Generalized coherent states and generalized squeezed coherent states

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Roy and Virendra Singh showed that the harmonic oscillator possesses an infinite string of exact shape-preserving coherent wave-packet states $|n,\alpha\rangle$ having classical motion. In this paper it is shown that the states $|n,\alpha\rangle$ could be obtained from the coherent state $|\alpha\rangle$ and it is also shown how a coherent state $|\alpha\rangle$ could be expanded in the basis of $|n,\alpha\rangle$'s. Further, the possibility of "squeezing" the state $|n\rangle$ is investigated and the "generalized squeezed coherent states" are obtained. The squeezed coherent states for the displaced oscillator are also defined. The physical meaning of squeezing is also pointed out.

I. INTRODUCTION

It is well known that Schrödinger's original motivation[1] for introducing coherent states was to look for those states with probability-density wave packet remaining unchanged in shape as time progresses and have the classical motion

$$\langle x\rangle = x_{\rm cl}(t)\equiv A\cos(\omega t+\varphi) \tag{1}$$

and

$$\langle p\rangle = p_{\rm cl}(t)\equiv M\dot{x}\ , \tag{2}$$

where

$$A=|\alpha|\left[\frac{2\hbar}{M\omega}\right]^{1/2} \tag{3}$$

and

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is a coherent state (CS).

Nieto[2] *et al.* have used Schrödinger's criterion to define coherent states for arbitrary potentials. (For the other criteria of defining coherent states see Ref. 3.)

Recently, Roy and Virendra Singh[4] showed, by adopting Schrödinger's criterion, i.e., to define coherent states as those with *undistorted normalizable wave packets with classical motion,* that the harmonic oscillator possesses an infinite string of coherent states, hitherto not thought of. Originally they were known as "semicoherent states" introduced by Boiteux and Levelut.[4] We briefly discuss their coherent states below.

Define

$$U(\alpha(t))\equiv\exp[\alpha(t)a^{\dagger}-\alpha^{*}(t)a]\ . \tag{4}$$

Then the "generalized coherent states" (GCS's) of the harmonic oscillator are

$$|n,\alpha\rangle\equiv U(\alpha(t))|n\rangle\exp[-i(n+\tfrac{1}{2})\omega t]\ ,\qquad n=0,1,2,\ldots\ , \tag{5}$$

where $|n\rangle$ is the nth state of the harmonic oscillator. It is easy to see that the state $|n,\alpha\rangle$ satisfies Schrödinger's criterion.

The Fock-space representation of $|n,\alpha\rangle$ is found to be

$$|n,\alpha\rangle=\exp[-\tfrac{1}{2}|\alpha(0)|^2]\sum_{m=0}^{\infty}\left[\frac{n!}{m!}\right]^{1/2}L_n^{(m-n)}(|\alpha(0)|^2)[\alpha(0)]^{m-n}|m\rangle\exp[-i\omega t(m+\tfrac{1}{2})]\ , \tag{6}$$

where $L_n^{(m-n)}(x)$ are Laguerre polynomials.

Also, the uncertainty in the state $|n,\alpha\rangle$ is given by

$$\Delta x\,\Delta p=(n+\tfrac{1}{2})\hbar\omega\ . \tag{7}$$

The above equation implies that the minimum uncertainty (i.e., $\hbar\omega/2$) is not necessary for the classical motion of a wave packet. This fact has been also noted by Ohnuki and Kamefuchi.[5]

The states $|n,\alpha\rangle$, though not explicitly stated, could be spotted in the literature.[6–11] The states $|n,\alpha\rangle$ could also be obtained when one considers the Hamiltonian[6]

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}q^2+qf(t)\ , \tag{8}$$

where $f(t)$ is an external force. If the driving term is taken to be linear in a and $a^{\dagger}$, then one obtains the states $|n,\alpha\rangle$.

If one considers $\langle m|n,\alpha\rangle$, then

$$|\langle m|n,\alpha\rangle|^2 = e^{-|\alpha|^2}\left[\frac{n!}{m!}\right]|\alpha|^{2(m-n)}[L_n^{(m-n)}(|\alpha|^2)]^2\ . \tag{9}$$

It has been shown by Koonin[7] that Eq. (9) is related to the S-matrix element S_{mn} by

$$| S_{mn} |^2 = | \langle m \mid n,\alpha \rangle |^2 \,. \tag{10}$$

S_{mn} gives the amplitude for excitation from the initial oscillator state $| n \rangle$ to the final $| m \rangle$.

Also, Hollenhorst[8] has proved that Eq. (9) gives the matrix element for a transition from the state $| n \rangle$ to the state $| m \rangle$ under the influence of a gravity wave.

Equation (9) is also known as Schwinger's formula,[9] and is also given by Feynman.[10] (See also Refs. 11—13.)

Section II discusses the relationship between GCS's and CS's and in Sec. III generalized squeezed coherent states are introduced.

II. RELATIONSHIP BETWEEN GCS AND CS

We are interested in obtaining $| n,\alpha \rangle$ from $| \alpha \rangle$. Let

$$| n,\alpha \rangle = A(a,a^\dagger,n) \, | \alpha \rangle \tag{11}$$

$$\equiv A(a,a^\dagger,n) U(\alpha) \, | 0 \rangle \,, \tag{12}$$

where $A(a,a^\dagger,n)$ is the operator to be determined. Also

$$| n,\alpha \rangle \equiv U(\alpha) \, | n \rangle \tag{13}$$

$$\equiv U(\alpha) \frac{(a^\dagger)^n}{\sqrt{n!}} \, | 0 \rangle \,. \tag{14}$$

From Eqs. (12) and (14)

$$A = U \frac{(a^\dagger)^n}{\sqrt{n!}} U^\dagger \,. \tag{15}$$

Since $U(\alpha)$ translates a and $a^\dagger$, it could be proved using operator calculus[14] that

$$A = \frac{(a^\dagger - \alpha^*)^n}{\sqrt{n!}} \,. \tag{16}$$

Therefore,

$$| n,\alpha \rangle = \frac{(a^\dagger - \alpha^*)^n}{\sqrt{n!}} \, | \alpha \rangle \,. \tag{17}$$

Now, the meaning of the state $| n,\alpha \rangle$ is very clear as the nth state of the oscillator whose ground state is $| \alpha \rangle$, a coherent state, *not* $| 0 \rangle$, as in the case of the usual oscillator. In other words, the GCS's are the excited states of the displaced oscillator.

The above result is clearly depicted in the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
| \alpha \rangle \arrow[d, "A"'] & | 0 \rangle \arrow[l, "U(\alpha)"'] \arrow[d, "\frac{(a^\dagger)^n}{\sqrt{n!}}"] \\
| n,\alpha \rangle & | n \rangle \arrow[l, "U(\alpha)"]
\end{tikzcd}$$

$| 0 \rangle$: ground state of the harmonic oscillator,

$| \alpha \rangle$: ground state of the displaced harmonic oscillator .

The above method of obtaining GCS's using Eq. (17) not only establishes the relationship between GCS's and CS's and also gives a simple algebraic way to obtain the GCS's.

One can also see that the GCS $| n,\alpha \rangle$ is the "generalized coherent state" in the sense of Perelemov[11], for whom the reference state could be an arbitrary vector in the Fock space.

Since

$$a^\dagger \, | \alpha \rangle = \left[\frac{\partial}{\partial \alpha} + \frac{\alpha^*}{2} \right] | \alpha \rangle \,, \tag{18}$$

Eq. (17) could be given a differential operator representation as

$$| n,\alpha \rangle = \frac{1}{\sqrt{n!}} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-\alpha^*)^{n-m} \left[\frac{\partial}{\partial \alpha} + \frac{\alpha^*}{2} \right]^m | \alpha \rangle \,. \tag{19}$$

Therefore, we observe that the GCS $| n,\alpha \rangle$ is related to the CS $| \alpha \rangle$ just the number state $| n \rangle$ is related to the vacuum state $| 0 \rangle$.

Now, we illustrate below an interesting use of GCS's.

The displacement operators $U(\alpha)$'s provide a complete and orthonormal basis for the adjoint group of the Weyl group formed by $a, a^\dagger, \mathbb{1}$ with a scalar product given by

$$(U(\alpha), U(\alpha')) = \mathrm{Tr}[U(\alpha) U^\dagger(\alpha')] = \pi \delta(\alpha - \alpha') \tag{20}$$

and

$$U(\alpha) U(\beta) = \exp[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)] U(\alpha+\beta) \,. \tag{21}$$

In view of Eqs. (20) and (21), we think of defining coherent states of the displaced oscillator as

$$| z,\alpha \rangle \equiv \exp[z(a^\dagger - \alpha^*) - z^*(a - \alpha)] \, | \alpha \rangle \tag{22}$$

$$= \exp[z^*\alpha - z\alpha^*] U(z) \, | \alpha \rangle$$

$$= \exp[z^*\alpha - z\alpha^*] e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} U(z) \, | n \rangle \,. \tag{23}$$

Using the relation (6), we get

$$| z,\alpha \rangle \equiv e^{(z^*\alpha - z\alpha^*)} e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} L_n^{(m-n)}(|\alpha|^2) \, | m \rangle \,. \tag{24}$$

Equation (21) could be written as

$$| z,\alpha \rangle \equiv \exp(z^*\alpha - z\alpha^*) U(z) U(\alpha) \, | 0 \rangle \,. \tag{25}$$

In view of Eq. (21), $| z,\alpha \rangle$ is just another element in the set of coherent states, which forms an invariant subspace of the Hilbert space. Equation (21) could also be written as

$$| z,\alpha \rangle \equiv e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, | \alpha,n \rangle \,. \tag{26}$$

From Eqs. (25) and (26), we note that any arbitrary coherent state could be expanded in terms of GCS's.

III. GENERALIZED SQUEEZED COHERENT STATES

First, we shall briefly discuss the squeezed coherent states (SCS's). The SCS is defined as[15,16]

$$|\alpha,z\rangle \equiv U(\alpha)S(z)|0\rangle , \tag{27}$$

where $U(\alpha)$ is the displacement operator given by

$$U(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^* a) \tag{28}$$

and

$$S(z)\equiv\exp\left[\frac{z}{2}a^\dagger a^\dagger-\frac{z^*}{2}aa\right] \tag{29}$$

is known as the squeeze operator. Also,

$$\begin{aligned} SaS^\dagger &= a\cosh r+e^{i\theta}a^\dagger \sinh r=b ,\\ Sa^\dagger S^\dagger &= a^\dagger\cosh r+e^{-i\theta}a\sinh r=b^\dagger , \end{aligned} \tag{30}$$

where $z=re^{i\theta}$. The squeezed states correspond to Gaussian wave packets with *widths distorted from that of the vacuum state and those states follow the classical motion though the uncertainties oscillate.*[17]

Recently there has been a lot of excitement regarding SCS's, since they are considered to be useful in the detection of gravity waves.[8] It has also been proved that these states are emitted in certain nonlinear optical processes.[18] (For a review on SCS's see Refs. 19 and 20.)

Now, we define generalized squeezed coherent states (GSCS's) as

$$|n,z,\alpha\rangle \equiv U(\alpha)S(z)|n\rangle . \tag{31}$$

We first compute $|n,z\rangle$:

$$|n,z\rangle \equiv S(z)|n\rangle \tag{32}$$

$$\equiv\sum_m |m\rangle\langle m|S(z)|n\rangle$$

$$\equiv\sum_m |m\rangle G_{mn}(z) . \tag{33}$$

Making use of a slightly modified form of the technique developed by Rashid,[21] we get the expansion coefficients $G_{mn}(z)$ to be

$$G_{mn}(z)=\begin{cases} e^{-i(n-m)\theta/2}(-1)^{m+n/2}\left[\dfrac{m!n!}{\cosh r}\right]^{1/2}\left[\dfrac{\tanh r}{2}\right]^{(m+n)/2}\displaystyle\sum_\lambda \frac{\left[-\dfrac{4}{\sinh^2 r}\right]^\lambda}{(2\lambda)!\left[\dfrac{m}{2}-\lambda\right]!\left[\dfrac{n}{2}-\lambda\right]!} & \text{for } m,n \text{ even}\\[2ex] e^{-i(n-m)\theta/2}(-1)^{m+n/2-3/2}\left[\dfrac{m!n!}{\cosh^3 r}\right]^{1/2}\left[\dfrac{\tanh r}{2}\right]^{(m+n)/2-1}\displaystyle\sum_\lambda \frac{\left[-\dfrac{4}{\sinh^2 r}\right]^\lambda}{(2\lambda+1)!\left[\dfrac{m-1}{2}-\lambda\right]!\left[\dfrac{n-1}{2}-\lambda\right]!} & \text{for } m,n \text{ odd}\\[2ex] 0, \text{ otherwise} . \end{cases} \tag{34}$$

Only odd-odd or even-even elements of $G_{mn}(z)$ survive due to the fact that $S(z)$ essentially creates two excitations every time it acts.

In the spirit of Refs. 17 and 22, we realize that $S(z)$ has a finite expectation value in the state $|n\rangle$ (for $n=0,1,2,\ldots$), i.e., squeezing of the states $|n\rangle$ is possible. We consider an interesting case below:

$$G_{00}(z)=\langle 0|S(z)|0\rangle \tag{35}$$

$$\equiv\frac{1}{(\cosh|z|)^{1/2}} . \tag{36}$$

[The authors of Refs. 17 and 19 have remarked that $G_{00}(z)$ sums as $\tanh|z|<1$ for $r<\infty$. See Eq. (4.2) of Ref. 14.] Now the GSCS $|n,z,\alpha\rangle$ is given by

$$|n,z,\alpha\rangle \equiv U(\alpha)|n,z\rangle \equiv U(\alpha)\sum_m |m\rangle G_{mm}(z) . \tag{37}$$

Using Eq. (6),

$$|n,z,\alpha\rangle = e^{-|\alpha|^2/2}\sum_{m,l} G_{mn}(z)\left[\frac{m!}{l!}\right]^{1/2} \times L_m^{(l-m)}(|\alpha|^2)\alpha^{l-m}|l\rangle . \tag{38}$$

Equation (38) gives the Fock-space representation for $|n,z,\alpha\rangle$. Also, $|n,z,\alpha\rangle$ could be expanded in terms of $|m,z\rangle$'s as given below.

Consider

$$\begin{aligned}\langle m,z \mid n,z,\alpha\rangle &= \langle m,z \mid U(\alpha) \mid n,z\rangle \\ &= \langle m \mid S^{\dagger}(z)U(\alpha)S(z) \mid n\rangle \\ &= \langle m \mid \exp(\alpha b^{\dagger}-\alpha^{*}b) \mid n\rangle \\ &= \langle m \mid \exp(\gamma a^{\dagger}-\gamma^{*}a) \mid n\rangle \\ &= \langle m \mid n,\gamma\rangle ,\end{aligned} \tag{39}$$

where $\gamma=(\alpha\cosh r-\alpha^{*}\sinh r)$ and we also note that $|n,\gamma\rangle$ is a GCS. From Eq. (26)

$$\begin{aligned}| n,z,\alpha\rangle &= \sum_{m} | m,z\rangle\langle m,z \mid n,z,\alpha\rangle \\ &= \sum_{m} | m,z\rangle\langle m \mid n,\gamma\rangle .\end{aligned} \tag{40}$$

Now $| m,z\rangle$ is given by

$$| m,z\rangle \equiv S(z) \mid m\rangle \tag{41}$$

and $\langle m \mid n,\gamma\rangle$ is given by

$$\langle m \mid n,\gamma\rangle = e^{-|\gamma|^2/2}\left[\frac{n!}{m!}\right]^{1/2} L_n^{(m-n)}(|\gamma|^2)(\gamma)^{m-n} . \tag{42}$$

Using Eqs. (41) and (42), we get GSCS as

$$| n,z,\alpha\rangle = e^{-|\gamma|^2/2}\sum_{m} | m,z\rangle \left[\frac{n!}{m!}\right]^{1/2} \times L_n^{(m-n)}(|\gamma|^2)(\gamma)^{m-n} . \tag{43}$$

As an example, we give below the GSCS $| 1,z,\alpha\rangle$:

$$| 1,z,\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{(\cosh|z|)^{3/2}}\sum_{k,m=0}^{\infty}\left[\frac{z}{2|z|}\tanh|z|\right]^{k}\frac{(2k+1)!}{\sqrt{k!m!}}L_{2k+1}^{(m-2k-1)}(|\alpha|^2)\alpha^{m-2k-1} | m\rangle . \tag{44}$$

Overcompleteness of $| n,z,\alpha\rangle$

Since $U(\alpha)$ and $S(z)$ are unitary, for a given α and z the set of states $| n,z,\alpha\rangle$, $n=0,1,2,\ldots,$ forms a complete set just like the set $| n\rangle$. For a given n and z, the set $| n,z,\alpha\rangle$ with all complex α's forms an overcomplete set. We can obtain the resolution of the identity as

$$\mathbb{1} = \int \frac{d^2\alpha}{\pi} | n,\alpha,z\rangle\langle n,\alpha,z | . \tag{45}$$

Using Eqs. (38), (39) and (43) the projection of $| n,z,\alpha\rangle$ on other states like $| m\rangle$ and $| m,z\rangle$ could be calculated. In the spirit of Eq. (21), we can define the squeezed coherent state of the displaced oscillator as

$$| z,\alpha\rangle_{\mathrm{DO}} \equiv \exp\left[\frac{z}{2}(a^{\dagger}-\alpha^{*})^2-\frac{z^{*}}{2}(a-\alpha)^2\right] | \alpha\rangle \tag{46}$$

$$\equiv (\cosh|z|)^{-1/2}\sum_{n=0}^{\infty}\left[\frac{z}{2|z|}\tanh|z|\right]^{n}\frac{[(2n)!]^{1/2}}{n!} | 2n,\alpha\rangle . \tag{47}$$

The physical interpretation of the GSCS is the same as that of the two-photon coherent state of the radiation field.[16] We can consider the GSCS as a coherent state formed due to two excitations on a particular state $| n\rangle$. It is a well-established fact that SCS's are employed in quantum nondemolition (QND) measurements to reduce the quantum noise. It is also hoped that GSCS's will find application in the QND measurements and quantum optics.

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