic form, where σ and $y-\bar{y}$ are the time and rapidity variables, respectively, defined in the same proper frame. Hence, without loss of generality we may set $\bar{y}=0$, provided that $A(y)$ and $B(y)$ are given appropriate forms consistent with the approximation already made.

In replacing the original equation (3.4) and $\mathcal{F}$ by the simplified equation (3.13) on f, whose initial condition is a δ function, we have made an approximation which has the effect of eliminating the explicit dependence on the initial quark distribution P from the Fokker-Planck equation. Since $P(\tau,\eta,y)$ is least dependent on η and y in the central region, we should restrict the applicability of (3.15) only to the central region also. In that region we may approximate $B(y)$ by a constant B, whose value will be estimated in the next section on the basis of some reasonable properties of $w(y,y')$. It sets the scale of time in the problem. If we now define

$$\theta=B\sigma,\quad a(y)=-A(y)/B\ , \qquad (3.16)$$

where θ is a dimensionless time variable, then (3.15) becomes

$$\frac{\partial}{\partial\theta}f(\theta,y)=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}+a(y)\right]f(\theta,y)\ . \qquad (3.17)$$

This is in the form of the usual Fokker-Planck equation. The initial condition is

$$f(0,y)=\delta(y)\ . \qquad (3.18)$$

The customary way of determining $a(y)$ is by the requirement that the collision operator C acting on an equilibrium distribution $f_0(y)$ leads to no change, i.e., $C[f_0(y)]=0$. We adopt the same method, and write in general

$$f_0(y)=N_0\exp[-L(y)] \ . \tag{3.19}$$

Then we have

$$a(y)=L'(y) \ , \tag{3.20}$$

where the prime denotes derivative.

If the equilibrium distribution is Maxwellian we would have $-\beta_0 k^\mu \bar{v}_\mu$ in the exponent, which implies, in the frame $\bar{y}=0$,

$$L_M(y)=\beta\cosh y, \quad \beta=m_T/T \ . \tag{3.21}$$

On the other hand, if f_0 is to be a Fermi-Dirac or Bose-Einstein distribution, then

$$L_{F,B}(y)=\ln(e^{\beta\cosh y}\pm 1) \ . \tag{3.22}$$

It should be emphasized that in solving (3.13) rather than (3.4), with (3.8) being the bridge between $\mathscr{F}(\tau,\eta,y)$ and $f(\tau,\eta,y-\bar{y})$, we have limited ourselves to the study of the first phase of the thermalization process only. For when the equilibrium distribution $f_0(y)$ is attained, the substitution of $f_0(y-\bar{y})$ into (3.8) does not render $\mathscr{F}(\tau,\eta,y)$ fully thermalized, i.e., in the form $\exp[-L(y-\eta)]$. A fully thermalized distribution for $\mathscr{F}(\tau,\eta,y)$ is what we shall call "global" thermalization, in contrast to $f_0(y)$ which we shall call "local" thermalization, where the qualifiers, global and local, refer to the rapidity space, and not to space-time coordinates. Those distributions are defined only in a local space-time cell. We shall see in the next section that $f_0(y)$ has a narrow width in rapidity, corresponding to a thermal distribution that can be expected from a dynamical interaction having only short-range correlation in rapidity. The convolution of $P(\tau,\eta,\bar{y})$ with $f_0(\tau,\eta,y-\bar{y})$ in (3.8) results in a locally thermalized distribution $\mathscr{F}(\tau,\eta,y)$, since, if $P(\tau,\eta,\bar{y})$ has a wide spread in $\bar{y}$, (3.8) cannot effect a correlation between y values that are farther apart than the width of $f_0(\tau,\eta,y-\bar{y})$. Of course, that is the first step in thermalization before it becomes global. Recall that (3.12) is an approximation valid for small τ. How global thermalization can be achieved is not treated in this paper. What we consider below refers only to local thermalization, except toward the end of the next section where some remarks will be made on global thermalization.

IV. THE THERMALIZATION TIME

There are two aspects in the problem of determining the (local) thermalization time. One is to determine the time scale set by B; the other is to calculate the relaxation time θ_0 in the scaled-time variable θ. The former requires a knowledge of the transition rate $w(y,y')$, while the latter involves solving (3.17). Both are difficult. We shall circumvent both in their details, while extracting their key characteristics. Consider first the second problem.

A. Thermalization time θ_0

The normalization of $f(\theta,y)$, i.e., its integral over all y, is independent of θ because the right-hand side of (3.17) is a total derivative in y. By virtue of (3.18) we have $\int_{-\infty}^{\infty} f(\theta,y)dy=1$ for all θ. As θ increases, $f(\theta,y)$ broadens. Let the width be measured by the quantity

$$D(\theta)=\int_{-\infty}^{\infty} dy\, y^2 f(\theta,y) \ . \tag{4.1}$$

The corresponding quantity for the equilibrium distribution $f_0(y)$ is

$$D_0=\frac{\int_{-\infty}^{\infty} dy\, y^2\exp[-L(y)]}{\int_{-\infty}^{\infty} dy\, \exp[-L(y)]} \ . \tag{4.2}$$

Our aim now is to determine the rate at which $D(\theta)$ approaches D_0. From (3.17) and (3.20) we obtain by partial integration

$$\dot{D}(\theta)\equiv\frac{d}{d\theta}D(\theta) = 2\left[1-\int_{-\infty}^{\infty} dy\, yL'(y)f(\theta,y)\right] , \tag{4.3}$$

so that at $\theta=0$, (3.18) implies

$$\dot{D}(0)=2 \ . \tag{4.4}$$

This result depends only on the validity of the Fokker-Planck equation (3.17) near $\theta=0$, and not on the details of $f_0(y)$. Note, however, that $\dot{D}(\theta)=0$, if $f_0(y)$ is substituted into (4.3). Thus the system of equations has the necessary property that $f(\theta,y)\to f_0(y)$ as $\theta\to\infty$ even though the equations themselves may not be an accurate description of reality when θ is large.

For the specific forms of $f_0(y)$ considered in (3.21) and (3.22), we have[8]

$$yL_s'(y)\geq\beta_s y^2, \quad s=M,F,B \ , \tag{4.5}$$

where

$$\beta_M=\beta, \quad \beta_{F,B}=\beta(1\pm e^{-\beta})^{-1} \ . \tag{4.6}$$

The upper bound on $\dot{D}(\theta)$ is therefore

$$\dot{D}(\theta)\leq 2[1-\beta_s D(\theta)] \ . \tag{4.7}$$

This implies that

$$D(\theta)\leq{\beta_S}^{-1}[1-\exp(-2\beta_s\theta)] \ . \tag{4.8}$$

This result is valid so long as (3.17) and (3.20) are valid.

Let us define the thermalization time θ_0 to be

$${\theta_0}^{-1}=\dot{D}(0)/D_0 \ . \tag{4.9}$$

This definition does not rely on the precise solution $D(\theta)$ for all θ, and is in accord with the θ dependence of the expected (but unproven) form

$$D(\theta)=D_0[1-\exp(-\theta/\theta_0)] \ . \tag{4.10}$$

Equation (4.10) satisfies the bound (4.8), since (4.5), when applied to (4.2), implies the bound

$$D_0\leq{\beta_S}^{-1} \ . \tag{4.11}$$

From (4.4) and (4.9) we have

$$\theta_0=\tfrac{1}{2}D_0 \ . \tag{4.12}$$

Evidently, our definition of θ_0 also satisfies the upper bound implied by (4.8) in which the relaxation time is

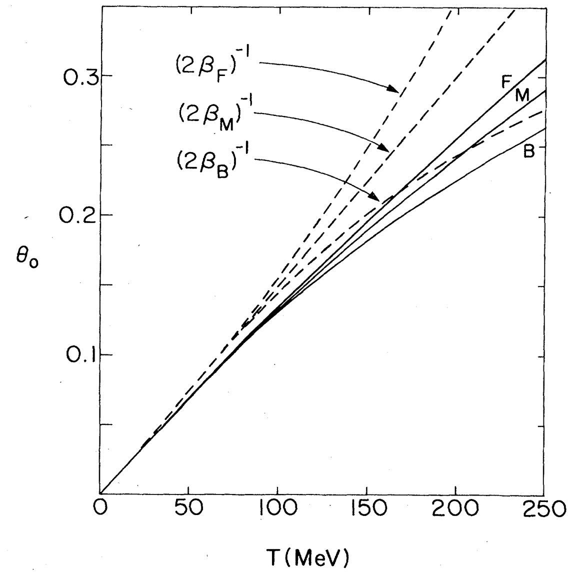

FIG. 2. Thermalization time θ_0 as a function of T. The solid lines are the results calculated from (4.9) while the dashed lines are the bounds $(2\beta_s)^{-1}$ for $s=M$, F, and B which stand for Maxwell, Fermi and Bose.

$(2\beta_s)^{-1}$. It is an upper bound because $\dot{D}(0)$ is fixed, while $D(\infty)$ is bounded from above.

In Fig. 2 are shown the values of θ_0 as functions of T, calculated using (4.12), (4.2), (3.21), and (3.22), for the equilibrium distributions of the M, F, and B types. They are not too far off from their respective bounds $(2\beta_s)^{-1}$. In those plots we have used $m_T=\frac{1}{3}$ GeV in relating β to T^{-1} [see (3.21)]. Clearly, for $T<250$ MeV, the thermalization time θ_0 is

$$\theta_0 \lesssim 0.3 \tag{4.13}$$

in all three cases. This is significant because without the transport equation there are essentially no physical guidelines on the basis of which one can make a realistic estimate of its value.

B. Time scale B^{-1}

We now consider the second problem concerning the time scale set by B in (3.16). The problem is, of course, very difficult. In the absence of a reliable method to treat QCD scattering in the nonperturbative domain we can only make qualitative arguments on the subject. Our aim is to give an order of magnitude estimate based on reasonable properties of soft interactions. As is evident from the definition given in (3.7), B is a measure of the rate of dispersion in rapidity due to scattering. The transition rate $w(y,y')$ describes the effect of the stochastic process which causes a test quark at y to change to $y+y'$ per unit time. Although its connection with the basic microscopic processes is not clear, one could argue that it should have the form

$$w(y,y') \propto \int dy_1 \frac{dN}{dy_1} W(y-y_1;y') , \tag{4.14}$$

where the proportionality constant is of order 1 $(\text{fm}/c)^{-1}$. In (4.14), $W(y-y_1;y')$ is the inclusive distribution for the reaction $q(y)+p(y_1)\to q(y+y')+X$, where p is a parton of any type (quark, antiquark, and gluon). The parton distribution dN/dy_1 is given by (2.20). Thus (4.14) represents the effect on a quark at y due to all the partons that it can interact with. It is expected that the quark interacts with many partons simultaneously and stochastically, and that the form in (4.14) is a gross simplification. In particular, $w(y,y')$ may not depend linearly on dN/dy_1. Nevertheless, the expression offers a framework to discuss the various issues involved.

There are several reasons to believe that $W(y-y_1;y')$ is strongly damped for large $|y-y_1|$. One is the short-range correlation in rapidity for pions produced in the central region.[9] It must imply that parton interaction should also be short ranged. Additional evidence is the negligible momentum degradation of quarks propagating through nuclei.[5] It has been shown that partons interact very ineffectively when they are very far apart in rapidity, such as in the situation where one belongs to the projectile, the other to the target, in a high-energy scattering process. Finally, in a more quantitative manner, one can consider a typical distribution of the type

$$f(y)=\frac{1}{2K_0(\beta)}\exp(-\beta\cosh y) , \tag{4.15}$$

where K_0 is the modified Bessel function. The full widths Δy of $f(y)$ at half maximum for $T=50,\ldots,250$ MeV at 50 MeV intervals are[3] $\Delta y=0.93$, 1.31, 1.59, 1.82, and 2.02. Since in a heavy-ion collision it is unlikely that we need to consider $T>250$ MeV, the above numbers lead us to believe that $W(y-y_1;y')$ is significant only for $|y-y_1|\lesssim 1$. As a consequence, the range of y' must also be bounded similarly due to momentum conservation. In the central region, the y dependence may be neglected; hence, we have approximately, from (3.7),

$$B\approx\frac{1}{2}\left[\frac{dN}{dy}\right]_0 \int dy'\, dy_1 y'^2 W(y_1;y') , \tag{4.16}$$

where $(dN/dy)_0$ is the rapidity distribution at $y\approx 0$. The double integrations are important only in the region $|y_1|\lesssim 1$ and $|y'|\lesssim 1$.

Having separated the different aspects about B in (4.16), we now comment on them in turn. Being of strong interaction with small momentum transfer, $W(y_1;y')$ should be of order one, satisfying the sum rule $\int dy'\, W(y_1;y')=1$ due to the conservation of the test quark. For the naive distribution $W(y_1;y')=\frac{1}{2}$, $|y'|\le 1$, one would have $\int dy' y'^2 W(y_1;y')=\frac{1}{3}$. Thus the double integral in (4.16) would yield a result of $O(1)$, which is likely to remain invariant under fine-tuning of the details of the dynamical input. From (2.20) we have $(dN/dy)_0=2NF(0)$. Hence, roughly one should expect $B\approx NF(0)$ $(\text{fm}/c)^{-1}$ if all N nucleons in each nucleus contribute to the thermalization of the partons in the space-time cell considered.

In Sec. II where the parton distribution $F(y)$ in a nucleon is introduced, we did not specify its form for the sake of brevity. We assert here that it is known,[3-5] and that $F(0)\approx 5$, which includes all types of partons. N is the number of nucleons in a longitudinal tube through the nucleus averaged over all impact parameters; for $12 < A < 238$, N ranges between 2 and 5. We thus arrive at a value $B \approx 25$ $(\mathrm{fm}/c)^{-1}$ for uranium-uranium collisions. Based on that one would have for the proper time of thermalization [cf. (3.16) and (4.13)]

$$\tau_0 = B^{-1}\theta_0 \approx 10^{-2} \text{ fm}/c \ . \tag{4.17}$$

This is too small to justify including the partons from all nucleons in the thermalization process. The reason is that the interaction region is not infinitely dense no matter how high the incident energy is. At finite Y the contracted spatial dimension is of order $L/\cosh Y$, as indicated in Fig. 1. At asymptotic Y the wee partons would still occupy a spatial volume of the order of 1 fm, as discussed in the last paragraph of Sec. II. In the short time interval of order 10^{-2} fm/c not all the partons in the volume of longitudinal extent not less than 1 fm can interact with a test quark. Since (2.20) is obtained after integrating over the entire longitudinal dimension, it gives an overestimate of the number of partons at $y \approx 0$ that can contribute to the thermalization process. If on the grounds of causality we assume that the number of partons interacting with a test quark is proportional to τ up to 1 fm/c (i.e., $B \propto \tau_0/1$ fm/c), then the proper time of thermalization is roughly the geometrical mean between 1 fm/c and the value in (4.17), i.e.,

$$\tau_0 \approx 0.1 \text{ fm}/c \ . \tag{4.18}$$

As we have remarked in the last paragraph of Sec. III, it is the time for local thermalization that we have determined in (4.18). How much longer the global thermalization time is depends on, among other factors, the rapidity range of the initial distribution $P(\tau,\eta,\bar{y})$ in the space-time cell at (τ,η). It is evident from (3.8) that if the range in $\bar{y}$ is not wide, there is not much difference between global and local thermalization. It is also clear from the geometry of the collision process, as illustrated in Fig. 1, that the larger is the value of τ relative to $L/\cosh Y$, the narrower is the range of $\bar{y}$. We therefore know that the global and local thermalization times are nearly equal at asymptotic Y. At finite Y, if τ is small enough so that the cell (τ,η) is inside the overlap region of the incident nuclei (see Fig. 1), then the range of $\bar{y}$ would be of order $2Y$, and global thermalization cannot be expected. Thus global thermalization time must be greater than $L/\sinh Y$, which differs from (4.18) in accordance with the experimental variables.

V. CONCLUSION

The thermalization problem has been investigated in this work in several stages. The quark distribution in a collisionless plasma is first determined for the kinematical conditions of a high-energy nucleus-nucleus collision. The effect of quark interactions is then taken into account by convoluting the collisionless distribution with an $f(\tau,\eta,y)$ which plays the role of an evolution function. This function $f(\tau,\eta,y)$ satisfies the relativistic Fokker-Planck equation with $\delta(y)$ as the initial condition. The broadening of $f(\tau,\eta,y)$ is a manifestation of the local-thermalization process, so its width provides a good measure of the time evolution of that process. While the determination of $f(\tau,\eta,y)$ would require a complete solution of the Fokker-Planck equation, the initial rate of change of its width does not. We have therefore been able to determine the local-thermalization time rather reliably, but only in terms of a scaled time variable. The scale itself is far more difficult to ascertain due to the complications associated with soft interactions. However, the physical factors that we have examined suggest that the local-thermalization time should be of order 0.1 fm/c. The global-thermalization time would be larger depending on collision parameters. Our estimate is that if the incident energy is high enough so that the overlap region of the contracted nuclei does not exceed 1 fm, the time for global thermalization (in rapidity) should not far exceed 1 fm/c. The result justifies the usual assumption about thermal equilibrium made in the phase-transition and hydrodynamical calculations.

A number of areas in this subject deserve further investigation. An obvious one is the problem of understanding better the transition rate $w(y,y')$ in QCD. It is related to the question of whether a connection can be established between soft interactions in QCD and stochastic processes, the phenomenological significance of which has already been pointed out by Carruthers.[10] Our result may allay any concerns for the need of nonequilibrium thermodynamics in treating certain problems in heavy-ion collisions. However, the Fokker-Planck equation does not provide a close link between the microscopic and macroscopic properties of the system. In treating Brownian motion it describes the approach to thermal equilibrium with the surrounding bath of a given temperature. For the quark-gluon system we have no thermal bath. The test quark is in the same system of partons that move toward thermal equilibrium with one another attaining a temperature that must be self-consistent. We have not addressed the question of what that temperature is. Finally, the detail dynamical mechanism of how a quark-gluon system can attain global thermalization in rapidity when the basic interaction is shortranged (in rapidity) remains an open question. Our crude estimate is based mainly on geometrical and kinematical considerations. A more reliable determination of the global-thermalization time can follow only upon a better understanding of the dynamics of the problem.

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