Energy distribution of leptons from $e^+e^- \to W^+W^-$

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We derive an analytic expression for the double-energy distribution of the final leptons in $e^+e^- \to W^+W^- \to l^+\nu l^-\bar{\nu}$. General values are taken for the magnetic moment and electric quadrupole moment at the γWW and ZWW vertices. An analytic expression for the energy distribution of just one of the leptons is also given. Numerical examples illustrate the feasibility of using these expressions to fix the three-gauge-particle interaction.

I. INTRODUCTION

Perhaps the primary *raison d'être* for e^+e^- colliders with beam energies capable of producing a pair of W bosons is to study the three-gauge-boson vertex. An indication of the energies required for such a study comes from the calculation of the total $e^+e^- \to W^+W^-$ cross section using the standard couplings but allowing the magnetic dipole (κ) and electric quadrupole (Q) moments to deviate from their canonical values of one and zero. The result of this calculation is rather discouraging; some anomalous integer values of κ and Q give values for the total cross section which differ from the canonical value by less than 20% until the energy is almost 30 GeV above threshold. This is partly due to the importance of the neutrino pole which does not depend on κ or Q, and partly due to cancellation between the effect of κ and the effect of Q. Values of κ and Q which differ from (1,0) by less than integers could give a total cross section which tracks the canonical one to even larger energies. The general expression for the cross section as a function of κ and Q will be given in the next section. We will also give a graph to demonstrate the above remarks.

The purpose of this paper is to give an analytic expression for the energy distribution of the leptons which come from the decays of the W's. That is, the energy distribution of l^+ and l^- from

$$
\begin{array}{l}
e^+ + e^- \to W^+ + W^- \\
\qquad\qquad\;\; \downarrow \qquad\; \hookrightarrow l^- + \bar{\nu} \\
\qquad\qquad\;\; \hookrightarrow l^+ + \nu \, .
\end{array}
$$

This energy distribution may be important if it can be used to study the three-gauge-boson vertex at lower beam energies than it seems possible to do by using the total cross section. The analytic expression will be given in Sec. III. Its implications will be discussed in Sec. IV.

Our expression is general in that we do not fix the values of κ and Q. We do not attempt to justify noncanonical values for these parameters by constructing theoretical models with other values;[1] our purpose is simply to study the feasibility of measuring κ and Q in order to fix the three-gauge-particle vertex.

II. THE TOTAL CROSS SECTION FOR $e^+e^- \to W^+W^-$

This cross section comes from three diagrams; neutrino exchange in the t channel, and photon exchange and Z-boson exchange in the s channel.[2] We will assume the interactions are as given in the standard model[3] with the exception of the γWW and ZWW vertices. For those we use[4]

$$
\begin{aligned}
\mathscr{L} = \frac{ig}{(g^2+g'^2)^{1/2}}(g'A^\nu - gZ^\nu)\Big[& (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)W^\mu - (\partial_\mu W_\nu - \partial_\nu W_\mu)W^{\mu+} - \kappa\partial_\mu(W^{\mu+}W_\nu - W^\mu W_\nu^+) \\
& - \frac{Q}{M_W{}^2}(\partial_\alpha\partial_\beta - \tfrac{1}{2}g_{\alpha\beta}\partial^2)(\partial_\nu W_\alpha W^{\beta+} - \partial_\nu W_\beta^+ W^\alpha)\Big] ,
\end{aligned} \tag{1}
$$

where g and g' are the usual gauge couplings

$$g' = g\tan\theta_W \; , \quad g = \frac{e}{\sin\theta_W} \; .$$

e is the electric charge, $\alpha = e^2/4\pi = \frac{1}{137}$. M_W is the mass of the W boson. Equation (1) is the standard coupling if κ is one and Q is zero. κ and Q could be different for the photon coupling than for the Z coupling and they will be kept separate below.[5]

The square of the neutrino-exchange graph gives the following contribution to the total cross section:

$$
\sigma_{\nu\nu} = \frac{\pi\alpha^2}{8s}\beta\frac{1}{\sin^4\theta_W}\left[2\frac{s}{M_W{}^2} - 4 + \frac{\beta^2}{12}\frac{s^2}{M_W{}^4} + \frac{4}{\beta}\left[1 - \frac{2M_W{}^2}{s}\right]\ln\left|\frac{1+\beta}{1-\beta}\right|\right] . \tag{2}
$$

s is the square of the c.m. energy; $s=4E^2$, where E is the beam energy. β is the velocity of the W's in the c.m. frame,

$$\beta=\left[1-\frac{4M_W{}^2}{s}\right]^{1/2} .$$

The cross term between the neutrino-exchange graph and the photon- plus Z-exchange graphs gives

$$\sigma_{\mathrm{CT}}=\frac{\pi\alpha^2}{8s}\beta\frac{1}{\sin^2\theta_W}\left[1-\frac{2\sin^2\theta_W-1}{2\sin^2\theta_W}\frac{s}{s-M_Z{}^2}\right]$$
$$\times\left[8\frac{M_W{}^2}{s}-4-\frac{s}{M_W{}^2}(2+\tfrac{4}{3}\beta^2)+\kappa\left[8+\frac{s^2}{M_W{}^4}(\tfrac{2}{3}\beta^2-1)\right]+\frac{16M_W{}^2}{\beta s}\left[\frac{M_W{}^2}{s}+1+\kappa\right]\ln\left|\frac{1+\beta}{1-\beta}\right|-Q\frac{\beta^2}{3}\frac{s^2}{M_W{}^4}\right] . \tag{3}$$

Different values of κ and Q can be used for the photon and for the Z by recognizing that the first term in the first set of large parentheses is the photon exchange, the second term is the Z exchange. M_Z is the mass of the Z.

The square of the photon plus Z exchange gives

$$\sigma_{\gamma Z}=\frac{\pi\alpha^2}{8s}\beta^3\left[1-\frac{4\sin^2\theta_W-1}{2\sin^2\theta_W}\frac{s}{s-M_Z{}^2}+\frac{(4\sin^2\theta_W-1)^2+1}{16\sin^4\theta_W}\frac{s^2}{(s-M_Z{}^2)^2}\right]$$
$$\times\left[-\tfrac{8}{3}-\frac{32M^2}{s}-16(\kappa+\kappa'+\tfrac{2}{3}\kappa\kappa')+4\frac{s}{M_W{}^2}(\tfrac{2}{3}+\kappa+\kappa')+\frac{2}{3}\frac{s^2}{M_W{}^4}\kappa\kappa'\right.$$
$$\left.+\tfrac{2}{3}\beta^2\left[\frac{s^2}{M_W{}^4}(\kappa Q'+\kappa' Q)-\frac{2s}{M_W{}^2}(Q+Q')+3\frac{s^2}{M_W{}^4}QQ'\right]\right] , \tag{4}$$

where, for the square of the photon exchange (the first term in the first large parentheses), $\kappa=\kappa'=\kappa_\gamma$, the magnetic moment of the photon coupling, and $Q=Q'=Q_\gamma$. For the last term in the first large parentheses (the square of the Z exchange) $\kappa=\kappa'=\kappa_Z$, $Q=Q'=Q_Z$. For the cross term use $\kappa=\kappa_\gamma$, $\kappa'=\kappa_Z$, $Q=Q_\gamma$, $Q'=Q_Z$.

The total cross section is the sum of (2), (3), and (4). For $\kappa=\kappa'=1$ and $Q=Q'=0$ it agrees with the standard result of Alles, Boyer, and Buras.[2]

Some values of the total cross section are shown in Fig. 1 for various values of κ and Q. For simplicity we have set $\kappa_\gamma=\kappa_Z$, $Q_\gamma=Q_Z$ in these plots. We used $\sin^2\theta_W=0.21$ with $M_W=82.42$ GeV and $M_Z=92.73$ GeV. The canonical values of κ and Q give a cross section which decreases for large energy as is required by unitarity. The curve shows that the beam energy must be greater than 120 GeV for the cross section to decrease 10% of its peak value. All other values of κ and Q give a total cross section which eventually rises with increasing energy but this violation of unitarity may not be observable until the energy gets reasonably large; for example, see the $\kappa=2$, $Q=0$ curve. It is for this reason that we turn now to a study of the energy distributions of the leptonic decay products of the W's.

III. ENERGY DISTRIBUTIONS

Consider the reaction where the produced W's decay into leptons, $W^\pm\to l^\pm+\nu$, where l is e, μ, or τ. We want to consider the double-energy distribution, $d^2\sigma/dE_+dE_-$, where $E_\pm$ are the energies of $l^\pm$. The dominant contribution to the process $e^+e^-\to W^+W^-\to l^+\nu l^-\bar{\nu}$ should occur when the W's are close to their mass shell—hence we make the narrow-width approximation of replacing the propagators by δ functions:

$$\left|\frac{1}{p^2-M_W{}^2+iM_W\Gamma}\right|^2\to\frac{\pi}{M_W\Gamma}\delta(p^2-M_W{}^2) , \tag{5}$$

where Γ is the total W width. With this approximation,

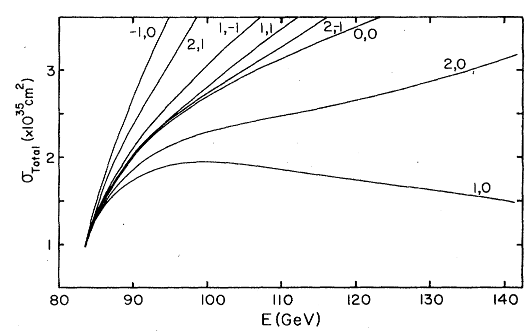

FIG. 1. The total cross section for $e^+e^-\to W^+W^-$ in units of 10^{-35} cm^2 as a function of the beam energy for various values of κ, Q.

the only diagrams are those considered in the previous section and the double-energy distribution can be given analytically. Because there is no interference among the leptons l^+ need not be the same flavor as l^-.

For convenience we write the energies of the leptons, $E_\pm$, as dimensionless variables $X_\pm$ which vary between zero and one,

$$X_\pm = \frac{1}{E\beta}\left[E_\pm - \frac{E}{2}(1-\beta)\right], \tag{6}$$

where, as before, E is the beam energy and β is the velocity of the W. The masses of the leptons are neglected everywhere. The contribution from the square of the neutrino exchange is

$$\begin{aligned}\left.\frac{d\sigma}{dX_+dX_-}\right|_{\nu\nu} = {} & \frac{\alpha^4\pi}{2^{12}\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\frac{E^2}{\beta^4}\\
&\times[-4\beta+12\beta^2-\tfrac{16}{3}\beta^3-12\beta^4+\tfrac{56}{3}\beta^5-12\beta^6-\tfrac{16}{3}\beta^7+12\beta^8-4\beta^9\\
&-L(-1+3\beta-\beta^2-4\beta^3+2\beta^4+2\beta^5+2\beta^6-4\beta^7-\beta^8+3\beta^9-\beta^{10})\\
&+(X_+ + X_-)(24\beta-48\beta^2-24\beta^3+80\beta^4-\tfrac{8}{3}\beta^5-16\beta^6+\tfrac{8}{3}\beta^7-16\beta^8)\\
&-(X_+ + X_-)L(6-12\beta-8\beta^2+24\beta^3+4\beta^4-16\beta^5-8\beta^6+8\beta^7+6\beta^8-4\beta^9)\\
&+(X_+^2+X_-^2)(-24\beta+36\beta^2+44\beta^3-68\beta^4-\tfrac{100}{3}\beta^5+28\beta^6+\tfrac{28}{3}\beta^7+4\beta^8+4\beta^9)\\
&-(X_+^2+X_-^2)L(-6+9\beta+13\beta^2-20\beta^3-12\beta^4+14\beta^5+10\beta^6-4\beta^7-6\beta^8+\beta^9+\beta^{10})\\
&+X_+X_-(-144\beta+144\beta^2+320\beta^3-272\beta^4-224\beta^5+112\beta^6+\tfrac{320}{3}\beta^7+16\beta^8-16\beta^9)\\
&-X_+X_-L(-36+36\beta+92\beta^2-80\beta^3-88\beta^4+56\beta^5+40\beta^6-16\beta^7-4\beta^8+4\beta^9-4\beta^{10})\\
&+(X_+X_-^2+X_+^2X_-)(144\beta-72\beta^2-360\beta^3+136\beta^4+296\beta^5-56\beta^6-\tfrac{392}{3}\beta^7-8\beta^8+8\beta^9)\\
&-(X_+X_-^2+X_+^2X_-)L(36-18\beta-102\beta^2+40\beta^3+104\beta^4-28\beta^5-44\beta^6+8\beta^7+4\beta^8-2\beta^9+2\beta^{10})\\
&+X_+^2X_-^2(-144\beta+360\beta^3-296\beta^5+\tfrac{392}{3}\beta^7-8\beta^9)\\
&-X_+^2X_-^2L(-36+102\beta^2-104\beta^4+44\beta^6-4\beta^8-2\beta^{10})]\,,\end{aligned} \tag{7}$$

where $L \equiv \ln[(1+\beta)^2/(1-\beta)^2]$.

The single-energy distribution is easily determined from the double distribution. (Set $X_- = \frac{1}{2}$, $X_-^2 = \frac{1}{3}$, and simplify.) The contribution from neutrino exchange is

$$\begin{aligned}\left.\frac{d\sigma}{dX_+}\right|_{\nu\nu} = {} & \frac{\alpha^4\pi}{2^{12}\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\frac{E^2}{\beta^2}\\
&\times\{\tfrac{8}{3}\beta(-1+2\beta+\tfrac{7}{3}\beta^2-4\beta^3-\tfrac{1}{3}\beta^4+2\beta^5-\beta^6)-\tfrac{2}{3}L(-1+2\beta-2\beta^2+2\beta^3-2\beta^4+2\beta^5-\beta^6)\\
&+X_+[16\beta(1-\tfrac{2}{3}\beta-\beta^2+\tfrac{4}{3}\beta^3+\tfrac{7}{9}\beta^4-\tfrac{2}{3}\beta^5-\tfrac{1}{3}\beta^6)-\tfrac{4}{3}L(3-2\beta-4\beta^2+2\beta^3-2\beta^4+2\beta^5+4\beta^6-2\beta^7-\beta^8)]\\
&+X_+^2[16\beta(-1+\beta^2-\tfrac{7}{9}\beta^4+\tfrac{1}{3}\beta^6)-\tfrac{4}{3}L(-3+4\beta^2+2\beta^4-4\beta^6+\beta^8)]\}\,.\end{aligned} \tag{8}$$

If we integrate both X_+ and X_- in (7) the cross section is

$$\sigma_{\nu\nu} = \frac{\alpha^4\pi}{9(2)^9\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2 E^2\left[\tfrac{4}{3}\beta(3+\beta^2-3\beta^4)+(1-\beta^2-\beta^4+\beta^6)\ln\frac{(1+\beta)^2}{(1-\beta)^2}\right]. \tag{9}$$

In these energy distributions, unlike the previous section, we express the magnetic-moment contribution in terms of its deviation y from the canonical value,

$$\kappa = 1 + y\,. \tag{10}$$

In terms of y and Q the cross term between the neutrino exchange and the photon plus Z exchange is

$$
\begin{aligned}
\left.\frac{d\sigma}{dX_+dX_-}\right|_{\mathrm{CT}} = {} & \frac{\alpha^4\pi}{2^{10}\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\frac{E^2}{\beta^2}\left[\sin^2\theta_W+(\tfrac{1}{2}-\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right] \\
& \times\{(X_++X_-)[-4\beta+12\beta^2-\tfrac{4}{3}\beta^3-28\beta^4-\tfrac{20}{3}\beta^5+20\beta^6+12\beta^7-4\beta^8 \\
& \quad -L(-1+3\beta-8\beta^3+6\beta^4+6\beta^5-8\beta^6+3\beta^8-\beta^9) \\
& \quad +y(-2\beta+6\beta^2-\tfrac{2}{3}\beta^3-14\beta^4-\tfrac{10}{3}\beta^5+10\beta^6+6\beta^7-2\beta^8) \\
& \quad -yL(-\tfrac{1}{2}+\tfrac{3}{2}\beta-4\beta^3+3\beta^4+3\beta^5-4\beta^6+\tfrac{3}{2}\beta^8-\tfrac{1}{2}\beta^9)] \\
& +(X_+{}^2+X_-{}^2)[5\beta-12\beta^2-\tfrac{10}{3}\beta^3+28\beta^4+12\beta^5-20\beta^6-\tfrac{38}{3}\beta^7+4\beta^8-\beta^9 \\
& \quad -L(\tfrac{5}{4}-3\beta-\tfrac{5}{4}\beta^2+8\beta^3-\tfrac{7}{2}\beta^4-6\beta^5+\tfrac{11}{2}\beta^6-\tfrac{7}{4}\beta^8+\beta^9-\tfrac{1}{4}\beta^{10}) \\
& \quad +y(2\beta-6\beta^2+\tfrac{2}{3}\beta^3+14\beta^4+\tfrac{10}{3}\beta^5-10\beta^6-6\beta^7+2\beta^8) \\
& \quad -yL(\tfrac{1}{2}-\tfrac{3}{2}\beta+4\beta^3-3\beta^4-3\beta^5+4\beta^6-\tfrac{3}{2}\beta^8+\tfrac{1}{2}\beta^9) \\
& \quad -Q(2\beta-\tfrac{22}{3}\beta^3+\tfrac{10}{3}\beta^5+2\beta^7)+QL(\tfrac{1}{2}-2\beta^2+3\beta^4-2\beta^6+\tfrac{1}{2}\beta^8)] \\
& +X_+X_-[40\beta-48\beta^2-\tfrac{272}{3}\beta^3+112\beta^4+32\beta^5-80\beta^6-16\beta^7+16\beta^8-8\beta^9 \\
& \quad -L(10-12\beta-26\beta^2+32\beta^3+20\beta^4-24\beta^5-4\beta^6+2\beta^8+4\beta^9-2\beta^{10}) \\
& \quad +y(24\beta-24\beta^2-56\beta^3+56\beta^4-\tfrac{8}{3}\beta^5-40\beta^6-8\beta^7+8\beta^8) \\
& \quad -yL(6-6\beta-16\beta^2+16\beta^3+12\beta^4-12\beta^5-2\beta^8+2\beta^9) \\
& \quad -Q(-16\beta+\tfrac{176}{3}\beta^3+16\beta^5-16\beta^7)+QL(-4+16\beta^2-24\beta^4+16\beta^6-4\beta^8)] \\
& +(X_+X_-{}^2+X_+{}^2X_-)[-42\beta+24\beta^2+100\beta^3-56\beta^4-\tfrac{128}{3}\beta^5+40\beta^6+\tfrac{52}{3}\beta^7-8\beta^8+10\beta^9 \\
& \quad -L(-\tfrac{21}{2}+6\beta+\tfrac{57}{2}\beta^2-16\beta^3-25\beta^4+12\beta^5+9\beta^6-\tfrac{9}{2}\beta^8-2\beta^9+\tfrac{5}{2}\beta^{10}) \\
& \quad +y(-24\beta+12\beta^2+56\beta^3-28\beta^4+\tfrac{8}{3}\beta^5+20\beta^6+8\beta^7-4\beta^8) \\
& \quad -yL(-6+3\beta+16\beta^2-8\beta^3-12\beta^4+6\beta^5+2\beta^8-\beta^9) \\
& \quad -Q(12\beta-44\beta^3-\tfrac{68}{3}\beta^5+12\beta^7)+QL(3-12\beta^2+18\beta^4-12\beta^6+3\beta^8)] \\
& +X_+{}^2X_-{}^2[42\beta-100\beta^3+\tfrac{128}{3}\beta^5-\tfrac{52}{3}\beta^7-10\beta^9-L(\tfrac{21}{2}-\tfrac{57}{2}\beta^2+25\beta^4-9\beta^6+\tfrac{9}{2}\beta^8-\tfrac{5}{2}\beta^{10}) \\
& \quad +y(24\beta-56\beta^3-\tfrac{8}{3}\beta^5-8\beta^7)-yL(6-16\beta^2+12\beta^4-2\beta^8) \\
& \quad -Q(-12\beta+44\beta^3+\tfrac{68}{3}\beta^5-12\beta^7)+QL(-3+12\beta^2-18\beta^4+12\beta^6-3\beta^8)]\}\ . \qquad (11)
\end{aligned}
$$

As in the previous section, if we have the $\sin^2\theta_W$ term in the first large parentheses then y and Q refer to the photon vertex, while for the second term, y and Q refer to the Z. s is $4E^2$.

The contribution to the single-energy distribution from this cross term is

$$
\begin{aligned}
\left.\frac{d\sigma}{dX_+}\right|_{\mathrm{CT}} = {} & \frac{\alpha^4\pi}{2^{10}\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\frac{E^2}{\beta^2}\left[\sin^2\theta_W+(\tfrac{1}{2}-\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right] \\
& \times\{\tfrac{1}{3}\beta(-1+6\beta-\tfrac{16}{3}\beta^2-14\beta^3+2\beta^4+10\beta^5+\tfrac{16}{3}\beta^7-2\beta^8-\beta^9) \\
& -\tfrac{1}{12}L(-1+6\beta-5\beta^2-16\beta^3+22\beta^4+12\beta^5-26\beta^6+11\beta^8-2\beta^9-\beta^{10}) \\
& +\tfrac{1}{3}\beta y(-1+3\beta-\tfrac{1}{3}\beta^2-7\beta^3-\tfrac{5}{3}\beta^4+5\beta^5+3\beta^6-\beta^7) \\
& -\tfrac{1}{12}Ly(-1+3\beta-8\beta^3+6\beta^4+6\beta^5-8\beta^6+3\beta^8-\beta^9) \\
& +Q(-\tfrac{2}{3}\beta+\tfrac{22}{9}\beta^3-\tfrac{10}{9}\beta^5-\tfrac{2}{3}\beta^7)-QL(-\tfrac{1}{6}+\tfrac{2}{3}\beta^2-\beta^4+\tfrac{2}{3}\beta^6-\tfrac{1}{6}\beta^8) \\
& +X_+[2\beta-4\beta^2-\tfrac{40}{3}\beta^3+\tfrac{28}{3}\beta^4-\tfrac{44}{9}\beta^5-\tfrac{20}{3}\beta^6+\tfrac{88}{9}\beta^7+\tfrac{4}{3}\beta^8-\tfrac{2}{3}\beta^9 \\
& \quad -L(\tfrac{1}{2}-\beta-\tfrac{7}{2}\beta^2+\tfrac{8}{3}\beta^3+\tfrac{23}{3}\beta^4-2\beta^5-7\beta^6+\tfrac{5}{2}\beta^8+\tfrac{1}{3}\beta^9-\tfrac{1}{6}\beta^{10}) \\
& \quad +Q(4\beta-\tfrac{44}{3}\beta^3-\tfrac{4}{9}\beta^5+4\beta^7)-QL(1-4\beta^2+6\beta^4-4\beta^6+\beta^8)
\end{aligned}
$$

$$
\begin{aligned}
&+y(2\beta-2\beta^2-10\beta^3+\tfrac{14}{3}\beta^4-\tfrac{34}{9}\beta^5-\tfrac{10}{3}\beta^6+\tfrac{14}{3}\beta^7+\tfrac{2}{3}\beta^8)\\
&-yL(\tfrac{1}{2}-\tfrac{1}{2}\beta-\tfrac{8}{3}\beta^2+\tfrac{4}{3}\beta^3+5\beta^4-\beta^5-4\beta^6+\tfrac{7}{6}\beta^8+\tfrac{1}{6}\beta^9)]\\
&+X_+^{\,2}[-2\beta+\tfrac{40}{3}\beta^3+\tfrac{44}{9}\beta^5-\tfrac{88}{9}\beta^7+\tfrac{2}{3}\beta^9-L(-\tfrac{1}{2}+\tfrac{7}{2}\beta^2-\tfrac{23}{3}\beta^4+7\beta^6-\tfrac{5}{2}\beta^8+\tfrac{1}{6}\beta^{10})\\
&+Q(-4\beta+\tfrac{44}{3}\beta^3+\tfrac{4}{9}\beta^5-4\beta^7)-QL(-1+4\beta^2-6\beta^4+4\beta^6-\beta^8)\\
&+y(-2\beta+10\beta^3+\tfrac{34}{9}\beta^5-\tfrac{14}{3}\beta^7)-yL(-\tfrac{1}{2}+\tfrac{8}{3}\beta^2-5\beta^4+4\beta^6-\tfrac{7}{6}\beta^8)]\}\ .
\end{aligned}
\tag{12}
$$

If we integrate over both X_+ and X_- we find

$$
\begin{aligned}
\sigma_{\rm CT}=&\frac{\alpha^4\pi}{2^{10}\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\frac{E^2}{\beta^2}\left[\sin^2\theta_W+(\tfrac{1}{2}-\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right]\\
&\times[-4\beta^3-\tfrac{4}{27}\beta^5+\tfrac{92}{27}\beta^7-\tfrac{4}{9}\beta^9+Q(-\tfrac{32}{27}\beta^5)-QL(-\tfrac{2}{9}\beta^2+\tfrac{2}{3}\beta^4-\tfrac{2}{3}\beta^6+\tfrac{2}{9}\beta^8)\\
&-L(-\beta^2+\tfrac{28}{9}\beta^4-\tfrac{10}{3}\beta^6+\tfrac{4}{3}\beta^8-\tfrac{1}{9}\beta^{10})+y(-\tfrac{16}{9}\beta^3-\tfrac{32}{27}\beta^5+\tfrac{16}{9}\beta^7)-yL(-\tfrac{4}{9}\beta^2+\tfrac{4}{3}\beta^4-\tfrac{4}{3}\beta^6+\tfrac{4}{9}\beta^8)]\ .
\end{aligned}
\tag{13}
$$

The square of the photon plus Z exchange gives

$$
\begin{aligned}
\left.\frac{d\sigma}{dX_+dX_-}\right|_{\gamma Z}=&\frac{\alpha^4\pi}{512\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\beta E^2\left[\left[4\sin^2\theta_W+(1-4\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right]^2+\left[\frac{s}{s-M_Z{}^2}\right]^2\right]\\
&\times\Bigg[(X_++X_-)\tfrac{1}{3}\beta^2(1-\beta^2)(4+2y+2y'+yy')\\
&+(X_+{}^2+X_-{}^2)\tfrac{1}{3}\beta^2[-\tfrac{7}{2}+3\beta^2+\tfrac{1}{2}\beta^4+2Q^2-2Q(1-\beta^2)-(2y+2y'+yy')(1-\beta^2)]\\
&+X_+X_-\left[\tfrac{2}{3}\beta^2(1+\beta^2)^2-Q(-Q+\beta^2-3)\frac{\beta^2}{3}\right]\\
&+(X_+{}^2X_-{}^2-X_+X_-{}^2-X_+{}^2X_-)\tfrac{1}{3}\beta^2(3+2\beta^2+3\beta^4+4Q(5-\beta^2)+12Q^2)\\
&+(X_+X_-+X_+{}^2X_-{}^2-X_+X_-{}^2-X_+{}^2X_-)\{\tfrac{4}{3}\beta^2[(1+\beta^2)(y+y'+yy')+2(Qy'+yQ')]\}\Bigg]\ .
\end{aligned}
\tag{14}
$$

As before the coefficient of $1/(s-M_Z{}^2)^2$ has $Q=Q'=Q_Z$, $y=y'=y_Z$ while the coefficient of $1/(s-M_Z{}^2)$ has $Q=Q_\gamma$, $Q'=Q_Z$, $y=y_\gamma$, and $y'=y_Z$. The remaining terms have $Q=Q'=Q_\gamma$, $y=y'=y_\gamma$.

Integrating over X_- gives

$$
\begin{aligned}
\left.\frac{d\sigma}{dX_+}\right|_{\gamma Z}=&\frac{\alpha^4\pi}{512\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\beta E^2\\
&\times\left[\left[4\sin^2\theta_W+(1-4\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right]^2+\left[\frac{s}{s-M_Z{}^2}\right]^2\right]\\
&\times\{\tfrac{1}{18}\beta^2[5-6\beta^2+\beta^4+(1-\beta^2)(2y+2y'+yy'+2Q+2Q')+4QQ']\\
&+\tfrac{1}{9}X_+(1-X_+)[12\beta^2-8\beta^4+4(y+y')\beta^2(2-\beta^2)+yy'\beta^2(5-\beta^2)\\
&+4\beta^2(2-\beta^2)(Q+Q'+y+y')+\beta^2(5-\beta^2)yy'+4(Qy'+Q'y)\beta^2]\}\ .
\end{aligned}
\tag{15}
$$

Integrating over both X_+ and X_- leaves

$$
\begin{aligned}
\sigma_{\gamma Z}=&\frac{\alpha^4\pi}{6(48)^2\sin^8\theta_W}\left[\frac{1}{M_W\Gamma}\right]^2\beta E^2\left[\left[4\sin^2\theta_W+(1-4\sin^2\theta_W)\frac{s}{s-M_Z{}^2}\right]^2+\left[\frac{s}{s-M_Z{}^2}\right]^2\right]\\
&\times[\tfrac{27}{2}\beta^2-13\beta^4+\tfrac{3}{2}\beta^6+(Q+Q')\beta^2(1+\beta^2)+QQ'6\beta^2+(y+y')\beta^2(7-5\beta^2)+2(yQ'+y'Q)\beta^2+2yy'\beta^2(2-\beta^2)]\ .
\end{aligned}
\tag{16}
$$

The total double-energy distribution, normalized by the total cross section, is (7), (11), and (14), divided by the sum of (9), (13), and (16):

$$
\frac{1}{\sigma}\frac{d\sigma}{dX_+dX_-}=\frac{1}{\sigma}\left[\left.\frac{d\sigma}{dX_+dX_-}\right|_{\nu\nu}+\left.\frac{d\sigma}{dX_+dX_-}\right|_{\rm CT}+\left.\frac{d\sigma}{dX_+dX_-}\right|_{\gamma Z}\right],
\tag{17}
$$

TABLE I. $(1/\sigma)d^2\sigma/dX_+dX_-$ for bins in X_+ and X_-. Each number is the percentage of events in that bin. The first number in each bin is for the $\kappa=1$, $Q=0$ case; the second number has $\kappa=2$, $Q=1$; and so on as given on the table. Thus 3.99% of the events have X_+ and X_- between 0.4 and 0.6 if $\kappa=Q=1$. The table is symmetric so only the values above the diagonal are given. This table is for a beam energy of $1.15M_W$. Because of roundoff errors a given set of numbers may not add to exactly 100.

X_+ \ X_-	0.2	0.4	0.6	0.8	1.0
1.0					
	1.70[a]	3.32	5.47	8.13	11.31
	1.74[b]	3.30	5.35	7.87	10.87
	1.66[c]	3.27	5.31	7.76	10.63
	1.69[d]	3.31	5.42	8.03	11.12
	1.70[e]	3.34	5.16	7.17	9.36
	1.70[f]	3.26	5.21	7.55	10.27
0.8					
	2.02	3.11	4.49	6.16	
	2.16	3.17	4.45	6.02	
	2.06	3.31	4.68	6.16	
	2.04	3.17	4.54	6.16	
	2.23	3.58	4.85	6.05	
	2.19	3.35	4.63	6.03	
0.6					
	2.89	2.95	3.65		
	2.56	3.06	3.69		
	2.44	3.24	3.99		
	2.41	3.03	3.75		
	2.60	3.58	4.33		
	2.59	3.32	4.00		
0.4					
	2.80	2.85			
	2.93	2.98			
	2.79	3.07			
	2.80	2.91			
	2.82	3.32			
	2.91	3.17			
0.2					
	3.25				
	3.27				
	3.11				
	3.21				
	2.89				
	3.14				

[a] $\kappa=1$, $Q=0$.
[b] $\kappa=2$, $Q=1$.
[c] $\kappa=1$, $Q=1$.
[d] $\kappa=1$, $Q=-1$.
[e] $\kappa=0$, $Q=0$.
[f] $\kappa=2$, $Q=0$.

where

$$\sigma=\sigma_{\nu\nu}+\sigma_{\rm CT}+\sigma_{\gamma Z} . \tag{18}$$

The total single-energy distribution is given by (8), (12), and (15):

$$\frac{1}{\sigma}\frac{d\sigma}{dX_+}=\frac{1}{\sigma}\left[\left.\frac{d\sigma}{dX_+}\right|_{\nu\nu}+\left.\frac{d\sigma}{dX_+}\right|_{\rm CT}+\left.\frac{d\sigma}{dX_+}\right|_{\gamma Z}\right] . \tag{19}$$

In the next section we give the results of (17) and (19) for some values of β, κ, and Q (Ref. 5).

IV. NUMERICAL RESULTS

Our main result is the double distribution given by Eq. (17). The single-energy distribution (19) is shown in Fig. 2 for a beam energy close to threshold, $E=1.15M_W$. The

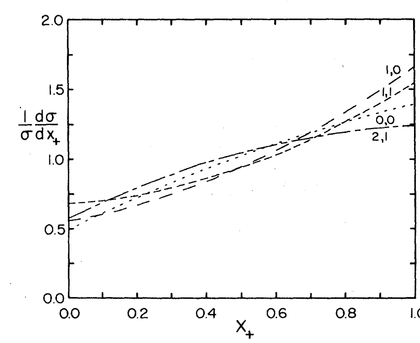

FIG. 2. $(1/\sigma)d\sigma/dX_+$ vs X_+ at a beam energy of $1.15M_W$ for various values of κ, Q. X_+ is the energy of one lepton as defined in (6) in the text.

TABLE II. $(1/\sigma)d^2\sigma/dX_+dX_-$ as in Table I, except that the beam energy here is $1.30M_W$.

X_+ \ X_-	0.2	0.4	0.6	0.8	1.0
1.0					
	1.38[a]	3.25	5.98	9.58	14.04
	1.45[b]	3.16	5.53	8.54	12.21
	1.26[c]	2.99	4.85	6.84	8.98
	1.29[d]	3.06	5.16	7.60	10.37
	1.48[e]	3.27	5.21	7.30	9.52
	1.42[f]	3.11	5.32	8.03	11.25
0.8					
	1.57	2.79	4.53	6.80	
	1.92	3.13	4.63	6.44	
	1.96	4.10	5.63	6.54	
	1.85	3.74	5.33	6.61	
	2.07	3.78	5.22	6.39	
	1.98	3.39	4.86	6.41	
0.6					
	1.83	2.46	3.36		
	2.26	3.00	3.79		
	2.32	4.36	5.47		
	2.18	3.84	4.89		
	2.38	3.80	4.74		
	2.33	3.37	4.22		
0.4					
	2.16	2.24			
	2.48	2.78			
	2.32	3.77			
	2.28	3.35			
	2.40	3.34			
	2.49	3.07			
0.2					
	2.55				
	2.57				
	1.98				
	2.14				
	2.15				
	2.44				

[a] $\kappa=1$, $Q=0$.
[b] $\kappa=2$, $Q=1$.
[c] $\kappa=1$, $Q=1$.
[d] $\kappa=1$, $Q=-1$.
[e] $\kappa=0$, $Q=0$.
[f] $\kappa=2$, $Q=0$.

curves are labeled by values of κ, Q to agree with the labeling of Fig. 1. The values of κ, Q which gave almost the same *total* cross section (see Fig. 1) as the total cross section for $\kappa=1$, $Q=0$ at this energy also have very similar energy distributions. Figure 3 shows the same thing for $E=1.30M_W$. As with the total cross section of Fig. 1, the distributions for values of κ, Q different from 1, 0 are becoming clearly distinguishable from the 1, 0 case at this energy. In particular, notice that all the curves are well below the 1, 0 curve for X_+ near 1. Notice however that nothing can be learned near $X_+=0.7$.

To study the double-energy distribution (17) we integrate X_+ and X_- over fixed intervals, thereby determining the percentage of events which fall into each bin. An example is shown in Table I where we have divided the range of X_+ and X_- into five regions. The beam energy assumed in Table I is the same as that of Fig. 2, $E=1.15M_W$. While the fraction of events is almost independent of κ and Q in some of the bins, it shows considerable variation for X_+ and X_- both large. Table II shows the same double distribution for the higher beam energy, $E=1.30M_W$. Table I seems to indicate that higher beam energies are not necessary for the determination of κ and Q if enough data can be taken at the lower

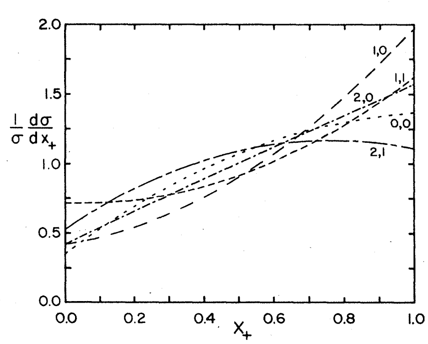

FIG. 3. $(1/d)d\sigma/dX_+$ vs X_+ at a beam energy of $1.30M_W$ for various values of κ, Q.

energies to fix sufficiently accurately the fraction of charged leptons in the largest energy bin.

V. SUMMARY

Our result is the analytic expression (17) for the double-energy distribution of leptons from W decay. Equation (17) is then used to derive easily the distribution in the energy of a single lepton, Eq. (19). These formulas are illustrated by the tables and by Figs. 2 and 3. The double distribution might be used, even at low energy, to fix the values of κ and Q and hence the three-gauge-particle interaction.

ACKNOWLEDGMENT

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[5]In the figures and in the tables we consider only the case where $\kappa=\kappa'$ and $Q=Q'$.