Gravitational Aharonov-Bohm effect in three dimensions

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We examine the effects of parallel transport of spinors and vectors around the known pointlike solutions of three-dimensional gravity. We point out that the massive, zero-spin solution corresponds, when suitably embedded in four dimensions, to a cosmic string.

Recently, several authors[1] have considered classical general relativity in three dimensions. The theory differs fundamentally from that of four dimensions: the vanishing of the Einstein tensor $G^{\mu\nu}$ implies the vanishing of the Riemann curvature $R^{\mu}{}_{\nu\alpha\beta}$, and Newton's constant G has dimensions of (length) rather than $(\text{length})^2$ (in units in which $\hbar=c=1$). If one takes Einstein's equations as the dynamical equations of motion, the former condition means that space is flat wherever $T^{\mu\nu}$ vanishes. However, this does not mean that a pointlike source has no gravitational effect: a light beam passing by a massive, pointlike source will be deflected,[1,2] and parallel transport in a closed circuit around such a source will in general give nontrivial results. The situation is analogous to the Aharonov-Bohm effect in electromagnetism.[2,3] Here we examine in detail the effects of such closed-path parallel transport. We also point out that the massive, spinless solution in three dimensions corresponds, in four dimensions, to a cosmic string.

Using a matrix notation for the spin connection and curvature, so that

$$\omega_{ab}\equiv\omega_{\mu ab}dx^{\mu}\ ,$$
$$R_{ab}\equiv\tfrac{1}{2}R_{\mu\nu ab}dx^{\mu}\wedge dx^{\nu}\ , \tag{1}$$

the Riemann curvature (in the tangent frame) may be written

$$R=d\omega+\omega\wedge\omega\ . \tag{2}$$

From our experience with gauge theories we know that R will vanish if and only if

$$\omega=UdU^{-1}\ , \tag{3}$$

where in an n-dimensional space $U(x)$ is an element of $SO(n-1,1)$. ω is then a "gauge transform of zero." Similarly, in the coordinate frame we can introduce the maxtrix-valued forms

$$\Gamma^{\mu}_{\nu}\equiv\Gamma^{\mu}_{\nu\alpha}dx^{\alpha}\ ,$$
$$R^{\mu}{}_{\nu}\equiv\tfrac{1}{2}R^{\mu}{}_{\nu\rho\alpha}dx^{\rho}\wedge dx^{\alpha}\ , \tag{4}$$

so that

$$R=d\Gamma+\Gamma\wedge\Gamma\ . \tag{5}$$

Clearly again R will vanish if and only if

$$\Gamma=SdS^{-1}\ , \tag{6}$$

for some matrix S. We can see what S is by writing the transformation rule for Γ:

$$\Gamma'^{\mu}_{\beta}(x')=\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\Gamma^{\nu}_{\alpha}(x)+\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\beta}\partial x'^{\gamma}}dx'^{\gamma}\ . \tag{7}$$

Since the space is flat, we can choose a frame (i.e., Cartesian coordinates) in which the first term on the right-hand side of (7) vanishes. Comparison with (6) then gives

$$S^{\mu}{}_{\rho}=\frac{\partial x'^{\mu}}{\partial x^{\rho}}\ . \tag{8}$$

Given U and S, it is straightforward to solve the equations for spinor and vector parallel transport. We imagine some curve with parameter τ, $x^{\mu}=x^{\mu}(\tau)$, $\tau_i\leq\tau\leq\tau_f$. Then the parallel-transported spinor at τ is given in terms of the spinor at $\tau=0$ by[4]

$$\frac{D\psi}{D\tau}=0=\frac{dx^{\mu}}{d\tau}\left(\frac{\partial}{\partial x^{\mu}}+\omega_{\mu}\right)\psi=\left(\frac{d}{d\tau}-\frac{dU}{d\tau}U^{-1}\right)\psi\ ,$$

so

$$\psi_a(\tau)=U_a{}^{b}(\tau)\psi_b(0)\ . \tag{9}$$

Similarly for a vector

$$V^{\mu}(\tau)=S^{\mu}{}_{\nu}(\tau)V^{\nu}(0)\ . \tag{10}$$

If, in particular, for a closed curve $U(\tau_f)\neq U(\tau_i)$ (similarly for S), we will find a gravitational Aharonov-Bohm effect.[2]

We turn now to the particular case of the pointlike solutions in three dimensions. There are, in general, two kinds of solutions:[1] a pointlike mass at rest with no angular momentum, with line element

$$ds^2=dt^2-d\rho^2-\rho^2\alpha^2d\theta^2\ ,$$
$$0\leq\theta<2\pi,\ \ \alpha=1-4Gm\ , \tag{11}$$

and a spinning, massless source with line element

$$ds^2=(dt-Ad\theta)^2-d\rho^2-\rho^2d\theta^2,\ \ A=4GJ\ . \tag{12}$$

Here m is the mass and J the angular momentum. In general, for a massive, spinning pointlike source,[1]

$$ds^2=(dt-Ad\theta)^2-d\rho^2-\rho^2\alpha^2d\theta^2\ , \tag{13}$$

but we prefer to deal with (11) and (12) separately.

The space part of the metric (11) is that of a plane with a wedge removed and edges identified, i.e., a cone. The angular defect gives rise to a mismatch of the components of spinors and vectors upon closed parallel transport around the (singular) cone tip. Choosing a frame in which the vielbein is

$$\begin{aligned} e^0 &= dt \ , \\ e^1 &= \cos\theta\, d\rho - \alpha\rho \sin\theta\, d\theta \ , \\ e^2 &= \sin\theta\, d\rho + \alpha\rho\cos\theta\, d\theta \ , \\ ds^2 &= e^0e^0 - e^1e^1 - e^2e^2 \ , \end{aligned} \tag{14}$$

the Cartan structure equations $de^a = -e^b V\omega^a_b$ give

$$\omega = -\frac{i}{2}(\alpha-1)\sigma^3 d\theta \ , \tag{15}$$

for the spin connection, so from (3),

$$U = e^{(i/2)(\alpha-1)\sigma^3\theta} \ .$$

(Here σ^3 is the Pauli matrix: we recall that Dirac spinors have two components in three dimensions.) Note that ω has the required property of vanishing when $\alpha=1$, i.e., when the space is flat. [Different choices of frame can give ω's that do not so vanish. For example, $e^0=dt$, $e^1=d\rho$, and $e^2=\rho\alpha d\theta$ gives $\omega=-i(\alpha/2)\sigma^3 d\theta$, and a spinor parallel transported in a closed circuit around the $\rho=0$ axis picks up an overall minus sign, which corresponds to an additional rotation of the spinor about its own axis by 2π. This minus sign is a consequence of the choice of frame, and not a physical effect associated with a solution of the field equations.] Thus a closed circuit, for the frame defined by (14), gives

$$\psi(2\pi) = [\cos(\alpha-1)\pi + i\sigma^3\sin(\alpha-1)\pi]\psi(0) \ , \tag{16}$$

and there will be no Aharonov-Bohm phase if and only if α is an odd integer. This condition is not empty because although α cannot be negative if the metric is to be regular everywhere outside the singularity, it can be arbitrarily positive (i.e., negative values for Gm are allowed).[1]

Similarly, the equations for parallel transport of a vector V_μ give

$$\begin{pmatrix} \dot V_\rho \\ \dot V_\theta \end{pmatrix} = \begin{pmatrix} 0 & \dot\theta/\rho \\ -\rho\alpha^2\dot\theta & \dot\rho/\rho \end{pmatrix} \begin{pmatrix} V_\rho \\ V_\theta \end{pmatrix} \ , \tag{17}$$

the solution to which is

$$\begin{pmatrix} V_\rho(\tau) \\ V_\theta(\tau) \end{pmatrix} = \begin{pmatrix} \cos\alpha\theta & (1/\rho_0\alpha)\sin\alpha\theta \\ -\rho\alpha\sin\alpha\theta & (\rho/\rho_0)\cos\alpha\theta \end{pmatrix} \begin{pmatrix} V_\rho(0) \\ V_\theta(0) \end{pmatrix} \ , \tag{18}$$

if at $\tau=0$, $\theta=0$ and $\rho=\rho_0$. The dots in (17) refer to differentiation with respect to τ. We have omitted V_t, which is constant. Thus for vectors there will be no Aharonov-Bohm effect if and only if α is an integer. Perhaps not surprisingly we have found that spinors are more sensitive than vectors for detecting such pointlike masses.

As regards the massless, spinning solution, Eq. (12), one can always choose a frame in which the spin connection vanishes, so spinor parallel transport is unaffected by the presence of such an object. It is straightforward to check that vectors are similarly unaffected.

Finally, we note that the solution (11) is a three-dimensional cross section of a cylindrically symmetric cosmic string, with line element[2]

$$ds^2 = dt^2 - d\rho^2 - \rho^2\alpha^2 d\phi^2 - dz^2 \ . \tag{19}$$

(Note that $R^\mu{}_{\nu\alpha\beta}$ vanishes everywhere outside such a string.)

Since $g_{\mu\nu}$ is independent of z, and g_{33} is a constant, the affine connection is also independent of z and has no z components: thus the nonvanishing components of the curvature are the same in four and three dimensions,

$${}^{(4)}R^\mu{}_{\nu\alpha\beta} = {}^{(3)}R^\mu{}_{\nu\alpha\beta} \ . \tag{20}$$

The only new component of Einstein's equations in four dimensions is therefore

$$-\tfrac{1}{2}g_{33}R = 8\pi{}^{(4)}G{}^{(4)}T_{33} = +\tfrac{1}{2}g_{00}R = -8\pi{}^{(3)}G{}^{(3)}T_{00} \tag{21}$$

(we have used ${}^{(3)}R_{00}=0$). Thus, since in three dimensions the only nonvanishing component of the stress tensor is[1]

$${}^{(3)}T_{00} = \frac{1}{\sqrt{g}} m\delta(x)\delta(y) \ , \tag{22}$$

we recover the nonvanishing components of the stress tensor of a cosmic string in four dimensions:[2]

$${}^{(4)}T_{33} = -{}^{(4)}T_{00} = -\frac{1}{\sqrt{-g}} m\delta(x)\delta(y) \ , \tag{23}$$

where

$$m \equiv \frac{1}{4G}(1-\alpha) \ . \tag{24}$$

Note that the dimensions of G are such that m is a mass in three dimensions, and a mass per unit length in four dimensions.

It is natural to ask whether one might construct a massless, spinning cosmic string by similarly embedding the solution (12) in four dimensions. However, while the Einstein tensor corresponding to the metric (11) contains the expected Dirac delta function, that corresponding to (12) contains squares of delta functions. Clement,[1] following a standard method,[5] has found R_{00}, R^i_0, and R^{ij}, for a completely general three-dimensional metric. From these one can construct $G^{\mu\nu}$ by using the identities

$$R = R_{00} - g_{0i}g_{0j}R^{ij} + g_{ij}R^{ij} \ , \tag{25}$$

$$R^{00} = R_{00} - g_{0i}g_{0j}R^{ij} - 2g_{0i}R^{0i} \ , \tag{26}$$

$$R^{0i} = R^i_0 - g_{0j}R^{ji} \ , \tag{27}$$

which are valid when $g_{00}=1$. If one then inserts the specific form (12) for $g_{\mu\nu}$, one finds a $G^{\mu\nu}$ whose only vanishing components are G^{ij}, $i\neq j$, and whose every nonvanishing component contains terms of the form $(\nabla^2\ln r)^2 \sim [\delta(\mathbf{r})]^2$. If one embeds in four dimensions by writing

$$ds^2 = (dt - A d\theta)^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2 \ , \tag{28}$$

one finds, by the arguments used above, the same singularities. The physical interpretation of these singularities, in three or four dimensions, is unclear.

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