PHYSICAL REVIEW D VOLUME 32, NUMBER 3 1 AUGUST 1985

Description of the energy dependence of proton-proton partial-wave amplitudes by means of O(4) expansions

J. Bystricky
Département de Physique des Particules Elémentaires, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France

P. LaFrance
Centre de Recherche de Mathématiques Appliquées, Université de Montréal, Case Postale 6128, Montréal, Québec, Canada H3C 3J7

F. Lehar
Département de Physique des Particules Elémentaires, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France

F. Perrot
Centre de Recherche de Mathématiques Appliquées, Université de Montréal, Case Postale 6128, Montréal, Québec, Canada H3C 3J7
and Département de Physique Nucléaire de Moyennes Energies, Centre d'Études Nucléaires de Saclay, Gif-sur-Yvette Cedex, France

P. Winternitz
Centre de Recherche de Mathématiques Appliquées, Université de Montréal, Case Postale 6128, Montréal, Québec, Canada H3C 3J7

(Received 23 July 1984)

The main content of this article is a phenomenological treatment of the energy dependence of proton-proton scattering in the energy region $0 \le E_{\text{lab}} \le 800$ MeV, using previously developed two-variable expansions of scattering amplitudes. Instead of turning directly to the experimental results, we make use of the recent Geneva-Saclay phase-shift analysis to calculate the proton-proton partial-wave amplitudes $A^{j}_{lsl's'}$ (together with their error bars). These amplitudes are expanded in terms of O(4) transformation matrices in a manner implied by the two-variable expansions. The expansion coefficients, i.e., the O(4) amplitudes, are then obtained by a χ^2 minimization, treating the phase-shift values of $A^{j}_{lsl's'}$ as data. We find that the O(4) expansions provide a good fit with a reasonable number of parameters, that the fit is quite stable with respect to the variation of the number of free parameters allowed, and that the expansions have a good threshold behavior. The results are sufficiently encouraging to warrant a future reanalysis of all nucleon-nucleon elastic-scattering data, using the O(4) expansions, and to extend the analysis to higher energies than those that have been treated by phase-shift analysis.

I. INTRODUCTION

The purpose of this article is to apply previously developed two-variable expansions[1–8] of scattering amplitudes for binary reactions to analyze proton-proton scattering from threshold to 800 MeV incident-particle kinetic energy. This is the first such application of the two-variable formalism and its aim is twofold. On the one hand we wish to provide a good and stable fit to the energy and angular dependence of the proton-proton scattering amplitudes in the mentioned energy region (for all scattering angles) in terms of a reasonably small number of free parameters—the expansion coefficients. On the other hand, we wish to test the appropriateness of these particular expansions, based on the representation theory of the rotation group O(4), as a means for reconstructing scattering amplitudes from data in the intermediate energy region. In principle we view these expansions as a tool that should complement and in some cases replace partial-wave analysis as a phenomenological reconstruction method.

As a brief historical remark we mention that two-variable expansions of scattering amplitudes for spinless particles were originally proposed[1,2] as an application of harmonic analysis on homogeneous spaces of the Lorentz group O(3,1). The scattering amplitude $f(s,t)$ for a reaction of the type $1+2\rightarrow3+4$ was expanded in terms of functions of both kinematic variables $s=(p_1+p_2)^2$ and $t=(p_1-p_3)^2$. The kinematic variables s and t ranged over the entire physical region, and the expansions were provided by the representation theory of the homogeneous Lorentz group O(3,1). Different choices of bases for the representations of O(3,1) made it possible to incorporate different types of single-variable expansions used in scattering theory, such as the partial-wave expansions (for fixed energy), Regge-pole expansions (for fixed momentum transfer), and eikonal expansion.[1–4] The envisaged applications were mainly theoretical ones: the expansions can serve as a tool for solving relativistic equations, and particle dynamics can be formulated in terms of properties of the expansion coefficients.

The most essential feature of the expansions is that they are "complete" expansions: the kinematic variables are displayed explicitly in known basis functions. The dynamics of any particular reaction is reflected in the properties and values of the expansion coefficients. The use of group-representation theory has made it possible to incorporate many desirable kinematic properties of the scattering amplitudes into the expansion functions. The latter have correct threshold behavior, and are orthogonal over the physical region; properties of amplitudes under parity conservation, the Pauli principle, etc., can be incorporated in a natural manner; various analyticity properties are manifest.

More recently, the expansions have been generalized to reactions with arbitrary spins[5] and the emphasis has shifted to the possibility of phenomenological applications. From this point of view, the simultaneous treatment of the entire physical region is a drawback, rather than an advantage. Indeed, the fact that the physical scattering region is infinite forces the expansions to involve integrals (over representations of the Lorentz group), in addition to sums (e.g., over angular momentum). What is more, experimental data are always available in finite-energy regions only. A way out of this dilemma is to restrict the treatment to a finite energy region $0 \le E \le E_{\max}$, where $E_{\max}$ is some chosen fixed energy (in this article $E_{\max}=800$ MeV in the laboratory system). It has been shown[6–8] that a restriction to a finite energy region makes it possible to replace the O(3,1) integral expansions by "discrete" expansions, based on the representation theory of the group O(4).

The O(4) expansions share many of the desirable properties of the original O(3,1) expansions. The expansion functions, which are now the O(4) group-transformation matrices, also have the correct threshold behavior. The expansions incorporate the usual O(3) partial-wave expansion (no longer, however, the Regge-pole or eikonal expansion). The basis functions are orthogonal, this time over an O(4) sphere, corresponding to the finite region that we are considering.

Since the O(4) expansions involve sums only (convergent series, to be more precise), they can be used as a tool for analyzing the energy dependence of scattering amplitudes. Indeed, the expansions involve two infinite summations, one over the total angular momentum j, conjugate to the scattering angle, the other over an O(4) label n, conjugate to the energy. In a phenomenological data treatment both sums are cut off at some maximal values and the expansion coefficients are calculated from a statistical best fit to the data. The coefficients, once determined, provide a complete description of the angular and energy dependence of the scattering amplitudes.

In this article, we do not return all the way to the proton-proton scattering data. Instead, we make use of the recent Geneva-Saclay phase-shift analysis[9] to calculate the proton-proton partial-wave amplitudes $A^{j}_{lsl's'}$ at 20-MeV intervals and we treat these amplitude values as "data." From them we calculate the O(4) amplitudes and ultimately the total amplitudes. The purpose of this exercise is mainly to test various features of the O(4) expansions. These include (1) the rapidity of convergence, i.e., the number of parameters needed to get a good global fit, as well as reasonable χ^2 values, (2) the stability of the expansions with respect to the choice of the cutoffs, and (3) the sensitivity with respect to structures in the amplitudes.

In Sec. II we present the O(4) expansions in a convenient form for our purposes. We establish some new properties of the O(4) transformation matrices needed to implement the consequences of time-reversal invariance and other symmetries on the O(4) amplitudes. In particular, we show that the O(4) expansions permit a convenient decoupling of the coupled triplet pp amplitudes. Section III is devoted to a numerical fit to the proton-proton "data," separately for singlet, uncoupled-triplet, and coupled-triplet states. The conclusions are contained in Sec. IV. Running ahead we state the main conclusion. The "hybrid" approach of this paper has shown that O(4) expansions provide a useful parametrization for describing nucleon-nucleon scattering. The next step is to perform a complete reanalysis of all pp scattering data, for all angles and all energies up to some $E_{\max}$, in terms of the O(4) expansions, taking all experimental points directly as they were measured (avoiding interpolations between the energies of different accelerators, etc.).

II. THE O(4) EXPANSIONS OF PROTON-PROTON SCATTERING AMPLITUDES

The O(4) expansions[7,8] in the "initial-spin—final-spin" formalism can be written as a partial-wave expansion,[10]

$$M_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=\sum_{ll'j}[4\pi(2l+1)]^{1/2}\frac{2l'+1}{2j+1}(l\sigma'-\sigma s\sigma\,|\,j\sigma')(l'0s'\sigma'\,|\,j\sigma')A^{j}_{lsl's'}(\alpha)Y_{l\sigma'-\sigma}(\theta,\phi) \tag{2.1}$$

supplemented by an energy expansion of the partial-wave amplitudes

$$A^{j}_{lsl's'}(\alpha)=\frac{1}{2j+1}\sum_{\lambda,\lambda'}(l0s\lambda\,|\,j\lambda)(l'0s'\lambda'\,|\,j\lambda')\sum_{n,\nu}[(n+1)^2-\nu^2]T^{n\nu}_{jss'\lambda'}d^{n\nu *}_{js\lambda}(\alpha)\ . \tag{2.2}$$

Here $T^{n\nu}_{jss'\lambda'}$ are the O(4) amplitudes and $d^{n\nu}_{js\lambda}(\alpha)$ are the reduced O(4) group-transformation matrices, given explicitly in the Appendix. The O(3) Clebsch-Gordan coefficients and spherical harmonics are as in Ref. 11.

The energy variable α is given by

$$\sin\alpha=\left[\frac{w-(m_1+m_2)^2}{w_M-(m_1+m_2)^2}\right]^{1/2},$$

$$(m_1+m_2)^2\le w\le w_M,\quad 0\le\alpha\le\pi/2\ , \qquad (2.3)$$

where $w=(p_1+p_2)^2$ is the total invariant energy and w_M is some fixed energy, up to which we plan to perform the data analysis. For the relation between the amplitudes $A^j_{lsl's'}$ and those of Stapp and coworkers,[12] usually used in phase-shift analysis, see Ref. 8.

As w varies over the considered region, the variables α, θ, and ϕ range over a hemisphere: $0\le\alpha\le\pi/2$, $0\le\theta\le\pi$, $0\le\phi<2\pi$. To be able to use the completeness of the O(4) d functions, we extend the definition of the partial-wave amplitudes to the entire sphere by symmetry:

$$A^j_{lsl's'}(\pi-\alpha)=A^j_{lsl's'}(\alpha)\ . \qquad (2.4)$$

The symmetry (2.4) will reduce the number of independent O(4) amplitudes in the expansions by a factor of 2.

In addition, time-reversal invariance implies

$$A^j_{lsl's'}(\alpha)=A^j_{l's'ls}(\alpha)\ . \qquad (2.5)$$

Below we shall obtain the implications of these two symmetries for the O(4) amplitudes $T^{n\nu}_{jss'\lambda'}$. Previously[7,8] we have shown that parity conservation

$$A^j_{lsl's'}=\eta(-1)^{l+l'}A^j_{lsl's'}$$

implies

$$T^{n\nu}_{jss'\lambda'}=\eta(-1)^{s-s'}T^{n\,-\nu}_{jss'-\lambda'} \qquad (2.6)$$

(η is the product of the four intrinsic parities) and that if the initial-state and final-state particles are identical, i.e.,

$$A^j_{lsl's'}=(-1)^{l'+s'}A^j_{lsl's'}$$

and

$$A^j_{lsl's'}=(-1)^{l+s}A^j_{lsl's'}\ ,$$

we obtain

$$T^{n\nu}_{jss'\lambda'}=(-1)^j T^{n\nu}_{jss'-\lambda'}=(-1)^j T^{n\,-\nu}_{jss'\lambda'}\ . \qquad (2.7)$$

From here on we restrict to the elastic scattering of two identical spin-$\frac{1}{2}$ particles, e.g., proton-proton scattering, and summarize the direct and inverse expansion formulas. We recall that parity conservation and the Pauli principle imply that l and l' are even in singlet states and odd in triplet ones.

Let us first consider the singlet ($s=s'=0$, $l=l'=j$ even) and uncoupled triplet ($s=s'=1$, $l=l'=j$ odd) states. We have

$$A^j_{j0j0}(\alpha)=\frac{1}{2j+1}\sum_{n=j}^{\infty}(n+1)^2 S^n_j d^{n0*}_{j00}(\alpha)\quad (j\text{ even})\ , \qquad (2.8)$$

$$A^j_{j1j1}(\alpha)=\frac{2}{(2j+1)^{1/2}}\sum_{n=j}^{\infty}n(n+2)T^{n1}_{j1}[d^{n1*}_{j11}(\alpha)-d^{n1*}_{j1-1}(\alpha)]\quad (j\text{ odd})\ . \qquad (2.9)$$

The inverse formulas are

$$S^n_j\equiv T^{n0}_{j000}=\begin{cases}\dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}A^j_{j0j0}(\alpha)d^{n0}_{j00}(\alpha)\sin^2\alpha\,d\alpha, & n\text{ even}\ ,\\ 0, & n\text{ odd}\ ,\end{cases} \qquad (2.10)$$

$$T^{n1}_{j1}\equiv\frac{T^{n1}_{j111}}{(2j+1)^{1/2}}=\begin{cases}0, & n\text{ even}\ ,\\ \dfrac{2}{3\pi(2j+1)^{1/2}}\displaystyle\int_0^{\pi/2}A^j_{j1j1}(\alpha)[d^{n1}_{j11}(\alpha)-d^{n1}_{j1-1}(\alpha)]\sin^2\alpha\,d\alpha\ , & n\text{ odd}\ .\end{cases} \qquad (2.11)$$

For the coupled triplet we have $s=s'=1$, $l=j\pm1$, $l'=j\pm1$, j even. Consider first the special case when $j=0$. The only nonvanishing triplet partial-wave amplitude with $j=0$ is

$$A^0_{1111}(\alpha)=\tfrac{1}{3}\sum_{n=1}^{\infty}(n+1)^2 T^{n0}_{00}d^{n0*}_{010}(\alpha) \qquad (2.12)$$

and the inverse formula is

$$T^{n0}_{00}\equiv T^{n0}_{0110}=\begin{cases}0, & n\text{ even}\ ,\\ \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}A^0_{1111}(\alpha)d^{n0}_{010}(\alpha)\sin^2\alpha\,d\alpha, & n\text{ odd}\ .\end{cases} \qquad (2.13)$$

In order to extract the full implications of time-reversal invariance (2.5) and the symmetry (2.4) for the coupled triplet with $j\ge2$ we first invert the expansion (2.2) for the case under consideration, using the orthogonality properties (A4) of the d functions.[13] We obtain

$$
\begin{aligned}
T_{j1}^{(2k)1} &= c\sqrt{2}\int_0^{\pi/2} B_1^j(\alpha) d_{j10}^{(2k)1}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j1}^{(2k+1)0} &= c\sqrt{2}\int_0^{\pi/2} B_1^j(\alpha) d_{j10}^{(2k+1)0}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j0}^{(2k)0} &= 2c\sqrt{2}\int_0^{\pi/2} B_4^j(\alpha) d_{j11}^{(2k)0}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j0}^{(2k+1)1} &= c\sqrt{2}\int_0^{\pi/2} B_4^j(\alpha)[d_{j11}^{(2k+1)1}(\alpha)+d_{j1-1}^{(2k+1)1}(\alpha)]\sin^2\alpha\, d\alpha\ ,\\
T_{j0}^{(2k)1} &= 2c\int_0^{\pi/2} B_3^j(\alpha) d_{j10}^{(2k)1}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j0}^{(2k+1)0} &= 2c\int_0^{\pi/2} B_3^j(\alpha) d_{j10}^{(2k+1)0}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j1}^{(2k)0} &= 2c\int_0^{\pi/2} B_2^j(\alpha) d_{j11}^{(2k)0}(\alpha)\sin^2\alpha\, d\alpha\ ,\\
T_{j1}^{(2k+1)1} &= c\int_0^{\pi/2} B_2^j(\alpha)[d_{j11}^{(2k+1)1}(\alpha)+d_{j1-1}^{(2k+1)1}(\alpha)]\sin^2\alpha\, d\alpha\ ,
\end{aligned}
\tag{2.14}
$$

where

$$
c=\frac{2}{3\pi(2j+1)^{5/2}}
\tag{2.15}
$$

and $k=j/2, j/2+1, j/2+2, \ldots$.

The new amplitudes $B_a^j(\alpha)$ $(a=1,\ldots,4)$ are convenient linear combinations of the usual partial-wave amplitudes (2.2), namely,

$$
\begin{aligned}
B_1^j(\alpha) &= [j(j+1)]^{1/2}[(2j-1)A^j_{(j-1)1(j-1)1}-(2j+3)A^j_{(j+1)1(j+1)1}]\\
&\quad +[(2j-1)(2j+3)]^{1/2}[jA^j_{(j-1)1(j+1)1}-(j+1)A^j_{(j+1)1(j-1)1}]\ ,\\
B_2^j(\alpha) &= (j+1)(2j-1)A^j_{(j-1)1(j-1)1}+j(2j+3)A^j_{(j+1)1(j+1)1}\\
&\quad +[j(j+1)(2j-1)(2j+3)]^{1/2}(A^j_{(j-1)1(j+1)1}+A^j_{(j+1)1(j-1)1})\ ,\\
B_3^j(\alpha) &= j(2j-1)A^j_{(j-1)1(j-1)1}+(j+1)(2j+3)A^j_{(j+1)1(j+1)1}\\
&\quad -[j(j+1)(2j-1)(2j+3)]^{1/2}(A^j_{(j-1)1(j+1)1}+A^j_{(j+1)1(j-1)1})\ ,\\
B_4^j(\alpha) &= [j(j+1)]^{1/2}[(2j-1)A^j_{(j-1)1(j-1)1}-(2j+3)A^j_{(j+1)1(j+1)1}]\\
&\quad +[(2j-1)(2j+3)]^{1/2}[-(j+1)A^j_{(j-1)1(j+1)1}+jA^j_{(j+1)1(j-1)1}]\ ,
\end{aligned}
\tag{2.16}
$$

where $j\geq 2$ is even.

Formulas (2.14) show, first of all, that the natural functions to expand in terms of the O(4) basis are the amplitudes $B_a^j(\alpha)$ rather than the coupled-triplet amplitudes $A^j_{l1l'1}(\alpha)$. They also indicate that half of the O(4) amplitudes $T_{j\lambda}^{n\nu}$ are actually redundant for a description of amplitudes satisfying the symmetry (2.4). This prompted us to derive the recursion relations (A5) and (A6) of the Appendix. Using (A5), (A6), and (2.14), we find that we can express all the odd-n amplitudes $T_{j\lambda}^{n\nu}$ in terms of even-n ones (for $j=2,4,\ldots$):

$$
\begin{aligned}
T_{j1}^{(2k+1)0} &= \frac{1}{2(k+1)[j(j+1)]^{1/2}}\{[(2k+1)(k+2)(2k+j+3)(2k-j+2)]^{1/2}T_{j1}^{(2k+2)1}\\
&\qquad -[k(2k+3)(2k+j+2)(2k-j+1)]^{1/2}T_{j1}^{(2k)1}\}\ ,\\
T_{j0}^{(2k+1)1} &= \frac{1}{[j(j+1)]^{1/2}}\left[\left[\frac{(k+2)(2k+j+3)(2k-j+2)}{2k+1}\right]^{1/2}T_{j1}^{(2k+2)1}\right.\\
&\qquad \left.-\left[\frac{k(2k+j+2)(2k-j+1)}{2k+3}\right]^{1/2}T_{j1}^{(2k)1}\right]\ ,\\
T_{j0}^{(2k+1)0} &= \frac{1}{2(k+1)[j(j+1)]^{1/2}}\{[(2k+1)(k+2)(2k+j+3)(2k-j+2)]^{1/2}T_{j0}^{(2k+2)1}\\
&\qquad -[k(2k+3)(2k+j+2)(2k-j+1)]^{1/2}T_{j0}^{(2k)1}\}\ ,\\
T_{j1}^{(2k+1)1} &= \frac{1}{2[j(j+1)]^{1/2}}\left[\left[\frac{(k+2)(2k+j+3)(2k-j+2)}{2k+1}\right]^{1/2}T_{j1}^{(2k+2)0}-\left[\frac{k(2k+j+2)(2k-j+1)}{2k+3}\right]^{1/2}T_{j1}^{(2k)0}\right].
\end{aligned}
\tag{2.17}
$$

It is now a simple matter to impose time-reversal invariance (2.5) which for nucleon-nucleon scattering at a fixed value of j reduces to a single relation:

$$A^j_{(j-1)1(j+1)1}(\alpha)=A^j_{(j+1)1(j-1)1}(\alpha)\,, \tag{2.18}$$

and hence

$$B^j_4(\alpha)=B^j_1(\alpha)\,. \tag{2.19}$$

Using (2.14) and (2.19) we find that time-reversal invariance implies

$$T^{(2k)0}_{j0}=2T^{(2k)1}_{j1}\,. \tag{2.20}$$

Finally, it follows from (2.17) and (2.20) that the free parameters in the coupled-triplet expansions for $j\geq 2$ (even) are

$$T^{n0}_{j1},\ T^{n1}_{j0},\ T^{n1}_{j1},\quad n \text{ even}\,. \tag{2.21}$$

Expanding the amplitudes $A^j_{1l'1}$ in (2.16) as in (2.2), eliminating the terms $(d^{n1}_{j11}+d^{n1}_{j1-1})$ and d^{n0}_{j10} by means of the recursion relations (A5) and (A6) and using (2.17) and (2.20) we "decouple" the coupled-triplet amplitudes and obtain the following extremely simple and uniform expansion formulas:

$$B^j_a(\alpha)=\sum_{2k=j}^{\infty} Z^j_{a2k}\,d^{2k0*}_{j11}(\alpha),\quad a=1,2,3\,, \tag{2.22}$$

where $B^j_a(\alpha)$ for $a=1,2,3$ and $j=2,4,\ldots$ are given by (2.16) once (2.18) is taken into account.

The new O(4) amplitudes Z^j_{an} $(a=1,2,3,\ n=2k)$ are linear combinations of the original independent coupled-triplet O(4) amplitudes (2.21). Indeed a straightforward calculation yields

$$\begin{aligned} Z^j_{1n}&=\frac{\kappa}{\sqrt{2}}(\rho^j_n T^{n1}_{j1}+\sigma^j_n T^{(n-2)1}_{j1}+\sigma^j_{n+2}T^{(n+2)1}_{j1})\,,\\ Z^j_{2n}&=\frac{\kappa}{2}(\rho^j_n T^{n0}_{j1}+\sigma^j_n T^{(n-2)0}_{j1}+\sigma^j_{n+2}T^{(n+2)0}_{j1})\,,\\ Z^j_{3n}&=\frac{\kappa}{2}(\rho^j_n T^{n1}_{j0}+\sigma^j_n T^{(n-2)1}_{j0}+\sigma^j_{n+2}T^{(n+2)1}_{j0})\,, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \kappa&=\frac{(2j+1)^{3/2}}{j(j+1)}\,,\\ \rho^j_n&=2(n+1)^2[n(n+2)+j(j+1)]\,,\\ \sigma^j_n&=-[(n-2)(n-1)(n+1)(n+2)(n-j-1)\\ &\quad\times(n-j)(n+j)(n+j+1)]^{1/2}\,, \end{aligned} \tag{2.24}$$

both j and n are even and $n\geq j\geq 2$.

We can now summarize the results of this section. The O(4) expansions for singlet, uncoupled-triplet, and coupled-triplet $j=0$ amplitudes for pp scattering are given by formulas (2.8), (2.9), and (2.12), respectively. For $j\geq 2$ the O(4) expansions lead to a complete decoupling of the "coupled"-triplet amplitudes, as manifested in the expansions (2.22). Notice that the coefficients Z^j_{a2k} for $a=1$, 2, and 3 are independent of each other and it is hence equally simple to treat the coupled triplet, as the other amplitudes.

The phenomenological analysis of Sec. III concerns the energy range up to $E_{\rm lab}=800$ MeV. This is well above the inelastic threshold and there is no point in imposing elastic unitarity. We allow the O(4) amplitudes $T^{n\nu}_{j\lambda}$ to have *a priori* arbitrary real and imaginary parts. The fit to the data, i.e., the partial-wave amplitudes calculated from a phase-shift analysis, will in itself impose the constraints due to unitarity. This means that below the threshold of pion production the phase shifts will be real, above this threshold they acquire non-negative imaginary parts, as they should.

III. THE O(4) ANALYSIS OF THE PARTIAL-WAVE AMPLITUDES

A. The proton-proton scattering data and the fitting procedure

As mentioned in the Introduction, the ultimate aim is to analyze simultaneously all available nucleon-nucleon scattering data. In this article our approach is more modest. We make use of the fact that existing phase-shift analyses already provide good fits to angular distributions of the observables. More specifically, we base ourselves on the recent phase-shift analysis[9] of all existing proton-proton elastic-scattering data for $0\leq E_{\rm lab}\leq 800$ MeV (and $0\leq\theta\leq\pi$). This phase-shift analysis is itself an energy-dependent one. The phase shifts were obtained by dividing the energy region into four overlapping intervals and then fitting the phase shifts in each interval by low-degree polynomials in the energy. The parametrization was such that a total of 57, 50, and 124 real parameters were introduced in the singlet ($l=j=0$, 2, 4, and 6), uncoupled triplet ($l=j=1$, 3, 5, and 7), and coupled triplet ($l\pm 1=j=0$, 2, 4, and 6), respectively. We have used these phase shifts, together with the corresponding error matrices, to calculate the amplitudes $A^j_{sl's'}(\alpha)$ at 20-MeV intervals. The uncertainties in the phase shifts are reflected in error bars on the "experimental points," i.e., the calculated values of $A^j_{sl's'}(\alpha)$. Correlations between errors are ignored.

The phase-shift analysis of Ref. 9 represents about 5300 independent data points: those summarized in Ref. 14 and numerous further experimental results obtained after this compilation was completed. It should also be mentioned that the method of "modified phase-shift analysis"[12(b)] was employed, i.e., higher partial waves were not set equal to zero, but rather to their values obtained by assuming that they are completely due to the exchange of one pion.

Our fitting procedure was a standard χ^2 least-squares minimization fit, one for each real or imaginary part of the partial-wave amplitudes, expanded in terms of the O(4) d functions as in (2.8), (2.9), (2.12), and (2.22).

The number of experimental points for each fit was $N_E=40$. Several criteria were used simultaneously to determine an acceptable minimum value of the number of real parameters N_T in each case. These were the following requirements.

(1) χ^2/DF must pass through a stable region or have a value close to 1 (DF$=N_E-N_T$ is the number of degrees of freedom).

(2) The overall fit to the data points must be visibly good.

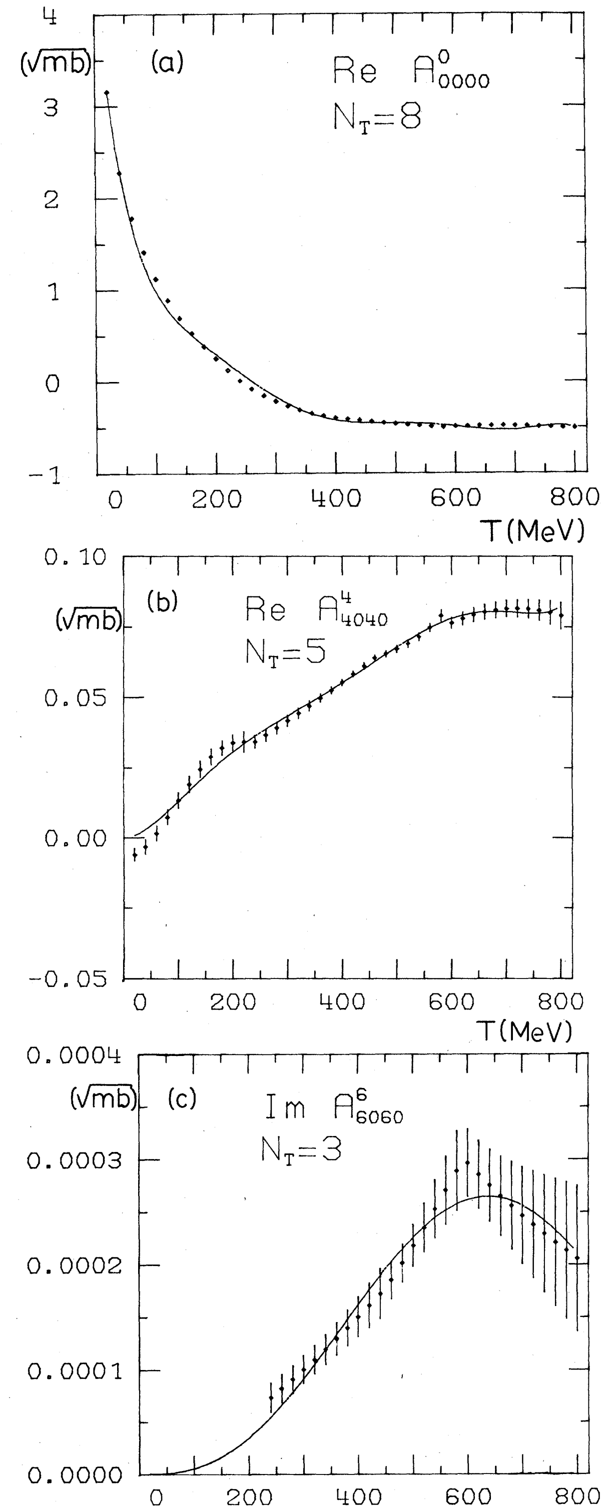

FIG. 1. Singlet O(4) fits to partial amplitudes. $\mathrm{Re}A^{0}_{0000}(\alpha)$, $\mathrm{Re}A^{4}_{4040}(\alpha)$, and $\mathrm{Im}A^{6}_{6060}(\alpha)$ in the laboratory kinetic-energy interval $0 \le T \le 800$ MeV. The χ^2/DF values are 88, 1.2, and 0.3, respectively. N_T is the number of parameters. The indicated error bars correspond to standard deviations calculated from the error matrix of the energy-dependent *pp* phase-shift analysis (Ref. 9).

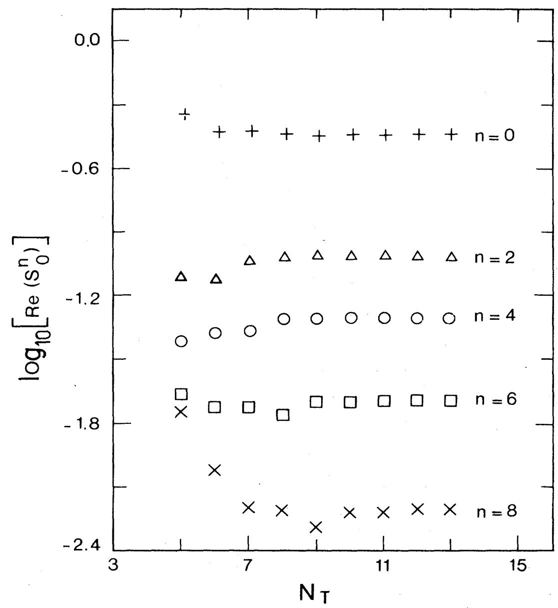

FIG. 2. Stability graph of the O(4) singlet amplitude $\mathrm{Re}S^n_0$. Variations of the absolute values of a sample amplitude are shown as a function of the number N_T of terms kept in the O(4) series for the fit. The logarithmic scale stresses the speed of convergence of these amplitudes with increasing values of n, indicated on the right.

(3) The values of O(4) amplitudes must be systematically smaller than their errors for $n > n_{max}$.

(4) The values of the parameters must become stable, i.e., the low-n terms do not change when further higher-n terms are added.

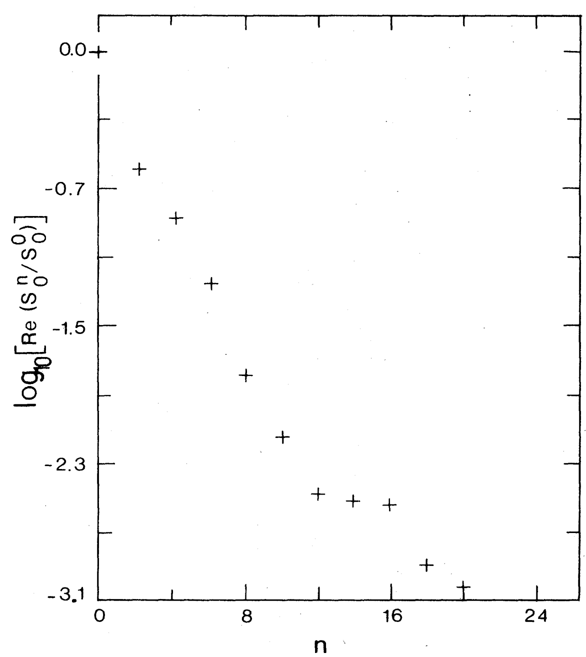

FIG. 3. Convergence graph of the O(4) amplitude $\mathrm{Re}S^n_0$. The sequence of ratios of the nth term to the first term is presented for $N_T = 11$.

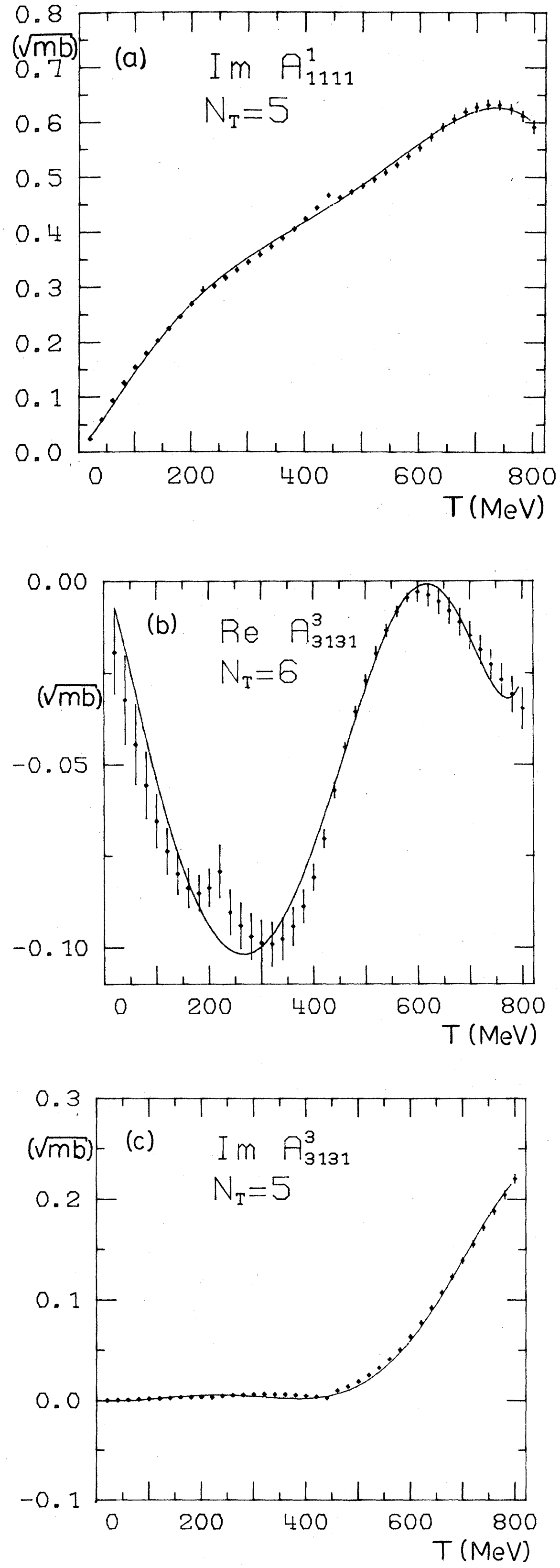

FIG. 4. Uncoupled triplet O(4) fits to partial amplitudes. $\mathrm{Im}A^1_{1111}(\alpha)$, $\mathrm{Re}A^3_{3131}(\alpha)$, and $\mathrm{Im}A^3_{3131}(\alpha)$. The χ^2/DF values are 3.7, 1.8, and 12.3, respectively.

We cannot restrict ourselves to a statistical criterion only, since the analysis involves an arbitrary number of input data (20-MeV steps) without taking their correlations into account. The χ^2 values are then biased by the somewhat artificial error bars.

B. The singlet amplitudes

We expand the singlet amplitude $A^j_{j0j0}(\alpha)$ for $l=j=0$, 2, 4, and 6 as in (2.8) and use the minimization procedure to determine the coefficients S^n_j. The ${}^1S_0$ amplitude $A^0_{0000}(\alpha)$ was by far the most difficult to fit in view, on one hand, of the very small "experimental errors" and, on the other hand, of the fact that its threshold behavior is not, *a priori*, known. The O(4) d functions $d^{n0}_{j00}(\alpha)$ for $j>0$ vanish at threshold $\alpha\to 0$ as $(\sin\alpha)^j$, i.e., more slowly than the physical amplitudes $A^j_{j0j0}(\alpha)$ should [namely, $(\sin\alpha)^{2j}$]. This can easily be corrected for by a slight modification of the expansions, which was indeed tried. It turned out to play no role in the considered energy region and we used the formalism as presented above. Some O(4) fits to $\mathrm{Re}A^j_{j0j0}$ and $\mathrm{Im}A^j_{j0j0}$ for $0\le T\equiv E_{\mathrm{lab}}\le 800$ MeV are shown on Figs. 1(a)—1(c). We see that for $j=0$ the visual fit is very good for $N_T=8$ even though the χ^2/DF is unacceptably large. For $N_T=10$ the curve already passes through all experimental points. Notice that in some cases, e.g., $\mathrm{Re}A^4_{4040}$ the O(4) curve corrects the threshold behavior of the reconstructed amplitudes (the phase shift analysis actually started at $T=10$ MeV, rather than $T=0$). For $\mathrm{Im}A^6_{6060}$ [Fig. 1(c)] the O(4) curve smoothly interpolates between the zero energy and the "experimental" points above 220 MeV. In the phase-shift analysis at low energies, this peripheral amplitude was replaced by the one-pion exchange amplitude, which does not contribute to $\mathrm{Im}A^j_{j0j0}$.

The stability of the singlet O(4) amplitudes can be judged by looking at Fig. 2, where we plot, in a logarithmic scale, the dependence of $\mathrm{Re}S^n_0$ on the number of parameters N_T used in the fit. We see that the values of $\mathrm{Re}S^n_0$ essentially stabilize at $N_T=6$, 8, 8, 9, and 10 for $n=0$, 2, 4, 6, and 8, respectively. We only give a sample graph here, the others are similar. Generally speaking the absolute values of $\mathrm{Re}S^n_j$ and $\mathrm{Im}S^n_j$ decrease as n increases. This is put in evidence on the example given in Fig. 3 again for $\mathrm{Re}S^n_0$.

C. The uncoupled-triplet amplitudes

The uncoupled-triplet amplitudes $A^j_{j1j1}(\alpha)$ were expanded as in (2.9) and the same minimization procedure was used to determine the O(4) amplitudes T^{n1}_{j1} ($j=1,3,5,7$, and $n\ge j$, odd). As examples, we show in Figs. 4(a)—4(c) the O(4) fits to $\mathrm{Im}A^1_{1111}$, $\mathrm{Re}A^3_{3131}$, and $\mathrm{Im}A^3_{3131}$ for $0\le T\le 800$ MeV. The same criteria were applied as for the singlet. Fits in general were easier to achieve, among other reasons because the experimental errors in the triplet amplitudes tend to be larger, than in the singlet ones. An exception is $\mathrm{Im}A^3_{3131}$, where the errors are quite small. We favored the cut off at $N_T=5$ parameters, even though the $\chi^2/\mathrm{DF}=12.3$ is large, since the visual fit is very good and the stability diagrams (not reproduced here) also point to this value. For higher j values

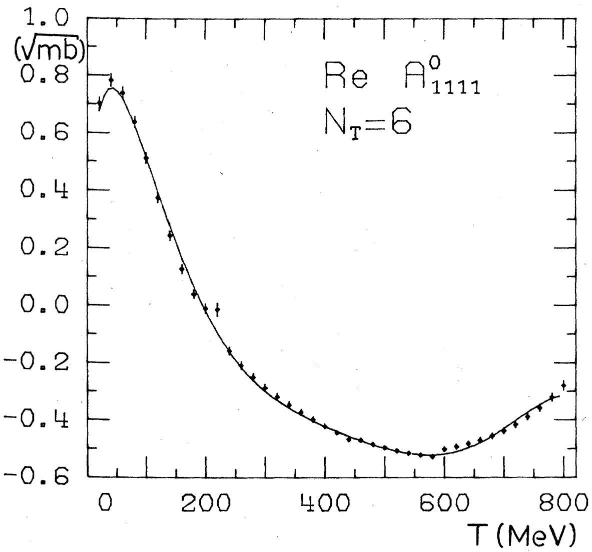

FIG. 5. Coupled triplet O(4) fit to the partial amplitude. $\mathrm{Re}A^0_{1111}(\alpha)$. $\chi^2/\mathrm{DF}=2.15$.

the relative errors are too large to permit a reliable determination of many parameters. The χ^2 fits are very good, the O(4) curves tend to the correct threshold values faster than the data points.

D. The coupled-triplet amplitudes

The only nonzero coupled-triplet amplitude for $j=0$ is $A^0_{1111}(\alpha)$. It is expanded as in (2.12). The χ^2 minimization provided a good fit with six real parameters ($n=1,3,\ldots,11$) each for $\mathrm{Re}A^0_{1111}$ (Fig. 5) and $\mathrm{Im}A^0_{1111}$.

The $j=2,4,6$ fits were performed to the decoupled version of the coupled-triplet amplitudes $A^j_{(j+1)1(j+1)1}$, $A^j_{(j-1)1(j+1)1}=A^j_{(j+1)1(j-1)1}$, and $A^j_{(j-1)1(j-1)1}$, namely, the three amplitudes $B^j_a(\alpha)$, $a=1,2,3$, as defined in (2.16) (with $B^j_4=B^j_1$). These were expanded using formulas (2.22) and the coefficients $Z^j_{a(2k)}$ were obtained from the usual χ^2/DF minimization. This provided us with numerical values of $Z^j_{a(2k)}$ and the O(4) amplitudes $T^{n\nu}_{j\lambda}$ [$n\geq j\geq 2$, n and j even, $(\nu,\lambda)=(1,1)$, $(0,1)$, or $(1,0)$] were calculated by inverting formulas (2.23) numerically. The fits for the amplitudes $\mathrm{Im}B^2_1$, $\mathrm{Re}B^4_2$, and $\mathrm{Re}B^6_1$, taken as examples, are shown on Figs. 6(a)–6(c). We note that low values of χ^2/DF and good visual fits are obtained with relatively few parameters (5 or 6 for each amplitude with $j=2$, between 2 and 5 for $j=4$ and $j=6$). This is facilitated by the overestimated error bars on the experimental amplitudes. Figure 6(c) shows an example of disagreements between some O(4) fits and phase-shift-analysis data in the low-energy region. Even though all the structures were described with a larger number of parameters, we have favored the lowest number of parameters sufficient to reproduce the amplitudes at intermediate energies. In any case the threshold behavior of the O(4) curves is preserved. We finally observe that in general the T amplitudes are more stable than the auxiliary Z amplitudes and that the absolute values of the former have the tendency to decrease more monotonically with increasing values of n than the latter.

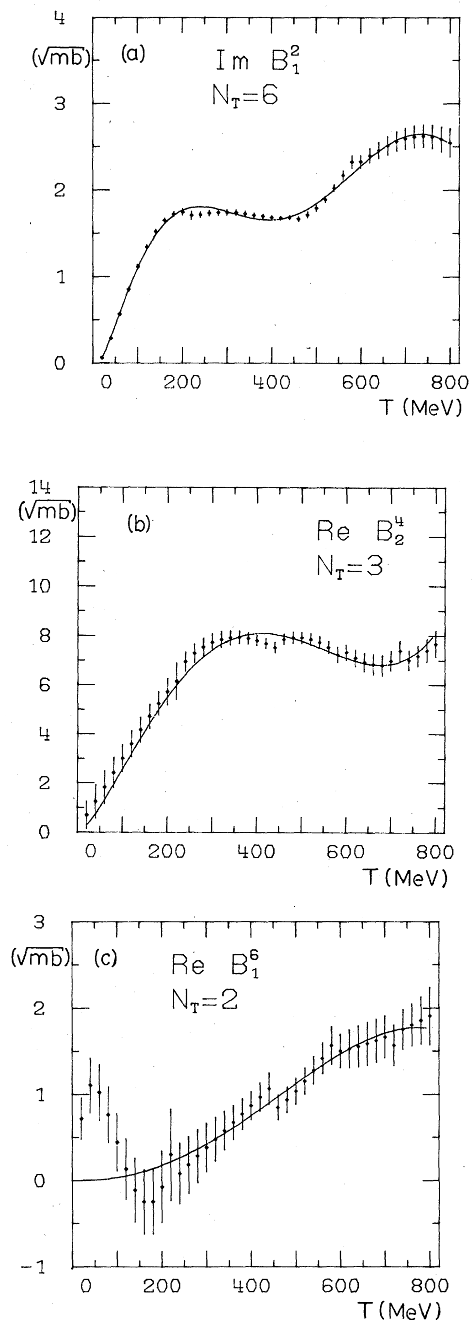

FIG. 6. Coupled triplet O(4) fits to partial amplitudes. $\mathrm{Im}B_1{}^2(\alpha)$, $\mathrm{Re}B_2{}^4(\alpha)$, and $\mathrm{Re}B_1{}^6(\alpha)$. The χ^2/DF values are 1.63, 0.79, and 1.22. The obtained Z^j_{an}, $a=1$, 2, and 3, lead to the original O(4) amplitudes T^{n1}_{j1}, T^{n0}_{j1}, and T^{n1}_{j0}, respectively. All the fits are performed separately since the new partial amplitudes $B^j_a(\alpha)$ are decoupled.

IV. CONCLUSIONS

The main conclusion that we draw is that the O(4) expansions have stood up to the test: they provide a good and stable fit to proton-proton partial-wave amplitudes $A^j_{lsl's'}(\alpha)$ for $0\leq T\leq 800$ MeV, using considerably fewer free parameters than conventional energy-dependent partial-wave analyses. Using the criteria of Sec. III A we have arrived at a total of 174 free real parameters: 46 for the singlet, 40 for the uncoupled-triplet, and 88 for the coupled-triplet amplitudes. While 174 is a larger number of parameters than we would have liked to use, it should be remembered that we are representing roughly 5300 independent data points, distributed over a large energy interval (800 MeV) and over 36 different curves (the real and imaginary parts of 18 partial-wave amplitudes). For comparison we mention that the original phase-shift analysis of Ref. 9 made use of 231 parameters (57 for the singlet, 50 and 124 for the uncoupled and coupled triplet, respectively).

We use the occasion to reiterate some of the advantages but also some of the problems of the O(4) expansions.

(1) Data points at all available energies and angles are treated simultaneously on an equal footing. No interpolation or extrapolation to chosen energy values is involved.

(2) The expansions incorporate and complement standard (energy-independent) partial-wave analysis. The partial-wave amplitudes have correct threshold behavior.

(3) The convergence of the O(4) expansions is guaranteed by group-theoretical arguments; the expansion is performed in terms of an orthonormal set of functions. As a consequence, a study of the O(4) amplitudes S_j^n and $T_{j\lambda}^{n\nu}$ as functions of n shows that they decrease rapidly and consistently and that their values are reasonably stable with respect to the cutoff value N_T. Typical examples of this behavior are shown on Figs. 2 and 3.

(4) Symmetries of the scattering amplitudes, such as parity conservation, the Pauli principle, and time-reversal invariance are readily imposed in terms of the O(4) amplitudes S_j^n and $T_{j\lambda}^{n\nu}$.

Several important questions remain open.

(a) The problem of the unitarity constraints on the scattering matrix in the O(4) formalism has only been touched upon at the end of Sec. II. These constraints, as mentioned above, are not important in the context of the present article.

In a direct fit of the scattering data, rather than of the partial waves, unitarity must be implemented, especially if the fit is restricted to energies below the inelastic threshold. It has been shown in an earlier article[7] that elastic unitarity implies that the O(4) amplitudes $T_{jss'\lambda}^{n\nu}$ must satisfy

$$T_{jss'\lambda'}^{n\nu} - (-1)^{2n-j-s} T_{jss'\lambda'}^{n\,-\nu *} = \sum_{n'\nu' n''\nu''} \sum_{\bar{s}\mu} Q_{js,\bar{s}\mu}^{n\nu n'\nu' n''\nu''} T_{js\bar{s}\mu}^{n'\nu' *} T_{j\bar{s}s'\lambda'}^{n''\nu''} , \tag{4.1}$$

where the quantities Q are expressed in terms of O(4) d functions as

$$Q_{js,\bar{s}\mu}^{n\nu n'\nu' n''\nu''} = \frac{2i}{\pi(2j+1)(2s+1)} [(n'+1)^2+\nu'^2][(n''+1)^2+\nu''^2] \times \sum_{\lambda} \int_0^{\pi} \sin^2\alpha \, d\alpha \, d_{js\lambda}^{n\nu}(\alpha) d_{js\lambda}^{n'\nu'}(\alpha) d_{j\bar{s}\mu}^{n''\nu'' *}(\alpha) . \tag{4.2}$$

These quantities can be expressed in terms of O(4) Clebsch-Gordan coefficients and simplified using the recursion relations of the present article. We shall return to these relations and their use in the implementation of elastic unitarity in connection with a fit to nucleon-nucleon scattering data.

(b) The O(4) expansions reflect the kinematics of the considered reaction; the dynamics is transferred to the O(4) scattering amplitudes. Dynamical singularities of partial-wave amplitudes, due to, e.g., the possible presence of dibaryons, or other structures (e.g., inelastic thresholds), are not automatically incorporated in the formalism. In a region where such effects are important, for instance, in an analysis of nucleon-nucleon data extended to above 1 GeV, an alternative approach should be adopted. On one hand, a straightforward O(4) fit along the lines of the present article should be performed; on the other, an O(4) fit to the background, supplemented by a Breit-Wigner-type fit to the resonances. A comparison of the two procedures would shed some light on the obscure problem of background subtraction and on the role of possible resonances.

A question not yet touched upon is the role of one-pion exchange in the O(4) formalism. By analogy with two-variable expansions for potential scattering,[15] we assume that large values of the parameter n correspond to large distances between the scattering nucleons. It follows that a modified approach, similar to modified phase-shift analysis, should be useful. It would consist of representing the values of the O(4) amplitudes $T_{j\lambda}^{n\nu}$ and S_j^n for $n > n_{\max}$ by their OPE values, rather than setting them equal to zero, as in the present paper. This should, on one hand, decrease the number of free parameters needed for a good fit, and, on the other, improve the stability of the fit with respect to a variation of the number of parameters used.

ACKNOWLEDGMENTS

The authors from each of the two participating research centers (Saclay and Montréal) are indebted to the other center for its hospitality in facilitating their mutual visits. The work of P.W. was supported in part by the Natural Sciences and Engineering Research Council of Canada and the "Fonds FCAC pour l'aide et le soutien á la recherche du Gouvernement dù Québec."

APPENDIX: SOME PROPERTIES OF THE O(4) REPRESENTATION MATRICES

We shall reproduce here some known formulas that we need in the text and also present some new results concerning recursion relations for the functions $d^{n\nu}_{j_1j_2\lambda}(\alpha)$.

A useful (and well-known) explicit expression is[7,13]

$$d^{n\nu}_{j_1j_2\lambda}(\alpha)=[(2j_1+1)(2j_2+1)]^{1/2}e^{-i\lambda\alpha}\sum_m \begin{bmatrix} \frac{n+\nu}{2} & \frac{n-\nu}{2} & j_1 \\ \lambda-m & m & -\lambda \end{bmatrix} \begin{bmatrix} \frac{n+\nu}{2} & \frac{n-\nu}{2} & j_2 \\ \lambda-m & m & -\lambda \end{bmatrix} e^{2im\alpha} . \tag{A1}$$

The threshold behavior of these functions is given by

$$d^{n\nu}_{j_1j_2\lambda}(\alpha)\underset{\alpha\to 0}{\sim}(\sin\alpha)^{|j_1-j_2|} . \tag{A2}$$

The symmetry properties are

$$\begin{aligned} &d^{n\nu}_{j_2j_1\lambda}(\alpha)=d^{n\nu}_{j_1j_2\lambda}(\alpha),\quad d^{n-\nu}_{j_1j_2-\lambda}(\alpha)=d^{n\nu}_{j_1j_2\lambda}(\alpha),\quad d^{n\nu*}_{j_1j_2\lambda}(\alpha)=d^{n\nu}_{j_1j_2\lambda}(-\alpha) ,\\ &d^{n\lambda}_{j_1j_2\nu}(\alpha)=d^{n\nu}_{j_1j_2\lambda}(\alpha),\quad d^{n-\nu}_{j_1j_2\lambda}(\alpha)=(-1)^{2n-j_1-j_2}d^{n\nu*}_{j_1j_2\lambda}(\alpha) ,\\ &d^{n\nu}_{j_1j_2\lambda}(\pi-\alpha)=e^{i\pi(n-\nu+j_1+j_2+\lambda)}d^{n\nu}_{j_1j_2-\lambda}(\alpha) . \end{aligned} \tag{A3}$$

The orthogonality and completeness relations[7] are

$$\begin{aligned} &\sum_{\lambda=-\min(j_1,j_2)}^{\min(j_1,j_2)}\int_0^\pi \sin^2\alpha\, d^{n\nu}_{j_1j_2\lambda}(\alpha)d^{n'\nu'*}_{j_1j_2\lambda}(\alpha)d\alpha=\frac{\pi}{2}\frac{(2j_1+1)(2j_2+1)}{(n+1)^2-\nu^2}\delta_{\nu\nu'}\delta_{nn'} ,\\ &\sum_{\nu=-\min(j_1,j_2)}^{\min(j_1,j_2)}\ \sum_{n=\max(j_1,j_2)}^{\infty}[(n+1)^2-\nu^2]d^{n\nu}_{j_1j_2\lambda}(\alpha)d^{n\nu*}_{j_1j_2\lambda'}(\alpha')=\frac{\pi}{2}\frac{(2j_1+1)(2j_2+1)}{\sin^2\alpha}\delta(\alpha-\alpha')\delta_{\lambda\lambda'} . \end{aligned} \tag{A4}$$

The representation theory of O(4) can be used to obtain a variety of recursion relations for the d functions. Here we only give two very special cases of these formulas which we have used to implement time-reversal invariance and the $\alpha\to\pi-\alpha$ symmetry in the coupled triplet. The formulas can be checked directly from the general expression (A1), and are, to our knowledge, new:

$$\begin{aligned} d^{n1}_{j11}(\alpha)+d^{n1}_{j1-1}(\alpha)=\frac{1}{[2j(j+1)]^{1/2}}\Bigg[&\left[\frac{(n+3)(n+j+2)(n-j+1)}{n}\right]^{1/2}d^{n+10}_{j11}(\alpha)\\ &-\left[\frac{(n-1)(n+j+1)(n-j)}{(n+2)}\right]^{1/2}d^{n-10}_{j11}(\alpha)\Bigg] \end{aligned} \tag{A5}$$

and

$$\begin{aligned} d^{n0}_{j10}(\alpha)=\frac{1}{(n+1)[2j(j+1)]^{1/2}}\{&[n(n+3)(n+j+2)(n-j+1)]^{1/2}d^{n+11}_{j10}(\alpha)\\ &-[(n-1)(n+2)(n+j+1)(n-j)]^{1/2}d^{n-11}_{j10}(\alpha)\} . \end{aligned} \tag{A6}$$

The only d functions needed for the O(4) expansions of pp scattering amplitudes are the following ones.

(1) Singlet and coupled triplet with $j=0$,

$$d^{n0}_{j00}(\alpha)=e^{-i(\pi/2)j}2^j j!\left[\frac{(2j+1)(n-j)!}{(n+1)(n+j+1)!}\right]^{1/2}(\sin\alpha)^j C^{j+1}_{n-j}(\cos\alpha) , \tag{A7}$$

where $C^{j+1}_{n-j}(\cos\alpha)$ are Gegenbauer polynomials.

(2) Uncoupled triplet,

$$d_{j11}^{(2k+1)1}(\alpha)-d_{j1(-1)}^{(2k+1)1}(\alpha)=i\left[\frac{6(2j+1)}{(k+1)(2k+1)(2k+3)}\right]^{1/2}(-1)^k$$
$$\times\left\{(-1)^{k+1}[(2k+1)(2k+2)]^{1/2}\begin{bmatrix}k & k+1 & j\\ k & -k-1 & 1\end{bmatrix}\sin(2k+1)\alpha\right.$$
$$+\sum_{m=1}^{k}(-1)^m\left[[(k-m+1)(k-m+2)]^{1/2}\begin{bmatrix}k & k+1 & j\\ -m & m-1 & 1\end{bmatrix}\right.$$
$$\left.\left.+[(k+m)(k+m+1)]^{1/2}\begin{bmatrix}k & k+1 & j\\ m-1 & -m & 1\end{bmatrix}\right]\sin(2m-1)\alpha\right\} \quad \text{(A8)}$$

(3) Coupled triplet with $j\neq 0$,

$$d_{j11}^{2k0}(\alpha)=-i\left[\frac{6(2j+1)}{k(k+1)(2k+1)}\right]^{1/2}\sum_{m=0}^{k-1}(-1)^m[(k-m)(k+m+1)]^{1/2}\begin{bmatrix}k & k & j\\ m & -m-1 & 1\end{bmatrix}\sin(2m+1)\alpha\ . \quad \text{(A9)}$$

[1]N. Ya. Vilenkin and Ya. A. Smorodinskiĭ, Zh. Eksp. Teor. Fiz. **46**, 1793 (1964) [Sov. Phys. JETP **19**, 1209 (1964)].

[2]P. Winternitz and I. Friš, Yad. Fiz. **1**, 889 (1965) [Sov. J. Nucl. Phys. **1**, 636 (1965)].

[3]M. Sheftel, J. Smorodinsky, and P. Winternitz, Phys. Lett. **26B**, 241 (1968).

[4]E. G. Kalnins, J. Patera, R. T. Sharp, and P. Winternitz, in *Group Theory and its Applications* (Academic, New York, 1965), Vol. 3 (a review containing extensive references to the original articles).

[5]M. Daumens, M. Perroud, and P. Winternitz, Phys. Rev. D **19**, 3413 (1979).

[6]J. Bystricky, F. Lehar, J. Patera, and P. Winternitz, Phys. Rev. D **13**, 1276 (1976).

[7]M. Daumens and P. Winternitz, Phys. Rev. D **21**, 1919 (1980).

[8]M. Daumens, E. Saintout, and P. Winternitz, Phys. Rev. D **26**, 1629 (1982).

[9]J. Bystricky, C. Lechanoine-Leluc, and F. Lehar, Saclay Report No. DPhPE-82-12, 1982 (unpublished).

[10]P. Moussa and R. Stora, in *Analysis of Scattering and Decay*, lectures at Herceg-Novi School, Yugoslavia (Gordon and Breach, New York, 1968).

[11]D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon, Oxford, 1968).

[12](a) H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957); (b) P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, *ibid.* **114**, 880 (1959).

[13]L. C. Biedenharn, J. Math. Phys. **2**, 433 (1961).

[14]J. Bystricky and F. Lehar, *Nucleon-Nucleon Scattering Data, Physics Data,* edited by H. Behrens and G. Ebel (Fachinformationszentrum, Karlsruhe), Nos. 11-1 (1978), 11-2 (1981), 11-3 (1981); J. Bystricky *et al.*, *Elastic and Charge Exchange Scattering of Elementary Particles, Landolt-Börnstein Tables, New Series,* edited by H. Schopper (Springer, Berlin, 1980), Vol. 9.

[15]E. G. Kalnins, J. Patera, R. T. Sharp, and P. Winternitz, Phys. Rev. D **8**, 2552 (1973); **8**, 3527 (1973).