Stability of Schwarzschild black holes in fourth-order gravity

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Both the classical and quantum stability of the Schwarzschild metric in fourth-order theories of gravity are investigated. Characterizing the theories by the mass squared of the two massive particles present in the linearized theory, we find that, provided both particles are nontachyonic, the black hole is classically stable; arbitrary initial perturbations which are regular at infinity and on the future event horizon cannot grow without bound. In the quantum case, where the Schwarzschild metric is unstable when in equilibrium with thermal radiation in the Einstein theory, we find that it is catastrophically unstable when the spin-two particle is tachyonic, but that when nontachyonic the fourth-order terms stabilize low-mass black holes; black holes do not evaporate away completely in these theories (an effect due to Hawking radiation of the negative-energy spin-two particle of the theory). We conclude that black holes appear well behaved in a quarter of the parameter space available to the fourth-order theories.

I. INTRODUCTION

It has long been known that any solution of the Einstein field equations is also a solution of the general fourth-order theory of gravity. The reverse is not the case, however, and so it is not obvious that the stable, highly symmetric black-hole solutions of the Einstein theory need remain stable in the fourth-order theory. Moreover, while it is known that black holes are unstable when in equilibrium with thermal radiation in quantum theory (due to Hawking radiation and, in particular, the black hole's negative specific heat), it is not obvious that they should remain so in the fourth-order theory, or that they could not be more seriously unstable than in the Einstein case.

In what follows, I shall treat separately the classical and quantum stability of the Schwarzschild black hole. A metric is to be considered classically stable if no initially small perturbation can grow unboundedly with time, while remaining a solution of the theory; a quantum stable metric is one whose (Euclidean) action is a local minimum in the space of all $R=0$ metrics.

The classical stability of the Schwarzschild metric has been investigated in a number of papers[1–6] by means of a frequency analysis, made possible by the staticity of the backround. A metric is stable if there are no modes with positive imaginary part of their frequency which are regular on the future event horizon and at infinity. The procedure followed has been to decompose each mode into spherical harmonics, separate these into odd- and even-parity perturbations, and then to use the gauge freedom to simplify the resulting set of coupled radial equations. So long as the solutions of these form a complete set, this is sufficient to prove stability for an arbitrary perturbation.[10]

This method carries over directly to the fourth-order problem. We shall find that it can be divided into two parts, one of which is identical to the Einstein case, and the other consisting of perturbations to the Ricci tensor. We find that we need both particles of the linearized theory to be nontachyonic in order to prove stability of the Schwarzschild black hole. Moreover, the formalism used makes it clear that the radial solutions form a complete set, and thus that the black hole is stable to arbitrary perturbations. This result is, perhaps, surprising, since there is evidently a bifurcation in the spherically symmetric solutions of the fourth-order theory, and one might expect therefore that the Schwarzschild metric would not be everywhere stable.

The quantum stability of the Schwarzschild black hole has been studied in the Einstein theory by Gross, Perry, and Yaffe,[7] by splitting arbitrary perturbations into a pure trace part, a longitudinal part, and a transverse trace-free part. It is found that only the transverse trace-free part can represent a physical quantum instability, and, decomposing by frequency and angular momentum, that black holes do exhibit such an instability. The extension of this method to the fourth-order theory is straightforward. We find that, provided the spin-two particle of the linearized theory is nontachyonic, black holes of sufficiently low mass are stabilized by the fourth-order terms.

II. FOURTH-ORDER GRAVITY THEORY

In this paper we shall be dealing with the stability of the Schwarzschild metric

$$ds^2=-V\,dt^2+V^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ , \qquad (2.1)$$

where

$$V=(1-2m/r)\ , \qquad (2.2)$$

in the general fourth-order theory (without a cosmological term), whose action is given[8] by

$$S=-\frac{1}{16\pi}\int(-R+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu})(-g)^{1/2}d^4x+\epsilon\chi\ , \qquad (2.3)$$

taking $G=1$. χ is given by

$$\chi=\frac{1}{32\pi^2}\int(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}-4R_{\mu\nu}R^{\mu\nu}+R^2)(-g)^{1/2}d^4x\ , \tag{2.4}$$

and can be ignored classically as its functional derivative with respect to the metric vanishes, while in quantum theory it is of significance only when topology changes are relevant (since χ, the Euler character, is a topological invariant). The field equations derived from (2.3) are then

$$-2\beta R^{\alpha\beta}R_{\alpha\mu\beta\nu}-2\alpha RR_{\mu\nu}-\beta\Box R_{\mu\nu}+R_{\mu\nu}+(2\alpha+\beta)R_{;\mu\nu}+g_{\mu\nu}[\tfrac{1}{2}\alpha R^2+\tfrac{1}{2}\beta R_{\sigma\tau}R^{\sigma\tau}-(2\alpha+\tfrac{1}{2}\beta)\Box R-\tfrac{1}{2}R]=0\ , \tag{2.5}$$

with trace

$$-2(3\alpha+\beta)\Box R-R=0\ . \tag{2.6}$$

We may note for future reference that the theory contains two mass scales, associated with the spin-0 and spin-2 particles present in the linearized theory.[8] (The former has significance even in the nonlinear sector.[9]) They are given, respectively, by

$$m_0{}^2=-1/(6\alpha+2\beta) \tag{2.7}$$

and

$$m_2{}^2=1/\beta\ , \tag{2.8}$$

so nontachyonic spin-0 and spin-2 particles require $(6\alpha+2\beta)$ to be negative and β to be positive, respectively.

III. CLASSICAL STABILITY

If we consider a perturbation, $h_{\mu\nu}$, of a metric $g_{\mu\nu}$ which satisfies the vacuum Einstein equation

$$R_{\mu\nu}(g_{\alpha\beta})=0\ , \tag{3.1}$$

such that the perturbed metric $(g_{\mu\nu}+h_{\mu\nu})$ still satisfies the fourth-order field equation (2.5), then we can derive a set of somewhat messy fourth-order partial differential equations for $h_{\mu\nu}$. It is simpler, however, to split the set of perturbations into two subsets: those which ensure that the perturbed metric still obeys the Einstein equations (Ricci-flat perturbations) and those which do not.

In the former case we have simply that

$$\begin{aligned}R_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta})&=R_{\mu\nu}(g_{\alpha\beta})+\delta R_{\mu\nu}(h_{\alpha\beta})\\&=\delta R_{\mu\nu}(h_{\alpha\beta})=0\ ,\end{aligned} \tag{3.2}$$

which is exactly the problem studied for the question of stability in the Einstein theory. Hence, no instabilities arise from such perturbations.

In the latter case we should consider not the perturbation in the metric but the resulting perturbation in Ricci tensor, $\delta R_{\mu\nu}(h_{\alpha\beta})$. (This is sufficient, since any non-Ricci-flat perturbation which is exponential with time corresponds to a Ricci perturbation which is exponential in time.) It is then easy to derive the equation

$$\begin{aligned}\Bigg\{\left[g_{\mu\alpha}g_{\nu\beta}+\left[\frac{2\alpha}{\beta}+\tfrac{1}{2}\right]g_{\alpha\beta}g_{\mu\nu}\right]\Box-\left[\frac{2\alpha}{\beta}+1\right]g_{\mu\nu}\nabla_\alpha\nabla_\beta\\+\left[2R_{\mu\alpha\nu\beta}-\frac{1}{\beta}g_{\mu\alpha}g_{\nu\beta}+\frac{1}{2\beta}g_{\alpha\beta}g_{\mu\nu}\right]\Bigg\}\delta R^{\mu\nu}=0\ .\end{aligned} \tag{3.3}$$

This has trace

$$\left[\left[\frac{6\alpha}{\beta}+2\right]\Box+\frac{1}{\beta}\right]\delta R=0\ . \tag{3.4}$$

This is just the scalar wave equation with mass $m_0{}^2$ on the Schwarzschild background. It is well known that if $m_0{}^2$ is positive there are no instabilities. (There would be instabilities, however, if the black hole were rotating.)

Let us turn for a moment to the boundary conditions which we shall need. It has been noted by Vishveshwara[2] that in applying boundary conditions at the Schwarzschild horizon one must work in a coordinate system which is nonsingular there, such as Kruskal coordinates, to ensure that they are physically meaningful. In Kruskal coordinates the line element is

$$\begin{aligned}ds^2=&(32m^2/r)\exp(-r/2m)(du^2-dv^2)\\&+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ .\end{aligned} \tag{3.5}$$

The coordinate transformation between this and (2.1) are given implicitly by

$$\left[\frac{r}{2m}-1\right]\exp(r/2m)=u^2-v^2 \tag{3.6}$$

and

$$t/4m=\mathrm{arctanh}(v/u)\ . \tag{3.7}$$

These may be rewritten as

$$\exp(\tilde{r}/2m)=u^2-v^2 \tag{3.8}$$

and

$$\exp(t/2m)=\frac{(u+v)}{(u-v)}\ . \tag{3.9}$$

In these coordinates the future horizon lies on $u=v$ and the past horizon on $u=-v$. The boundary conditions appropriate to our problem cannot involve the past horizon, as we are imposing an arbitrary perturbation at $t=0$ and are interested in its future evolution. Thus we need only have regularity on the future event horizon, $u=v$. Now any function of the asymptotic form $\exp(\alpha\tilde{r})\exp(\beta t)$ transforms to

$$(u+v)^{(\alpha+\beta)/2m}(u-v)^{(\alpha-\beta)/2m}\ .$$

Thus, for regularity on the future event horizon, a function which is exponentially increasing with time must be sufficiently exponentially damped as $\tilde{r}\to-\infty$ to satisfy our boundary conditions: we need Re $(\alpha-\beta)\geq 0$ for stability. (We shall see that the equations make this equivalent to the requirement that α should equal β.)

Turning back to (3.3), we find that we can greatly simplify the problem by decomposing the perturbation into trace-free and pure trace parts:

$$\delta R_{\mu\nu}=\tfrac{1}{4}\delta R g_{\mu\nu}+\delta\overline{R}_{\mu\nu}\ , \tag{3.10}$$

where

$$g^{\mu\nu}\delta\overline{R}_{\mu\nu}=0\ . \tag{3.11}$$

Then we can write (3.3) as

$$\left[g_{\mu\alpha}g_{\nu\beta}\Box+2R_{\mu\alpha\nu\beta}-\frac{1}{\beta}g_{\mu\alpha}g_{\nu\beta}\right]\delta\overline{R}^{\mu\nu}+\tfrac{1}{4}\left[\left[\frac{8\alpha}{\beta}+3\right]g_{\alpha\beta}\Box-\left[\frac{8\alpha}{\beta}+4\right]\nabla_\alpha\nabla_\beta+\frac{1}{\beta}g_{\alpha\beta}\right]\delta R=0\ . \tag{3.12}$$

Since the operator acting on $\delta\overline{R}^{\mu\nu}$ is a linear differential operator, the first term is exponential with time according to whether $\delta\overline{R}^{\mu\nu}$ is, and similarly the second term is exponential according to whether $\delta\overline{R}$ is. Thus, either each term vanishes separately, or else $\delta R_{\mu\nu}$ is stable or unstable according to whether δR is stable or unstable. We are therefore led to study the following equation:

$$\left[-(\Delta_L)_{\mu\alpha\nu\beta}-\frac{1}{\beta}g_{\mu\alpha}g_{\nu\beta}\right]\delta\overline{R}^{\mu\nu}=0\ , \tag{3.13}$$

where

$$(\Delta_L)_{\mu\alpha\nu\beta}=-(g_{\mu\alpha}g_{\nu\beta}\Box+2R_{\mu\alpha\nu\beta})\ .$$

In doing so, we follow Regge and Wheeler[1] in analyzing the perturbations into modes of definite frequency, angular momentum, and parity. We may ignore the ϕ dependence, considering only the $M=0$ harmonics, as all physics will be independent of M. We write

$$\delta\overline{R}_{ab}=\delta\overline{R}_{ab}^{\text{odd}}+\delta\overline{R}_{ab}^{\text{even}}\ , \tag{3.14}$$

where

$$\delta\overline{R}_{ab}^{\text{odd}}=\begin{bmatrix}0&0&0&\rho_0\dfrac{\partial}{\partial\theta}\\ 0&0&0&\rho_1\dfrac{\partial}{\partial\theta}\\ 0&0&0&\rho_2\left[\dfrac{\partial^2}{\partial\theta^2}-\cot\theta\dfrac{\partial}{\partial\theta}\right]\\ \text{sym}&\text{sym}&\text{sym}&0\end{bmatrix}Y_L^0(\theta)\exp(kt) \tag{3.15}$$

and

$$\delta\overline{R}_{ab}^{\text{even}}=\begin{bmatrix}P_0&P_1&\rho_0\dfrac{\partial}{\partial\theta}&0\\ \text{sym}&P_2&\rho_1\dfrac{\partial}{\partial\theta}&0\\ \text{sym}&\text{sym}&K+G\dfrac{\partial^2}{\partial\theta^2}&0\\ 0&0&0&K+G\cot\theta\dfrac{\partial}{\partial\theta}\end{bmatrix}Y_L^0(\theta)\exp(kt)\ , \tag{3.16}$$

where "sym" indicated a component which may be found from the symmetry of $\delta\overline{R}_{\mu\nu}$. A few points should be noted here: the perturbations are written in the orthonormal basis, not coordinate basis; the functions $\rho_0,\rho_1,\rho_2,P_0,P_1,P_2,K,G$ depend on r only; k is the imaginary frequency, so when it has a positive real part there is instability.

The contracted Bianchi identities tell us that the perturbations are transverse, and these, together with the tracelessness of $\delta\overline{R}_{\mu\nu}$, provide us with constraint equations which can be used to simplify (3.13) in the two cases (3.15) and (3.16).

It is worth noting that since $\Box$ and ∇_μ do not commute, transverse perturbations will not be eigenstates of angular momentum. However, the field equations will be found to reduce to purely radial equations, as Δ_L (on transverse perturbations) does commute with the angular momentum operator.

If we look at the $k=0$ form of (3.13) on a Schwarzschild background we know[7] that only when $1/\beta=0.19/m^2$ is there a normalizable solution. That mode is spherically symmetric ($L=0$) and so indicates a bifurcation of the spherically symmetric solutions of the fourth-order theory. (That is, we are led to believe that there is another family of spherically symmetric solutions, which becomes identical with the Schwarzschild solutions at a critical mass 0.44β.) We might expect, therefore, that we will find an $L=0$ instability for $1/\beta<0.19/m^2$. In what follows I shall mostly assume $k\neq0$. (A brief treatment of $k=0$ may be found in Appendix E.)

We shall first deal only with $L\geq 2$ perturbations. Then in the odd-parity case the Bianchi identities yield a single constraint,

$$\left[r^{-3}\frac{\partial}{\partial r}(V^{1/2}r^3\rho_1)+kV^{-1/2}\rho_0 + \left[\frac{\partial}{\partial\theta}+2\cot\theta\right]\frac{\rho_2}{r}\right]Y_L^0=0\ , \quad (3.17)$$

which may be used to reduce the three independent equations in three variables resulting from (3.13) to the following two equations:

$$\left[D_2-\left[k^2+\frac{L(L+1)V}{r^2}+\frac{V}{\beta}+\frac{9V^2}{4r^2}-\frac{5V}{2r^2}+\frac{1}{4r^2}\right]\right]\rho_0 +\frac{(V-1)k}{r}\rho_1=0\ ,$$
$$\left[D_4-\left[k^2+\frac{L(L+1)V}{r^2}+\frac{V}{\beta}+\frac{5V^2}{4r^2}-\frac{7V}{2r^2}+\frac{1}{4r^2}\right]\right]\rho_1 +\frac{(3V-1)k}{r}\rho_0=0\ , \quad (3.18)$$

where

$$D_n\equiv\frac{\partial^2}{\partial\tilde{r}^2}+\frac{nV}{r}\frac{\partial}{\partial\tilde{r}}\ .$$

For even-parity perturbations, we have three constraints

$$\left[\left[\frac{\partial}{\partial\theta}+\cot\theta\right]\frac{\rho_0}{r}+kV^{-1/2}P_0+V^{-1/2}r^{-2}\frac{\partial}{\partial r}(Vr^2P_1)\right]Y_L^0=0\ ,$$
$$\left[\left[\frac{\partial}{\partial\theta}+\cot\theta\right]\frac{\rho_1}{r}+kV^{-1/2}P_1+\frac{V^{-1/2}-V^{1/2}}{2r}P_0+r^{-2}\frac{\partial}{\partial r}(V^{1/2}r^2P_2)-\frac{V^{1/2}}{r}[2K-L(L+1)G]\right]Y_L^0=0\ , \quad (3.19)$$
$$\left[\frac{K}{r}\frac{\partial}{\partial\theta}+\frac{1-L(L+1)}{r}G\frac{\partial}{\partial\theta}+kV^{-1/2}\rho_0+r^{-3}\frac{\partial}{\partial r}(V^{1/2}r^3\rho_1)\right]Y_L^0=0\ .$$

These, together with the trace-free condition,

$$P_2-P_0+2K-L(L+1)G=0\ , \quad (3.20)$$

may be used to reduce the seven independent equations obtained from (3.13) to the following three coupled equations:

$$\left[D_2-\left[k^2+\frac{L(L+1)V}{r^2}+\frac{V}{\beta}+\frac{3V^2}{2r^2}-\frac{2V}{r^2}+\frac{1}{2r^2}\right]\right]P_0+\left[\frac{2kV}{r}-\frac{2k}{r}\right]P_1+\left[\frac{5V^2}{2r^2}-\frac{2V}{r^2}-\frac{1}{2r^2}\right]P_2=0\ ,$$
$$\left[D_4-\left[k^2+\frac{L(L+1)V}{r^2}+\frac{V}{\beta}-\frac{V^2}{r^2}-\frac{2V}{r^2}+\frac{1}{r^2}\right]\right]P_1+\left[\frac{3kV}{r}-\frac{k}{r}\right]P_0+\left[\frac{kV}{r}-\frac{k}{r}\right]P_2=0\ , \quad (3.21)$$
$$\left[D_6-\left[k^2+\frac{L(L+1)V}{r^2}+\frac{V}{\beta}-\frac{5V^2}{2r^2}-\frac{4V}{r^2}+\frac{1}{2r^2}\right]\right]P_2+\left[\frac{6kV}{r}-\frac{2k}{r}\right]P_1+\left[\frac{-3V^2}{2r^2}-\frac{1}{2r^2}\right]P_0=0\ .$$

Now in each case, the resulting equations, (3.18) and (3.21), may be written in the matrix form

$$\left[I\frac{\partial^2}{\partial\tilde{r}^2}+\Lambda\frac{\partial}{\partial\tilde{r}}-M\right]X=0\ , \quad (3.22)$$

where X is a function-valued column vector (ρ_0,ρ_1) or (P_0,P_1,P_2) as appropriate, and Λ is diagonal. Moreover, asymptotically the equations become, as $\tilde{r}\to\infty$,

$$\left[I\frac{\partial^2}{\partial\tilde{r}^2}-\left[k^2+\frac{1}{\beta}\right]I\right]X=0\ , \quad (3.23)$$

and at the horizon ($\tilde{r}\to\infty$) they become

$$\left[I\frac{\partial^2}{\partial\tilde{r}^2}-M_{-\infty}\right]X=0\ , \quad (3.24)$$

where for odd-parity perturbations

$$M_{-\infty}=\begin{bmatrix} k^2+\frac{1}{16m^2} & \frac{k}{2m} \\ \frac{k}{2m} & k^2+\frac{1}{16m^2}\end{bmatrix}\ , \quad (3.25)$$

and for even-parity perturbations

$$M_{-\infty}=\begin{bmatrix} k^2+\frac{1}{8m^2} & \frac{k}{m} & \frac{1}{8m^2} \\ \frac{k}{2m} & k^2+\frac{1}{4m^2} & \frac{k}{2m} \\ \frac{1}{8m^2} & \frac{k}{m} & k^2+\frac{1}{8m^2}\end{bmatrix}\ . \quad (3.26)$$

Now Wald[10] has shown that is is sufficient to prove stability to show that there are no modes with k real and positive or zero, so for $L\geq 2$ we shall consider only this case. So long as we have $m_2{}^2 \geq 0$, these matrices are positive definite for non-negative k^2, and so (3.23) and (3.24) admit only exponential solutions. We discard the divergent solutions at both infinity and the future horizon. (The latter boundary condition must again be examined in Kruskal coordinates, using the appropriate transformations for tensor components. Exponentially divergent solutions remain divergent in the nonsingular coordinate system.) So we are looking for solutions of (3.18) and (3.21) with at least two zeros on $[2m, \infty]$.

We use the following identity,

$$D_n = r^{-n} V \frac{\partial}{\partial r}\left[r^n V \frac{\partial}{\partial r} \right], \tag{3.27}$$

to write our equations in Sturm-Liouville self-adjoint form:

$$\left[\frac{\partial}{\partial r}\left[P \frac{\partial}{\partial r} \right] - V^{-2} P M \right] X = 0 , \tag{3.28}$$

where P is positive definite. There are no solutions of this equation with more than one zero on an interval $[a,b]$ if and only if the functional

$$I[P,M;\eta] = \int_{a_0}^{b_0} \left[\left[\frac{\partial}{\partial r}\eta \right]^T P \left[\frac{\partial}{\partial r}\eta \right] + \eta^T (V^{-2} P M)\eta \right] dr \tag{3.29}$$

is positive definite, for all $[a_0, b_0] \subset [a,b]$ and $\eta(a_0) = \eta(b_0) = 0$.

Now if M is positive definite for any given value of L, it will also be positive definite for any greater value, so we need only consider the $L=2$ perturbations. Similarly, we see that if it is positive definite when $1/\beta = 0$, then it will be for all positive $m_2{}^2$. Then we can evaluate the sums of products of the eigenvalues of M for both the even- and odd-parity perturbations to find that in both cases M is positive definite when k^2 is non-negative (see Appendix A). Now since P is positive definite and diagonal, both terms in the integral (3.29) are positive definite, and hence there are no solutions which satisfy our boundary conditions.

When $L=1$ the constraints simplify, becoming for even-parity perturbations [since G can then be absorbed into K in (3.16)]

$$\begin{aligned}
&\left[\left[\frac{\partial}{\partial \theta} + \cot\theta \right] \frac{\rho_0}{r} + k V^{-1/2} P_0 + V^{-1/2} r^{-2} \frac{\partial}{\partial r}(V r^2 P_1) \right] Y_1^0 = 0 , \\
&\left[\left[\frac{\partial}{\partial \theta} + \cot\theta \right] \frac{\rho_1}{r} + k V^{-1/2} P_1 + \frac{V^{-1/2} - 3V^{1/2}}{2r} P_0 + r^{-3} \frac{\partial}{\partial r}(V^{1/2} r^3 P_2) \right] Y_1^0 = 0 , \\
&\left[\frac{P_0 - P_2}{2r} \frac{\partial}{\partial \theta} + k V^{-1/2} \rho_0 + r^{-3} \frac{\partial}{\partial r}(V^{1/2} r^3 \rho_1) \right] Y_1^0 = 0 ,
\end{aligned} \tag{3.30}$$

and in the odd case [since now ρ_2 vanishes in (3.15)],

$$r^{-3} \frac{\partial}{\partial r}(V^{1/2} r^3 \rho_1) + k V^{-1/2} \rho_0 = 0 . \tag{3.31}$$

The constraints (3.30) provide a differential equation in P_0, P_1, P_2,

$$V^2 \frac{\partial^2}{\partial r^2} P_2 + \left[\frac{11V^2}{2r} + \frac{3V}{2r} \right] \frac{\partial}{\partial r} P_2 + \left[\frac{-3V^2}{2r} + \frac{V}{2r} \right] \frac{\partial}{\partial r} P_0 + (2kV) \frac{\partial}{\partial r} P_1 + \left[\frac{5V^2}{r^2} + \frac{3V}{r^2} \right] P_2 + \left[\frac{-3V^2}{r^2} + \frac{V}{r^2} + k^2 \right] P_0 + \left[\frac{5kV}{r} + \frac{k}{r} \right] P_1 = 0 , \tag{3.32}$$

and the even-parity equations (3.21) are unchanged. By examining the asymptotic forms of (3.21) and (3.32) and applying the boundary conditions, they can be shown to admit no normalizable modes. (See Appendix B.) In the odd-parity case, the equations resulting from (3.13) differ from (3.18), being

$$\begin{aligned}
&\left[D_2 - \left[k^2 + \frac{V}{\beta} + \frac{9V^2}{4r^2} - \frac{V}{2r^2} + \frac{1}{4r^2} \right] \right] \rho_0 + \frac{(V-1)k}{r} \rho_1 = 0 , \\
&\left[D_2 - \left[k^2 + \frac{V}{\beta} + \frac{25V^2}{4r^2} - \frac{V}{2r^2} + \frac{1}{4r^2} \right] \right] \rho_1 + \frac{(V-1)k}{r} \rho_0 = 0 ,
\end{aligned} \tag{3.33}$$

which again have no nondivergent solutions (Appendix C).

Finally, for $L=0$, the form of (3.15) shows that there are no odd-parity perturbations at all, while for even-parity perturbations, ρ_0, ρ_1, and G must vansih. Then the Bianchi identities yield only two constraints,

$$kV^{-1}P_0+V^{-1/2}r^{-2}\frac{\partial}{\partial r}(Vr^2P_1)=0\ ,$$

$$kV^{-1/2}P_1+\frac{V^{-1/2}-3V^{1/2}}{2r}P_0+r^{-3}\frac{\partial}{\partial r}(V^{1/2}r^3P_2)=0\ , \tag{3.34}$$

on the three equations

$$\left[D_2-\left[k^2+\frac{V}{\beta}+\frac{3V^2}{2r^2}-\frac{2V}{r^2}+\frac{1}{2r^2}\right]\right]P_0+\left[\frac{2kV}{r}-\frac{2k}{r}\right]P_1+\left[\frac{5V^2}{2r^2}-\frac{2V}{r^2}-\frac{1}{2r^2}\right]P_2=0\ ,$$

$$\left[D_2-\left[k^2+\frac{V}{\beta}+\frac{V^2}{r^2}+\frac{1}{r^2}\right]\right]P_1+\left[\frac{kV}{r}-\frac{k}{r}\right]P_0+\left[\frac{kV}{r}-\frac{k}{r}\right]P_2=0\ , \tag{3.35}$$

$$\left[D_2-\left[k^2+\frac{V}{\beta}+\frac{15V^2}{2r^2}-\frac{2V}{r^2}+\frac{1}{2r^2}\right]\right]P_2+\left[\frac{2kV}{r}-\frac{2k}{r}\right]P_1+\left[\frac{9V^2}{2r^2}-\frac{2V}{r^2}-\frac{1}{2r^2}\right]P_0=0\ .$$

Once more the system admits no nontrivial solutions (Appendix D).

Let us now summarize what we have proved. We first showed that the general perturbations could be split into Ricci-flat and Ricci perturbations, and that only the latter could cause instability. We then further subdivided the Ricci perturbations into pure trace and trace-free parts showing that the former could not cause instability, provided $m_0{}^2$ is non-negative. Finally we split the trace-free Ricci perturbations into odd- and even-parity classes, and found that no instabilities could arise here provided $m_2{}^2$ is non- negative. We may also note that we have inadvertently also solved the original Regge-Wheeler problem, as, when $m_2{}^2$ vanishes, (3.13) is the same stability equation as they tackled but calculated in a different gauge. We have, moreover, shown stability under arbitrary perturbations, since our equations are in Sturm-Liouville self-adjoint form. (Working to higher order in the perturbations, we are likely to find instabilities due to interactions between the positive-energy and negative-energy modes. These would, however, require a initial perturbation of finite magnitude to excite them.)

One might wonder why we found no $L=0$ instabilities. As fas as the structure of the equations (3.34) and (3.35) is concerned, it is clear that the nature of the constraints when $k=0$ is quite different from when $k\neq 0$ and is responsible for the normalizable $k=0$ mode (Appendix E). (In terms of the complex k plane, the $k=0$ mode must move into the the left-hand plane as m moves away from the critical mass.) Physically, there seem to be two possibilities: either the bifurcation is an artifact of linearization, and no other spherically symmetric solution exists, or the bifurcation is real and (since the form of the $k=0$, $L=0$ mode would lead us to believe that any second family of solutions would have a smaller horizon size for a given total mass, rising with $1/\beta$ until when $1/\beta=0.19/m^2$ it merges with the Schwarzschild solution), the second law of black-hole thermodynamics ensures the stability of the Schwarzschild solution.

We can extend the above analysis to cover those theories in which R term is removed from (2.3). Then the last term of (3.4) and (3.13) disappears, ensuring stability for all values of α and β.

IV. QUANTUM STABILITY

It is clear from the path-integral formulation of quantum theory that quantum stability is determined by the sign of the second variation of the (Euclidean) action about a background solution of the theory, since if this is positive it indicates that one is at a local minimum of the action in the space of all $R=0$ metrics, and thus that there is some lower bound on the size of quantum fluctuations needed to escape from that stationary point to any lower action solution.

In the case of (2.3), the second variation about a Schwarzschild background may be written

$$\delta^2 I=-\frac{1}{8\pi}\int[\tfrac{1}{2}H^T\Delta_L(\beta\Delta_L+1)H+f(h)+g(h_{\mu\nu}{}^{;\nu})]g^{1/2}d^4x\ , \tag{4.1}$$

where H is the traceless part of the metric perturbation considered as a 16-component vector, and Δ_L is the Lichnerowicz operator $(-\Box g_{\mu\alpha}g_{\nu\beta}-2R_{\mu\alpha\nu\beta})$ considered as a 16×16 matrix. If we fix the gauge by setting $h_{\mu\nu}{}^{;\nu}=0$, the terms which depend only on the divergence of the perturbation [denoted generically by $g(h_{\mu\nu}{}^{;\nu})$] will vanish. The term $f(h)$, which depends only on $\Box h$ also vanishes, since we are only interested in $\delta R=0$ perturbations, and in our gauge $\delta R=-\Box h$. Thus, the quantum stability of the Schwarzschild black hole depends on the positivity of

$$H^T\Delta_L(\beta\Delta_L+1)H\ . \tag{4.2}$$

If we consider the eigenvalue equation

$$\Delta_L\eta_n=\lambda_n\eta_n\ , \tag{4.3}$$

then we can write (4.2) in the form

$$\sum_n\lambda_n(\beta\lambda_n+1)\eta_n{}^2\ , \tag{4.4}$$

assuming $H=\sum_n\lambda_n\eta_n$, where η_n are a complete set of orthonormal eigenfunctions. We know[7] that there is one and only one negative eigenvalue,

$$\lambda_0=-0.19m^{-2}\ , \tag{4.5}$$

taking $G=1$. So if β vanishes we have a single negative mode.

When ${m_2}^2$ is negative, the $n=0$ mode remains negative, but is joined by all modes for which $\lambda_n \geq |{m_2}^2|$. Since λ_n increases without bound with L, there are an infinite number of negative modes in this case, which is catastrophic.

In the positive-${m_2}^2$ case, however, we see that only the $n=0$ mode can be negative, but it is in fact positive provided the black-hole mass is less than $0.44{m_2}^{-1}$. Thus, low-mass black holes are stabilized by the fourth-order terms. (If we take $\beta \approx 1$, as it is a dimensionless coupling constant, black holes lighter than about the Planck mass will stable.)

This last result may seem puzzling: since in Einstein theory low-mass black holes are less quantum stable than high-mass ones, it is surprising that it is the former which are stabilized.

This problem can be dealt with by a small calculation. The second variation in the action may be written, noting the Ricci-flat background, as

$$\begin{aligned}\delta^2 I &= \frac{1}{8\pi}\int (\delta R_{\mu\nu}h^{\mu\nu}+2\beta\delta R_{\mu\nu}\delta R^{\mu\nu})g^{1/2}d^4x \\ &= \frac{1}{8\pi}\int [(\tfrac{1}{2}\Delta_L h_{\mu\nu})h^{\mu\nu} \\ &\qquad +2\beta(\tfrac{1}{2}\Delta_L h_{\mu\nu})(\tfrac{1}{2}\Delta_L h^{\mu\nu})]g^{1/2}d^4x\ ,\end{aligned} \tag{4.6}$$

and when $h^{\mu\nu}$ is the negative mode, η_0,

$$\delta^2 I = \frac{1}{16\pi}\int \left[\frac{-0.19}{m^2}+\frac{0.36}{m^4}\beta\right](\eta_0^{\mu\nu}\eta_{0\mu\nu})g^{1/2}d^4x\ . \tag{4.7}$$

So we see that, while Einstein term does indicate greater instability for smaller masses, the stabilizing effect of the Ricci-squared term increases even faster as the black-hole mass decreases.

We can understand the stabilizing effect in the following way. In the Einstein case the negative specific heat and the instability are just due to the fact that when a black hole loses (gains) mass its temperature increases (decreases) and so the Hawking radiation is greater (less) than the incoming thermal radiation, resulting in further mass loss (gain). In the fourth-order theory we do not only have gravitons, which follow the above picture, but a spin-two "poltergeist"[8], a negative-energy propagating particle which, because of its negative energy, will have a stabilizing effect: when Hawking radiation decreases (increases) the black hole will lose (gain) net mass.

This would explain why the mass of the spin-two particle needs to be low for stabilization; it must be thermally excitable. It also explains why the black-hole mass should be low; it must have a temperature high enough to thermally excite the poltergeist. Even so, one might wonder why the stabilizing energy flux due to the poltergeist should ever exceed the destabilizing effect of the graviton. This may be due to the fact that a massive spin-two particle has five degrees of freedom, as opposed to the graviton's two. This could be called into question in the real world, however, in which other fields might contribute to black-hole instability.

Again we can extend the above analysis to theories without a pure R term, the last term of (4.6) dropping out and so guaranteeing quantum stability of the pure gravity theory for positive β.

V. CONCLUSION

In studying the classical and quantum stability of the Schwarzschild metric in the class of theories with action given by (2.3), we have effectively ruled out three-quarters of the parameter-space available to those theories from further consideration. For classical stability we found that we needed both β and $-(3\alpha+\beta)$ positive. In the quantum case, we needed β positive to avoid a catastrophic instability. It was also found that the existence of a negative-energy particle in the fourth-order pure gravity theories can stabilize low-mass black-holes. We have therefore, a class of theories which appear well behaved, have no tachyonic particles in the linearized regime, and which can represent the real world.

APPENDIX A

In this appendix we shall see that the matrices M, which are defined by (3.18), (3.21), and (3.22), are positive definite. We shall consider only the $L=2$, $1/\beta=0$ case, as if M is positive definite in this case, it will be for all positive β and all $L\geq 2$. In the even-parity case then,

$$M=\begin{bmatrix} \dfrac{3V^2+8V+1}{2r^2} & \dfrac{2k(1-V)}{r} & \dfrac{-5V^2+4V+1}{2r^2} \\ \dfrac{k(1-3V)}{r} & \dfrac{-V^2+4V+1}{r^2} & \dfrac{k(1-V)}{r} \\ \dfrac{3V^2+1}{2r^2} & \dfrac{2k(1-3V)}{r} & \dfrac{-5V^2+4V+1}{2r^2} \end{bmatrix}+k^2 I\ . \tag{A1}$$

Now, noting that $0\leq V\leq 1$ on $[2m,\infty]$, we may write

$$|M-\lambda I| = -\lambda^3+\lambda^2\left[3k^2+\frac{10V-2V^2}{r^2}\right]-\lambda\left[3k^4+k^2\left[\frac{36V-16V^2}{r^2}\right]+\left[\frac{1+12V+30V^2-20V^3+V^4}{r^4}\right]\right]$$
$$+\left[k^2\left[k^2-\frac{1}{r^2}\right]^2+k^4\left[\frac{26V-14V^2}{r^2}\right]+k^2\left[\frac{4V+58V^2-4V^3-35V^4}{r^4}\right]\right.$$
$$\left.+\left[\frac{2V(1-V)(1+5V)(1+4V-V^2)}{r^6}\right]\right], \tag{A2}$$

and thus see that, since each of the polynomials in V is positive definite, the matrix M is positive definite on the range of interest.

Similarly, in the odd-parity case we have

$$M=\begin{bmatrix} \dfrac{9V^2+14V+1}{4r^2} & \dfrac{k(1-V)}{r} \\ \dfrac{k(1-3V)}{r} & \dfrac{5V^2+10V+1}{4r^2} \end{bmatrix}+k^2 I, \tag{A3}$$

and thus

$$|M-\lambda I| = \lambda^2-\lambda\left[2k^2+\left[\frac{1+12V+7V^2}{2r^2}\right]\right]$$
$$+\left[\left[k^2-\frac{1}{4r^2}\right]^2+k^2\left[\frac{20V+V^2}{2r^2}\right]+\left[\frac{24V+154V^2+160V^3+45V^4}{16r^4}\right]\right], \tag{A4}$$

which again is manifestly positive definite.

APPENDIX B

For the $L=1$ even-parity perturbations we have Eq. (3.21) and (3.32), which can be shown to be consistent by eliminating the second derivative of P_2 from (3.32), using (3.21), to give

$$\left[\frac{-3V^2+V}{2r}\right]\frac{\partial}{\partial r}P_0+\left[\frac{V^2+V}{2r}\right]\frac{\partial}{\partial r}P_2+2kV\frac{\partial}{\partial r}P_1$$
$$+\left[\frac{-3V^2+2V+1}{2r^2}+k^2\right]P_0$$
$$+\left[\frac{5V^2+2V+1}{2r^2}+\frac{V}{\beta}+k^2\right]P_2+k\left[\frac{3-V}{r}\right]P_1=0, \tag{B1}$$

differentiating this and using (3.21) to eliminate second derivatives to give another first-order equation (with twice as many terms, so I shall not give it here), and repeating this, yielding a third first-order equation (with twice as many terms again) which can be shown to be a linear combination of the first two.

However, by considering solutions near the horizon we shall find there are no nondivergent solutions. Asymptotically (3.21) becomes at the horizon

$$\left[\frac{\partial^2}{\partial\tilde{r}^2}-\left[k^2+\frac{1}{8m^2}\right]\right]P_0-\frac{k}{m}P_1-\frac{1}{8m^2}P_2=0,$$
$$\left[\frac{\partial^2}{\partial\tilde{r}^2}-\left[k^2+\frac{1}{4m^2}\right]\right]P_1-\frac{k}{2m}P_0-\frac{k}{2m}P_2=0, \tag{B2}$$
$$\left[\frac{\partial^2}{\partial\tilde{r}^2}-\left[k^2+\frac{1}{8m^2}\right]\right]P_2-\frac{k}{m}P_1-\frac{1}{8m^2}P_0=0,$$

and (B1) becomes (when $1/\beta\neq 0$)

$$\frac{1}{4m}\frac{\partial}{\partial\tilde{r}}(P_0+P_2)+\left[k^2+\frac{1}{8m^2}\right](P_0+P_2)+2k\frac{\partial}{\partial\tilde{r}}P_1$$
$$+\frac{3k}{2m}P_1+\frac{V}{\beta}(P_2-P_0)=0. \tag{B3}$$

Now (B2) has solutions (provided $k^2\neq 1/4m^2, 0$)

$$P_1+\tfrac{1}{2}(P_0+P_2)=A_+\exp\left[\left[k+\frac{1}{2m}\right]\tilde{r}\right]$$
$$+A_-\exp\left[-\left[k+\frac{1}{2m}\right]\tilde{r}\right],$$
$$(P_0-P_2)=C_+\exp(k\tilde{r})+C_-\exp(-k\tilde{r}), \tag{B4}$$
$$-P_1+\tfrac{1}{2}(P_0+P_2)=B_+\exp\left[\left[k-\frac{1}{2m}\right]\tilde{r}\right].$$
$$+B_-\exp\left[-\left[k-\frac{1}{2m}\right]\tilde{r}\right],$$

but (B3), noting that $V \approx \exp(\tilde{r}/2m)$ near the horizon, restricts these to give

$$C_+ = \beta \left[4k^2 + \frac{3k}{m} + \frac{1}{2m^2} \right] A_+ ,$$
$$C_- = \beta \left[4k^2 - \frac{3k}{m} + \frac{1}{2m^2} \right] B_- . \tag{B5}$$

At this point we must apply our boundary conditions in detail. We use

$$\delta R^k_{00} = f^2(r)(u^2-v^2)^{-1}(u^2P_0+v^2P_2-2uvP_1)\exp(kt) ,$$
$$\delta R^k_{01} = f^2(r)(u^2-v^2)^{-1}[(u^2+v^2)P_1-uv(P_0+P_2)] \times \exp(kt) ,$$
$$\delta R^k_{11} = f^2(r)(u^2-v^2)^{-1}(v^2P_0+u^2P_2-2uvP_1)\exp(kt) , \tag{B6}$$
$$\delta R^k_{02} = 4m(u^2-v^2)^{-1}(1-2m/r)^{1/2}(u\rho_0-v\rho_1)\exp(kt) ,$$
$$\delta R^k_{12} = -4m(u^2-v^2)^{-1}(1-2m/r)^{1/2}(v\rho_0-u\rho_1)\exp(kt) ,$$

where

$$f^2(r) = \frac{32m^2}{r}\exp(-r/2m) ,$$

the superscript k denotes components in Kruskal coordinates, and angular dependence has been suppressed. Applying (B4) to the first three of these components, we see that the A_-,B_-,C_- parts of the solutions behave independently as $(u-v)^{-4mk}$ near the future event horizon and so must vanish for Re $k>0$. Furthermore, the B_+ part goes as $(u-v)^{-2}$ and so it too must vanish. Now we know from (3.30) that

$$\rho_0 \approx V^{-1/2}\left[kP_0 + \frac{1}{2m}P_1 + \frac{\partial}{\partial \tilde{r}}P_1 \right] ,$$
$$\rho_1 \approx V^{-1/2}\left[kP_1 + \frac{1}{4m}(P_0+P_2) + \frac{\partial}{\partial \tilde{r}}P_2 \right] \tag{B7}$$

near the horizon, and so, from the last two components of (B6), we see that the A_+ part behaves as $(u-v)$, while the C_+ part behaves as $(u-v)^{-1}$ near the future event horizon. But due to the constraint (B5) we cannot set C_+ to zero without also setting A_+ to zero. Thus for all Re$k>0$, there are no regular $L=1$ even-parity perturbations.

When $1/\beta = 0$, the last term in (B3) becomes

$$\frac{V^2}{2m}(P_2-P_0) ,$$

and the solutions are as (B4) but with $C_+ = C_- = A_+ = B_- = 0$. The same conclusions may be drawn.

When $k=1/2m$, the $B_\pm$ solutions of (B4) no longer apply, but are replaced by a solution linear in $\tilde{r}$; the second line of (B5) changes appropriately. In any case, it is clear from (B6) that such solutions are divergent at the future horizon.

Thus there are no solutions which are regular at the future event horizon for Re$k>0$ in the $L=1$ even-partiy case. [In the $L\geq 2$ case, the crucial difference is the absence of (B3) which was needed to force the A_+ part to vanish, which in the $L\geq 2$ case are perfectly acceptable solutions.]

APPENDIX C

In the $L=1$ odd-parity case, we can use (3.31) to eliminate ρ_0 from each of (3.33), yielding

$$\left[V^2\frac{\partial^2}{\partial r^2} + \frac{2V}{r}\frac{\partial}{\partial r} - \frac{35V^2}{4r^2} + \frac{5V}{2r^2} + \frac{1}{4r^2} - k^2 + \frac{V}{\beta} \right]\rho_1 = 0 , \tag{C1}$$

and an equation which is the result of

$$\left[-\frac{V}{k}\frac{\partial}{\partial r} + \frac{1-7V}{2kr} \right]$$

acting on this, so they are consistent. The solution near the horizon is

$$\rho_1 = A\exp\left[\left[-\frac{1}{4m} + k \right]\tilde{r} \right] + B\exp\left[\left[-\frac{1}{4m} - k \right]\tilde{r} \right] . \tag{C2}$$

Now (3.31) yields $\rho_0 \approx -\rho_1/4mk$ near the future event horizon, and so

$$\delta R^k_{03} \approx (u^2-v^2)^{-1}(1-2m/r)^{1/2}(-u/4mk - v) \times \rho_1\exp(kt) . \tag{C3}$$

Thus, regularity on the future event horizon demands that $B=0$ and $A=0$ for Re$k > -1/4m$.

APPENDIX D

The $L=0$ even-partiy equations (3.34) and (3.35) may be shown to be consistent; using the constraints to eliminate all P_1,P_2 derivatives from (3.35) and then the second equation of (3.35) to eliminate the P_1 derivatives from the other two equations, a single algebraic relation results:

$$\left[k^2 + \left[\frac{3V^2-2V-1}{4r^2} \right] \right] P_0 - \left[k^2 + \left[\frac{-V^2+2V-1}{4r^2} \right] \right] P_2 + \frac{2kV}{r}P_1 = 0 . \tag{D1}$$

We may examine the behavior near the horizon, as we did in Appendix B for the $L=1$ perturbations, and again (B4) applies. The additional information obtained from the constraints (3.34) however, which near the horizon become

$$kP_0 + \frac{1}{2m}P_1 + \frac{\partial}{\partial \tilde{r}}P_1 = 0 ,$$
$$kP_1 + \frac{P_0+P_2}{4m} + \frac{\partial}{\partial \tilde{r}}P_2 = 0 \tag{D2}$$

restricts the solutions to

$$P_0=P_2=A_-\exp\left[-\left(k+\frac{1}{2m}\right)\tilde{r}\right]+B_+\exp\left[\left(k-\frac{1}{2m}\right)\tilde{r}\right],$$
$$P_1=A_-\exp\left[-\left(k+\frac{1}{2m}\right)\tilde{r}\right]-B_+\exp\left[\left(k-\frac{1}{2m}\right)\tilde{r}\right], \tag{D3}$$

but, as in the $L=1$ even-parity case, we see from (B6) that regularity on the future event horizon demands that $A_-=B_+=0$ for $\mathrm{Re}k>0$. (Again we find that the $k=1/2m$ case must be treated separately, but yields similar results.) Thus, there are no unstable $L=0$ perturbations.

APPENDIX E

When $k=0$ the equations simplify, some of the radial functions decouple, and the effects of the constraints change. For odd-parity perturbations, ρ_0 and ρ_1 decouple [see (3.18) and (3.33)]. At infinity each will still be exponential if $1/\beta>0$, but if $1/\beta=0$ they will either diverge or fall off as a power of r (faster than $1/r$). In each case we still find that ρ_0 and ρ_1 must go to zero at the boundaries if it is not to diverge: Sturm-Liouville theory can be used to rule out any solutions to (3.18) or (3.33).

For even-parity perturbations, P_1 decouples from P_0,P_2 [see (3.21), (3.35)]. In the $L\geq 2$ case, we again have either exponential behavior at infinity or power-law behavior according to whether $1/\beta=0$, and again this allows us to use Sturm-Liouville theory to prohibit solutions. In the $L=0$ and $L=1$ cases, however, Sturm-Liouville theory does allow solutions, and such have been found previously.[1,7]

The only physically significant solution is found when $L=0$ and $1/\beta=0.19/m^2$ (Ref 7). In this case the first constraint of (3.34) implies that P_1 vanishes if it is not to diverge. Near the horizon the remaining constraint gives

$$\frac{P_0+P_2}{4m}+\frac{\partial}{\partial\tilde{r}}P_2=0\,, \tag{E1}$$

and the equations (3.35) become

$$\left[\frac{\partial^2}{\partial\tilde{r}^2}-\frac{1}{8m^2}\right]P_0-\frac{1}{8m^2}P_2=0\,,$$
$$\left[\frac{\partial^2}{\partial\tilde{r}^2}-\frac{1}{8m^2}\right]P_2-\frac{1}{8m^2}P_0=0\,, \tag{E2}$$

which together give as the only nondivergent solution $P_0=-P_2=\text{const}$. From (B6) we see that δR^k_{00}, $\delta R^k_{01},\delta R^k_{11}$ are then regular at the future event horizon, and from (B7) that ρ_0 and ρ_1 vanish. Thus in this case the constraints no longer force nondivergent solutions to vanish at the horizon. In particular, the first constraint of (3.34) in the $L=0,k\neq 0$ case near the horizon may be written

$$\left[\frac{\partial}{\partial\tilde{r}}+\frac{1}{2m}\right]\{[P_1+\tfrac{1}{2}(P_0+P_2)]-[-P_1+\tfrac{1}{2}(P_0+P_2)]\}+\frac{k}{2}\{(P_0-P_2)+[P_1+\tfrac{1}{2}(P_0+P_2)]+[-P_1+\tfrac{1}{2}(P_0+P_2)]\}=0 \tag{E3}$$

which for $k\neq 0$ implies that P_0-P_2, which is the only nondivergent mode, (B4) vanishes, while for $k=0$ this does not follow.

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