Color confinement and the quantum-chromodynamic vacuum. III

Jose Luis Rodriguez Marrero and Arthur R. Swift
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
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In earlier papers it was shown that if soft gluons do not propagate in the vacuum state the vacuum expectation value of the modified Coulomb interaction develops a singularity at zero momentum that leads to the confinement of color. The consistency of this picture is established with the demonstration that if the interaction is singular soft gluons do not propagate. This conclusion is shown to be independent of most approximations. Taken together the three papers in the series provide a framework for calculating the most important aspects of color confinement in quantum chromodynamics.

I. INTRODUCTION

This paper is the third in a series which seeks to establish an analytic connection between the properties of the vacuum in quantum chromodynamics (QCD) and the confinement of color. In the first paper[1] (I) it was established that, if the vacuum expectation value (VEV) of the modified Coulomb interaction in the Coulomb gauge is infrared singular, color is confined. An infrared-singular interaction is one which grows with distance in configuration space or which diverges sufficiently rapidly near the origin in momentum space. In both the quark and gluon sectors only color-singlet states have finite energy. There are infrared divergences in the bound-state equations. Both the interaction and the self-energy contributions are singular. For a color-singlet state there is an exact cancellation between the two. If the state carries color, there is no cancellation and the energy is infinite. The second paper[2] (II) examined the mechanism for producing the postulated infrared singularity. It was shown that if the gluon propagator is softened at low momentum, the modified Coulomb interaction naturally develops an infrared singularity. In other words, the failure of low-momentum gluons to propagate in the physical vacuum leads to a long-range confining interaction.

In this paper we demonstrate that the nonpropagation of gluons is, in turn, a consequence of the singular interaction. In QCD there is a regime, distinct from the perturbation-theory sector, in which a confining interaction is generated in a self-consistent manner. The vacuum modifies gluon propagation to produce a singular Coulomb interaction. This interaction, in turn, produces the modification of gluon propagation. This regime is characterized by a single dimensionful parameter which cannot be determined within the theory. The results of this paper provide a theoretical justification for the special properties that were postulated in I and II.

Two arguments are presented to establish the softening of the gluon propagator. In the next section, the Tamm-Dancoff[3] formalism of I is applied to vacuum-vacuum amplitudes. In the third section the VEV of the Hamiltonian is minimized as a functional of the gluon propagator. Both procedures produce the same integral equation for the VEV of the gluon propagator. There is a digression in Sec. II that is needed to establish an essential property of the VEV of gluon field operators. As in earlier papers, all calculations are in the Coulomb gauge. The necessary approximations have been used before. The zero-momentum solution of the integral equation is discussed in Sec. IV. All but the most dedicated readers will want to skip the fifth section which analyzes the effects of the most important approximations. The conclusion is that if the modified Coulomb interaction is infrared singular, the gluon propagator approaches a constant at $p=0$. Perturbation theory predicts $1/p$. The final section discusses the significance of our results and suggests an explicit model of the vacuum state which appears to incorporate those properties of the Coulomb-gauge vacuum essential for confinement.

II. THE GLUON PROPAGATOR IN THE TAMM-DANCOFF FORMALISM

In the Tamm-Dancoff[3] formalism one calculates the expectation value of the commutator of field operators with the Hamiltonian,

$$\langle 0\,|\,[O(\phi),H]\,|\,\psi\rangle=\langle E_\psi-E_0\rangle\langle 0\,|\,O(\phi)\,|\,\psi\rangle\ . \tag{1}$$

The states $|\,\psi\rangle$ and $|\,0\rangle$ are eigenstates of the total Hamiltonian. Thus, $|\,0\rangle$ is the physical vacuum state. The operator $O(\phi)$ is a product of field operators ϕ, and the commutator on the left of (1) is evaluated with the equal-time, canonical commutation rules. Here, we take $|\,\psi\rangle=|\,0\rangle$, and the right-hand side vanishes. The gluon propagator is a function of the vacuum expectation value of two-gluon field operators,

$$\langle\, A_i^a(\mathbf{p})A_j^b(\mathbf{k})\,\rangle=\frac{\delta_{ab}\delta(\mathbf{p}+\mathbf{k})P_{ij}(\mathbf{p})}{2(2\pi)^3}A(\mathbf{p})\ , \tag{2}$$

where $P_{ij}(\mathbf{p})=\delta_{ij}-p_ip_j/p^2$ and a,b are color indices. In the Coulomb gauge[4] the gluon field $A_i^a(\mathbf{p})$ has only spatial indices and is transverse $[\mathbf{p}\cdot\mathbf{A}^a(\mathbf{p})=0]$. The δ functions and the spin-projection operator in (2) are fixed by the requirement that the vacuum state is colorless, spinless, and translation invariant. The factor of $1/[2(2\pi)^3]$ is chosen so that in the perturbation-theory limit $A(p)=1/p$. We intend to show that in the interacting theory

$$\lim_{p\to 0} A(p)=1/m \ , \tag{3}$$

where m is an arbitrary constant with dimensions of mass.

The canonically conjugate operator to $A_i^a(\mathbf{p})$ is $E_i^a(\mathbf{p})$,

$$[A_i^a(\mathbf{p}),E_j^b(\mathbf{k})]=\frac{i}{(2\pi)^3}\delta_{ab}\delta(\mathbf{p}+\mathbf{k})P_{ij}(\mathbf{p}) \ . \tag{4}$$

Since $E_i^a(\mathbf{p})$ is a transverse vector operator,

$$\langle E_i^a(\mathbf{p})E_j^b(\mathbf{k})\rangle=\frac{\delta_{ab}\delta(\mathbf{p}+\mathbf{k})P_{ij}(\mathbf{p})}{2(2\pi)^3}E(p) \ . \tag{5}$$

In the perturbation-theory limit $E(p)=p=1/A(p)$. The VEV's of the two other bilinear field operators are fixed by the equal-time commutation relations together with the parity and time-reversal invariance of the vacuum state,

$$\begin{aligned}\langle A_i^a(\mathbf{p})E_j^b(\mathbf{k})\rangle &= -\langle E_i^a(\mathbf{p})A_j^b(\mathbf{k})\rangle \\ &= i\frac{\delta_{ab}\delta(\mathbf{p}+\mathbf{k})P_{ij}(\mathbf{p})}{2(2\pi)^3} \ .\end{aligned} \tag{6}$$

To use (1) we need the QCD Hamiltonian in the Coulomb gauge,[5]

$$H=H_0+H_C+H'+H'' \ . \tag{7}$$

The free Hamiltonian is

$$H_0=\frac{(2\pi)^3}{2}\int d\mathbf{p}[\mathbf{E}^a(\mathbf{p})\cdot\mathbf{E}^a(-\mathbf{p})+p^2A^a(\mathbf{p})\cdot A^a(-\mathbf{p})] \ . \tag{8}$$

The Coulomb Hamiltonian is

$$\begin{aligned}H_C=\frac{g^2(2\pi)^3}{2}\int d\mathbf{1}\,d\mathbf{2}\,d\mathbf{3}\,d\mathbf{4}\,f_{ahc}f_{bde}\mathbf{E}^h(\mathbf{1})\cdot\mathbf{A}^c(\mathbf{2})\\ \times F_{ab}(-\mathbf{1}-\mathbf{2},-\mathbf{3}-\mathbf{4})\mathbf{E}^d(\mathbf{3})\cdot\mathbf{A}^e(\mathbf{4}) \ .\end{aligned} \tag{9}$$

In I the modified Coulomb operator $F_{ab}(\mathbf{p},\mathbf{k})$ was assumed to have a VEV of the form

$$\langle F_{ab}(\mathbf{p},\mathbf{k})\rangle=\delta_{ab}\delta(\mathbf{p}+\mathbf{k})F(p) \ , \tag{10}$$

with $F(\mathbf{p})\to p^{-2n}$ as $p\to 0$. When $n>\frac{3}{2}$, color is confined. In lowest-order perturbation theory $F(\mathbf{p})=1/p^2$. In II it was shown that if $A(p)\to p^{-2\beta}$ as $p\to 0$, with $\beta<\frac{1}{2}$, $F(\mathbf{p})$ has the required behavior. The operator $F_{ab}(\mathbf{p},\mathbf{k})$ is defined in terms of the QCD Coulomb Green's function,[1]

$$\begin{aligned}F_{ab}(\mathbf{p},\mathbf{k}) &= \int d\mathbf{q}\,D_{ac}(\mathbf{p},\mathbf{q})q^2D_{cb}(-\mathbf{q},\mathbf{k})\\ &= \frac{d}{dg}[gD_{ab}(\mathbf{p},\mathbf{k})] \ .\end{aligned} \tag{11}$$

The second expression for $F_{ab}(\mathbf{p},\mathbf{k})$ comes from converting the integral equation[5] for the operator $D_{ab}(\mathbf{p},\mathbf{k})$,

$$\begin{aligned}D_{ab}(\mathbf{p},\mathbf{k}) &= \frac{\delta_{ab}\delta(\mathbf{p}+\mathbf{k})}{p^2}\\ &\quad +ig\frac{f_{ade}}{p^2}\int d\mathbf{q}\,\mathbf{p}\cdot\mathbf{A}^d(\mathbf{p}-\mathbf{q})D_{eb}(\mathbf{q},\mathbf{k}) \ ,\end{aligned} \tag{12}$$

into a power series in the coupling constant g and inserting that series into the defining equation for $F_{ab}(\mathbf{p},\mathbf{k})$.

The three- and four-gluon interaction terms are contained in H',

$$\begin{aligned}H' &= ig(2\pi)^3f_{abc}\int d\mathbf{1}\,d\mathbf{2}\,d\mathbf{3}\,\delta(\mathbf{1}+\mathbf{2}+\mathbf{3})\mathbf{A}^a(\mathbf{1})\cdot\mathbf{A}^c(\mathbf{3})\mathbf{1}\cdot\mathbf{A}^b(\mathbf{2})\\ &\quad +\frac{g^2(2\pi)^3}{4}f_{abc}f_{ade}\int d\mathbf{1}\,d\mathbf{2}\,d\mathbf{3}\,d\mathbf{4}\,\delta(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})\mathbf{A}^b(\mathbf{1})\cdot\mathbf{A}^d(\mathbf{3})\mathbf{A}^c(\mathbf{2})\cdot\mathbf{A}^e(\mathbf{4}) \ .\end{aligned} \tag{13}$$

The final piece of the Hamiltonian, H'', contains renormalization counterterms and the Schwinger[4] term.

To apply the Tamm-Dancoff formalism, we need to make two major approximations. Both have been made before.[2] In Sec. V we consider the validity of these approximations, or at least the conditions under which they are valid. In H_C, $F_{ab}(\mathbf{p},\mathbf{k})$ is replaced by its VEV as it was in I. We are interested in the infrared limit where the VEV is assumed to dominate H_C. The practical effect is to replace an operator by a c-number. H_C becomes quartic in gluon field operators. The second assumption is that the VEV of a product of gluon field operators is to be evaluated in terms of all possible VEV of pairs of fields. For example,

$$\begin{aligned}\langle A_i^a(\mathbf{1})A_j^b(\mathbf{2})E_m^c(\mathbf{3})E_n^d(\mathbf{4})\rangle=\frac{1}{[2(2\pi)^3]^2}[&\delta_{ab}\delta(\mathbf{1}+\mathbf{2})P_{ij}(\mathbf{1})\delta_{cd}\delta(\mathbf{3}+\mathbf{4})P_{mn}(\mathbf{3})A(\mathbf{1})E(\mathbf{3})\\ &-\delta_{ac}\delta(\mathbf{1}+\mathbf{3})P_{im}(\mathbf{1})\delta_{bd}\delta(\mathbf{2}+\mathbf{4})P_{jn}(\mathbf{2})\\ &-\delta_{ad}(\mathbf{1}+\mathbf{4})P_{in}(\mathbf{1})\delta_{bc}\delta(\mathbf{2}+\mathbf{3})P_{jm}(\mathbf{2})] \ .\end{aligned} \tag{14}$$

This procedure may not be an approximation if the VEV is interpreted in terms of propagators. The implicit assumption is that gluon number does not change when a gluon scatters off the vacuum. In II the method was used to convert the operator equation for $D_{ab}(\mathbf{p},\mathbf{k})$ into a "Dyson" equation for $\langle D_{ab}(\mathbf{p},\mathbf{k})\rangle$. A third, less significant approximation is the neglect of H' and H''. We return to these corrections in Sec. V.

The derivation of the key equation is almost trivial. We use (1) with $O(\phi)$ equal to one of the three bilinear products of $A_i^a(\mathbf{p})$ and $E_j^b(\mathbf{k})$. If $O(\phi)$ is $A_i^a(\mathbf{p})A_j^b(\mathbf{k})$ or $E_i^a(\mathbf{p})E_j^b(\mathbf{k})$, the result is just $0=0$. The interest is in

$$\langle [A_i^a(\mathbf{p})E_j^b(\mathbf{k}),H_0+H_C]\rangle=0$$

$$=\frac{i}{2(2\pi)^3}\delta(\mathbf{p}+\mathbf{k})\delta_{ab}P_{ij}(\mathbf{p})\times\left[E(p)-p^2A(p)+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{q}\,F(\mathbf{p}+\mathbf{q})\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})][E(p)A(q)-A(p)E(q)]\right], \tag{15}$$

where N refers to SU(N). This equation can be rewritten in the form

$$\frac{A(p)}{E(q)}=\frac{1+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{q}\,F(\mathbf{p}+\mathbf{q})A(q)\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})]}{p^2+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{q}\,F(\mathbf{p}+\mathbf{q})E(q)\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})]}. \tag{16}$$

The trace produces an angular factor $1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2$. If there is a second equation that relates $A(p)$ and $E(p)$, Eq. (16) becomes a nonlinear integral equation for $A(p)$.

Several comments must be made. The function $F(\mathbf{p}+\mathbf{q})$ is infrared singular. However, the singularity can be subtracted with the procedure discussed in I. Let

$$F(\mathbf{p}+\mathbf{q})=\rho\delta(\mathbf{p}+\mathbf{q})+\bar{F}(\mathbf{p}+\mathbf{q}), \tag{17}$$

with ρ a divergent constant. $\bar{F}(\mathbf{p}+\mathbf{q})$ is infrared finite. If (17) is used in (15), the divergent constant ρ cancels out. Hence in (16) we can replace $F(\mathbf{p}+\mathbf{q})$ by $\bar{F}(\mathbf{p}+\mathbf{q})$ to render the integrals finite at $\mathbf{p}+\mathbf{q}=0$. Next we note that asymptotic freedom implies that as $p\to\infty$, all functions approach their asymptotic values: $A(p)\to 1/p$, $E(p)\to p$, and $F(\mathbf{p})\to 1/p^2$. Hence, the integrals in (16) are ultraviolet divergent. The divergence is cancelled by counterterms in H''. Since we are interested in the $p\to 0$ properties of $A(p)$, we imagine the integrals are cut off at large momentum through an unspecified renormalization prescription.

Further progress requires a second relation between $A(p)$ and $E(p)$. In the Coulomb gauge the color charge operator is[5]

$$Q_a=(2\pi)^3 f_{abc}\int d\mathbf{p}\,\mathbf{A}^b(\mathbf{p})\cdot\mathbf{E}^c(-\mathbf{p}). \tag{18}$$

We calculate the color charge density of the vacuum state using the prescription for the VEV of products of field operators

$$\frac{\langle Q^2\rangle}{V}=\frac{N(N^2-1)}{2(2\pi)^3}\int d\mathbf{p}[E(p)A(p)-1], \tag{19}$$

where V is the volume of space. If the vacuum state is a color singlet, this density should vanish. Using

$$\int d\mathbf{p}[E(p)A(p)-1]=0,$$

we find

$$\frac{\langle Q^4\rangle}{V}=\frac{5}{8}\frac{N^2(N^2-1)}{(2\pi)^3}\int d\mathbf{p}[E(p)A(p)-1]^2. \tag{20}$$

This VEV should also vanish if the vacuum state is locally colorless. Thus we have

$$E(p)=A(p)^{-1}. \tag{21}$$

In perturbation theory, this condition is satisfied, but it is true more generally. The VEV $A(p)$ satisfies the equation

$$A(p)^2=\frac{1+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{q}\,\bar{F}(\mathbf{p}+\mathbf{q})A(q)\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})]}{p^2+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{q}\,\frac{\bar{F}(\mathbf{p}+\mathbf{q})}{A(q)}\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})]}. \tag{22}$$

When $g=0$, the perturbation limit is recovered. In Sec. IV we study this equation and show that if $F(\mathbf{p}+\mathbf{q})$ is infrared singular, $A(0)=1/m$, where m is a finite constant with dimensions of mass.

III. VACUUM ENERGY DENSITY

An alternate derivation of Eq. (22) starts with the VEV of the Hamiltonian. Using the approximations of the previous section, we find that

$$\frac{\langle H\rangle}{V}=\frac{N^2-1}{2(2\pi)^3}\left[\int d\mathbf{p}[E(p)+p^2A(p)]+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{p}\,d\mathbf{k}\,\mathrm{Tr}[P(\mathbf{p})P(\mathbf{k})]F(\mathbf{p}+\mathbf{k})[E(p)A(k)-1]\right]. \tag{23}$$

If $E(p)=1/A(p)$, the vacuum energy density is a functional of $A(p)$. [It is also infrared finite; $F(\mathbf{p}+\mathbf{k})$ can be replaced by $\bar{F}(\mathbf{p}+\mathbf{k})$.] When the energy density is minimized, we discover

$$\frac{\delta}{\delta A(p)}\frac{\langle H\rangle}{V}=\frac{N^2-1}{2(2\pi)^3}\left[p^2-\frac{1}{A(p)^2}+\frac{g^2N}{4(2\pi)^3}\int d\mathbf{k}\,\mathrm{Tr}[P(\mathbf{p})P(\mathbf{k})\bar{F}(\mathbf{p}+\mathbf{k})\left[\frac{1}{A(k)}-\frac{A(k)}{A(p)^2}\right]\right]=0. \tag{24}$$

Upon rearrangement this expression becomes the integral equation for $A(p)$ obtained from the Tamm-Dancoff method. Unfortunately, this minimization procedure cannot easily be extended to encompass the analysis of correction terms. The implicit dependence of $F(\mathbf{p}+\mathbf{k})$ on $A(p)$ makes the approach too cumbersome. However, the identity of the results in the two approaches suggests that there is a strong connection between confinement and a vacuum state of minimum energy.

IV. INFRARED BEHAVIOR OF $A(p)$

It is possible to extract the $p\to 0$ behavior of $A(p)$ from Eq. (22). First we consider a generic integral:

$$I(p)=\int \frac{d\mathbf{k}}{k^{2\beta}[(\mathbf{p}+\mathbf{k})^2+\mu^2]^n[\lambda^2+k^2]^s} . \tag{25}$$

If $n>\frac{3}{2}$ and $\mu=0$, $I(p)$ is divergent due to the infrared singularity at $\mathbf{k}=-\mathbf{p}$. The factor of $(\lambda^2+k^2)^{-s}$ controls the large-k behavior of the integral but is irrelevant for the small-k limit. When $n>\frac{3}{2}$, $\mu\to 0$, and $p\to 0$, $I(p)$ becomes

$$\begin{aligned} I(p)=&\frac{\pi^{3/2}\Gamma(n-\frac{3}{2})}{\Gamma(n)}(\mu)^{3-2n}\frac{p^{-2\beta}}{(\lambda^2+p^2)^s}\\ &+\pi^{3/2}\frac{\Gamma(n+\beta-\frac{3}{2})\Gamma(\frac{3}{2}-n)\Gamma(\frac{3}{2}-\beta)}{\Gamma(n)\Gamma(\beta)\Gamma(3-n-\beta)}\frac{p^{3-2n-2\beta}}{\lambda^{2s}}\\ &+2\pi\frac{\Gamma(n+\beta+s-\frac{3}{2})\Gamma(\frac{3}{2}-n-\beta)}{\Gamma(s)}\lambda^{3-2n-2\beta-2s} . \end{aligned} \tag{26}$$

The term which diverges as $\mu\to 0$ contains the infrared singularity. This infinity cancels out of all integrals. The second term dominates as $p\to 0$ as long as $\beta\neq 0$. If $\beta=0$, the finite part of the integral is independent of p as $p\to 0$. Equation (26) suggests an alternate prescription for obtaining the dominant, infrared-finite part of $I(p)$. First set $k=0$ in the integrand except where the $k,p\to 0$ limit of the integrand would be altered. Then do the integration for values of n and β for which it converges. The result is analytically continued to interesting values of n and β. It is not necessary to maintain the distinction between $F(\mathbf{p})$ and $\bar{F}(\mathbf{p})$; the infrared singularity never appears. The power of p that results is exactly that which is calculated from scaling $\mathbf{k}$ by p in the integrand. The large-k behavior of the integrand is irrelevant in the $p\to 0$ limit as evidenced by the factor of $1/\lambda^{2s}$ in (26).

Applying this lesson to the integrals in the equation for $A(p)$, we try a solution of the form

$$A(p)=\frac{1}{m}\left[\frac{p^2}{m^2}\right]^{-\beta} . \tag{27}$$

The $p\to 0$ limit of (22) is

$$\frac{1}{m^2}\left[\frac{p^2}{m^2}\right]^{-2\beta}=\frac{1+C(n,\beta)\dfrac{\lambda}{m^{1-2\beta}}p^{3-2n-2\beta}}{p^2+C(n,-\beta)\dfrac{\lambda}{m^{2\beta-1}}p^{3-2n+2\beta}} . \tag{28}$$

We used $F(\mathbf{p}+\mathbf{k})=\lambda[(\mathbf{p}+\mathbf{k})^2]^{-n}$ when $p,k\to 0$. If $n+\beta>\frac{3}{2}$ and $n-\beta>\frac{1}{2}$, the integral terms dominate both the numerator and denominator. (A linear confining potential corresponds to $n=2$.) Consistency of the ansatz (27) requires

$$\frac{1}{m^2}\left[\frac{p^2}{m^2}\right]^{-2\beta}=\frac{C(n,\beta)}{C(n,-\beta)}\frac{1}{m^2}\left[\frac{p^2}{m^2}\right]^{-2\beta} , \tag{29}$$

or $C(n,\beta)/C(n,-\beta)=1$. Clearly, $\beta=0$ is a solution. Explicit calculation shows that there are no other solutions for reasonable values of β. In particular, the perturbation-theory value $\beta=\frac{1}{2}$ is not a solution if $n>\frac{3}{2}$. When $\beta=0$, the character of the $p\to 0$ integrals changes [see (26)], and (28) is replaced by

$$\frac{1}{m^2}=\frac{1+[C(n)+p^2C_N'(n)]/m}{p^2+m[C(n)+p^2C_D'(n)]} , \tag{30}$$

where $C_N'(n)$ and $C_D'(n)$ are not equal because they depend on the order-p^2 corrections to $A(p)$. The inconsistency that appears when (30) is rearranged,

$$\frac{p^2}{m^2}[1+mC_D'(n)]=1+\frac{p^2}{m^2}C_N'(n) , \tag{31}$$

is not a problem. The correction terms in the numerator differ from those in the denominator.

V. CORRECTION TERMS

Heretofore we have ignored the H' and H'' parts of the full Hamiltonian. In addition we replaced the modified Coulomb operator by its VEV. Our analysis of the effect of these approximation centers on the question of whether a more accurate calculation would change our conclusion about the zero-momentum limit of $A(p)$. The effect of H' is easily computed,

$$\begin{aligned}\langle [A_i^a(\mathbf{p})E_j^b(\mathbf{k}),H']\rangle=&\frac{i\delta_{ab}\delta(\mathbf{p}+\mathbf{k})}{2(2\pi)^3}P_{ij}(\mathbf{p})A(p)\\ &\times\left[-\frac{8}{3}\frac{g^2N}{2(2\pi)^3}\int d\mathbf{q}\,A(q)\right] . \end{aligned}\tag{32}$$

The expression in large parentheses appears in the denominator of (22) as an additional momentum-independent term. If $n-\beta>\frac{3}{2}$ in (28), it has no effect on our discussion of the zero-momentum limit of $A(p)$. If the inequality is violated, the H' term dominates the denominator and any solution other than $\beta=0$ is impossible. Thus, consideration of H' strengthens our conclusions about $A(p)$.

The effect of H'' is more nebulous since we have carefully avoided specifying the renormalization counterterms in detail. However, those terms, related as they are to the infinite-momentum limit, will not contain the infrared singularity associated with $F(\mathbf{p}+\mathbf{k})$. Thus, to the extent that H'' contributes at all as $p\to 0$, when $\beta\to 0$, its effect will be reduced by at least p^{2n-3} compared to the terms retained in (22). The Schwinger term[4] does not have an infrared divergence,

$$H_S''=\frac{g^2}{8(2\pi)^3}\int d1\,d2\,d3\,d4\,\delta(1+2+3+4)1\cdot 3 f_{abc}\times f_{ade}D_{bd}(1,2)D_{ec}(3,4)\ . \tag{33}$$

It affects only the denominator of (22) and is negligible as $p\rightarrow 0$ (order $p^{4\beta}$) compared to the dominant part of H_C.

The effect of replacing the operator $F_{ab}(\mathbf{p},\mathbf{k})$ by its VEV is more difficult to estimate. We must investigate

$$\langle [A_i^a(\mathbf{p})E_j^b(\mathbf{k}),H_C]\rangle=\frac{ig^2}{2(2\pi)^3}P_{is}(\mathbf{p})P_{jt}(\mathbf{k})f_{fhc}f_{gde}\times\int d1\,d2\,d3\,d4\frac{d}{dg}[g(\ )]\ , \tag{34}$$

where

$$\begin{aligned}(\ )=&-\delta_{cb}\delta(\mathbf{k}+2)\langle A_s^a(\mathbf{p})E_t^h(1)D_{fg}(-1-2,-3-4)\mathbf{E}^d(3)\cdot\mathbf{A}^e(4)\rangle\\&-\delta_{eb}\delta(\mathbf{k}+4)\langle \mathbf{E}^h(1)\cdot\mathbf{A}^c(2)D_{fg}(-1-2,-3-4)A_s^a(\mathbf{p})E_t^d(3)\rangle\\&+\delta_{ah}\delta(\mathbf{p}+1)\langle A_s^c(2)E_t^b(\mathbf{k})D_{fg}(-1-2,-3-4)\mathbf{E}^d(3)\cdot\mathbf{A}^e(4)\rangle\\&+\delta_{ad}\delta(\mathbf{p}+3)\langle \mathbf{E}^h(1)\cdot\mathbf{A}^c(2)D_{fg}(-1-2,-3,-4)A_s^e(4)E_t^b(\mathbf{k})\rangle\\&-ig\langle \mathbf{E}^h(1)\cdot\mathbf{A}^c(2)A_s^a(\mathbf{p})\int d\mathbf{q}\,D_{fx}(-1-2,-\mathbf{q})f_{xby}q_tD_{yg}(\mathbf{q}+\mathbf{k},-3-4)\mathbf{E}^d(3)\cdot\mathbf{A}^e(4)\rangle\ .\end{aligned} \tag{35}$$

The integral equation for $D_{ab}(\mathbf{p},\mathbf{k})$ has been used to evaluate the commutator $[E_j^b(\mathbf{k}),D_{ac}(\mathbf{p},\mathbf{q})]$. The derivative with respect to coupling constant allows us to deal with $D_{ac}(\mathbf{p},\mathbf{k})$ rather than $F_{ac}(\mathbf{p},\mathbf{k})$. Since $D_{ac}(\mathbf{p},\mathbf{k})$ contains all powers of $A_i^a(\mathbf{p})$, an accurate treatment of the VEV's requires contractions between the explicit field operators and the implicit operators in $D_{ac}(\mathbf{p},\mathbf{k})$. The calculation is lengthy, but possible with the use of an ansatz that incorporates our rules for evaluating the VEV of a product of field operators. We set

$$A_i^a(\mathbf{p})=\frac{A(p)^{1/2}}{[2(2\pi)^3]^{1/2}}\epsilon_i(\mathbf{p})[a^a(\mathbf{k})+a^{\dagger a}(-\mathbf{p})]\ , \tag{36a}$$

$$E_j^b(\mathbf{k})=\frac{-i}{[2(2\pi)^3]^{1/2}}\frac{\epsilon_j(\mathbf{k})}{[A(k)]^{1/2}}[a^b(\mathbf{k})-a^{\dagger b}(-\mathbf{k})]\ . \tag{36b}$$

Polarization indices have been suppressed. The polarization vector $\epsilon_i(\mathbf{p})$ is transverse. The creation and annihilation operators $a^{\dagger a}(\mathbf{p})$ and $a^a(\mathbf{p})$ obey free-field commutation relations, and $a^a(\mathbf{p})$ annihilates the physical vacuum state.

The first VEV in (35) becomes

$$(\ )_1=-\frac{\delta_{cb}\delta(\mathbf{k}+2)}{[2(2\pi)^3]^2}\epsilon_s(\mathbf{p})\epsilon_t(1)\epsilon(3)\cdot\epsilon(4)\langle a^a(\mathbf{p})a^h(1)D_{fg}(-1-2,-3-4)a^{\dagger d}(-3)a^{\dagger e}(-4)\rangle\ . \tag{37}$$

When the explicit field operators are commuted to annihilate the vacuum, a variety of terms appear,

$$\begin{aligned}\langle\ \rangle_1=&\langle D_{fg}(-1-2,-3-4)\rangle[\delta_{ad}\delta_{he}\delta(\mathbf{p}+3)\delta(1+4)+\delta_{ae}\delta_{hd}\delta(\mathbf{p}+4)\delta(1+3)]\\&+\langle[[a^h(1),D_{fg}(-1-2,-3-4)],a^{\dagger d}(-3)]\rangle\delta_{ae}\delta(\mathbf{p}+4)+\cdots\\&+\langle[[[a^a(\mathbf{p}),[a^h(1),D_{fg}(-1-2,-3-4)]],a^{\dagger d}(-3)],a^{\dagger e}(-4)]\rangle\ .\end{aligned} \tag{38}$$

The first line is the part we retained in Sec. II. The three dots stand for three additional VEV's of double commutators. We proceed to evaluate a double commutator VEV,

$$\begin{aligned}&\langle[[a^h(\mathbf{p}),D_{fg}(-1-2,-3-4)],a^{\dagger d}(-\mathbf{k})]\rangle\\&=\frac{-g^2}{2(2\pi)^3}[A(p)A(k)]^{1/2}\\&\quad\times\int d\mathbf{q}\,d\mathbf{q}'[\langle D_{fx}(-1-2,-\mathbf{q})f_{xhy}\mathbf{q}\cdot\epsilon(\mathbf{p})D_{yz}(\mathbf{q}+\mathbf{p},-\mathbf{q}')f_{zdw}\mathbf{q}'\cdot\epsilon(\mathbf{k})D_{wg}(\mathbf{q}'+\mathbf{k},-3-4)\rangle+(\mathbf{p},h)\leftrightarrow(\mathbf{k},d)]\ .\end{aligned} \tag{39}$$

Equation (39) is exact. To evaluate the VEV we use the diagrammatic expansion introduced in II. The leading term in powers of g is obtained when each $D_{xy}(1,2)$ is replaced by its VEV,

$$\langle D_{xy}(1,2)\rangle=\delta_{xy}\delta(1+2)D(1)\ . \tag{40}$$

Higher orders introduce vertex functions which are implicit functions of $D(p)$. The quadruple commutator term in (38) is of order g^4.

There are a very large number of correction terms. Fortunately it is not necessary to discuss all of them in order to estimate their effect on the infrared behavior of $A(p)$. A representative term is the g^4 addition to the numerator in (22). The numerator becomes

$$1+\frac{g^2}{4(2\pi)^3}\int d\mathbf{q}\,\bar{F}(\mathbf{p}+\mathbf{q})A(q)\mathrm{Tr}[P(\mathbf{p})P(\mathbf{q})]$$
$$+\frac{g^2N^2}{16(2\pi)^6}\int\int d1\,d2\,A(1)A(2)[(\mathbf{p}+\mathbf{4})\cdot P(1)\cdot P(\mathbf{p})\cdot P(\mathbf{4})\cdot(\mathbf{p}+\mathbf{1})]$$
$$\times[\bar{F}(\mathbf{1}+\mathbf{p})gD(\mathbf{4}+\mathbf{p})gD(\mathbf{1}+\mathbf{4}+\mathbf{p})+gD(\mathbf{1}+\mathbf{p})\bar{F}(\mathbf{4}+\mathbf{p})gD(\mathbf{1}+\mathbf{4}+\mathbf{p})$$
$$+gD(\mathbf{1}+\mathbf{p})gD(\mathbf{4}+\mathbf{p})\bar{F}(\mathbf{1}+\mathbf{4}+\mathbf{p})]\,. \quad (41)$$

The double integral is infrared finite; hence, $\bar{F}(\mathbf{p})$ appears rather than $F(\mathbf{p})$. In the limit $p\to 0$ the p dependence of the correction is exactly that of the g^2 term. The rule for extracting the $p\to 0$ limit of an integral is to evaluate all functions in the integrand in that limit. In II it was shown that if $A(p)\approx(p/m)^{-2\beta}/m$, then $gD(p)\approx(p/m)^{\beta-5/2}/m^2$. The scale mass m is the same in both functions. Thus, if $F(p)\approx p^{-2n}$, and all internal momentum are scaled by p, the g^2 and g^4 parts of (41) are both proportional to $p^{3-2n-2\beta}$. On dimensional grounds we expect that all corrections, of any order in g, in either the numerator or denominator of (22), will have the same power dependence as $p\to 0$. There will be but one factor of $F(\mathbf{p})$ in the integrand of each integral—a consequence of the single derivative with respect to coupling constant. The factors of $A(p)$ and $D(p)$ either in the integrand or outside the integral will always compensate explicit momentum factors to maintain the power dependence.

We conclude that the correction terms arising from the fact that $F_{ac}(\mathbf{p},\mathbf{k})$ is properly an operator function of $A_i^a(p)$ do not change the zero-momentum behavior of $A(p)$. Equations (28) and (29) are still valid. We have verified that, to order g^4, the dominant numerator and denominator integrals are identical when $\beta\to-\beta$. Explicit evaluation of correction terms in II suggests that there may be a ten-percent shift in the coefficient functions.

Finally we mention the "approximation" inherent in our method for evaluating the VEV of a product of field operators. It is actually an ansatz for the nature of the physical vacuum state. To the extent that interactions actually dress the field operators in accordance with Eq. (36), there is no approximation. If our approach to QCD is self consistent, the validity of (36) constitutes another probe of the vacuum state.

VI. CONCLUSIONS

In order to establish color confinement in QCD one should derive a mechanism for confinement and then prove that it works.[6] In our investigations of the canonical field theory in the Coulomb gauge, we have been working backwards. First we proposed the mechanism and showed that it confined color. Next, in II and this paper, we provided the derivation. The mechanism is an infrared-singular, modified Coulomb interaction. Confinement is observed in the Tamm-Dancoff[1] and the Bethe-Salpeter[7] treatments of bound states. The derivation of the singular interaction invokes a self-consistency argument. The Coulomb-gauge Green's function is determined by the physical, transverse gluon field. In II we argued that if soft gluons do not propagate, the Green's function, or equivalently, the running coupling constant, becomes singular at $p=0$. The modified Coulomb interaction is essentially the derivative of the Green's function. It is even more singular. Self-consistency is provided by the results of this paper. The singular interaction is responsible for the low-momentum cutoff on gluon propagation. In I and II we showed that if soft gluons do not propagate, color is confined. Here we proved that if color is confined, soft gluons do not propagate. The singular interaction and the modification of the propagator are different manifestations of the screening properties of the vacuum.

Are there any loopholes in our arguments? Our analysis of the correction terms makes us confident that the propagator tends to a constant as $p\to 0$ rather than diverging as $1/p$. The analysis in II which established the connection between $A(p)$ and the running coupling constant is equally rigorous. If $A(p)\to$constant, $gD(p)=g(p)/p^2\to p^{-5/2}$. The correction terms are again subject to dimensional analysis; the connection between the infrared power dependence of $A(p)$ and $g(p)$ is maintained in higher orders. The quantitative connection between the infrared singularity in the modified Coulomb interaction and $A(p)$ is moderately sensitive to higher-order terms. In II it was shown that the lowest-order Dyson equation for $F(\mathbf{p})$ predicts that there are no solutions for $F(p)$ if $A(p)\to p^{-2\beta}$ and $\beta<0.0095$. If β takes this minimum value, $F(p)\to p^{-3.991}$. [A linear configuration space potential corresponds to $F(p)\to p^{-4}$.] In this paper we predict $\beta=0$. Full consistency and a derivation of a linear potential are tantalizingly close. Hopefully, there are small corrections which, when calculated in detail, will shift the limit on β in the desired direction.

An important extension of this work will be the construction of an explicit model of the vacuum. It is easy to find a unitary operator which dresses bare gluon field operators to produce the ansatz in (36). That unitary operator, acting on the bare vacuum, creates a state consisting of a coherent superposition of color-singlet gluon pairs.[8] The existence of an explicit vacuum state paves the way for investigation of the gauge dependence of our conclusions. At a minimum it should be possible to demonstrate invariance of the vacuum under a combination of Lorentz boost and gauge transformations which do not leave the Coulomb gauge. Ultimately the whole canonical, Coulomb-gauge treatment should be translated into the language of path integrals and freed from the choice of a particular gauge.

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