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# Continuous observations in quantum mechanics: An application to gravitational-wave detectors

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A two-stage measuring apparatus (detector and meter) for detecting gravitational waves is analyzed using a recently developed formalism for continuous observations in quantum mechanics. Detector and meter are modeled as quantum-mechanical dissipative linear systems; a large class of couplings between them is considered, so obtaining a unified treatment of various models which have appeared in the literature. In the simplest cases, explicit expressions for uncertainties in the estimate of the gravitational force are obtained. In the general case it is shown how the problem of the estimation of the force can be treated and general results are given.

## I. INTRODUCTION

A detector for gravitational waves can be crudely described as a harmonic oscillator which couples to the gravitational field; by monitoring some characteristic of the oscillator one obtains information about the wave. The gravitational wave is estimated to behave essentially as a classical force on the oscillator; however, this force is so weak that uncertainties in the measurement due to the Heisenberg principle can completely mask it. The need to avoid these uncertainties has led to the development of the concept of "quantum nondemolition (QND) measurement" (see Refs. 1—7 and references contained therein). Essentially, one chooses properly the measured observables of the detector in order that from the data obtained one can deduce the value and the time behavior of the classical force with an uncertainty as small as desired.

The chosen observables (QND or not) are often continuously monitored, so that we have to treat continuous measurements in quantum mechanics (QM). This is a delicate concept: only in the framework of a generalized formalism for QM (Refs. 8—12) can continuous observations be introduced in a well-founded and mathematically consistent way.

A theory of continuous observations, suitable for the quantum analysis of a gravitational-wave detector, was developed by our research group in Milan.[13–21] We succeeded in giving a quantum-mechanical description of a situation in which some observables (such as position, or position and momentum, for a particle) are continuously followed in their time evolution.

The aim of this paper is twofold. First, using the theory we have developed, I give a unified treatment of some models for gravitational-wave detection which have been introduced in the literature. I think that the physical ideas underlying the QND concept (see, for instance, Ref. 2) are essentially right; I reconsider these models here because, as questions of principle are raised, the analysis must be well founded in measurement theory in QM. Moreover, our formalism allows for an easy treatment of dissipative phenomena, such as diffusion and damping.

Second, the simple models introduced in the literature in connection with gravitational-wave detection provide a framework for a nontrivial application of the formalism of continuous observations, so allowing one to test the possibilities of this theory and to illuminate the underlying physical concepts.

The plan of the paper is as follows.

In Sec. II the formalism of continuous observations in QM is presented.

In Sec. III we discuss a quantum-mechanical model for a two-stage measuring apparatus (detector, coupled to the gravitational force, and meter) and introduce continuous observations for this model. Detector and meter are treated as linear dissipative one-dimensional systems and various couplings between them are considered. The strategy is to obtain information about the gravitational force starting from continuous measurements of the canonical position and momentum of the meter.

In Sec. IV, in order to test the possibilities of the continuous-measurement theory, we treat the simplest version of the introduced model. Mean values and correlations are calculated and compared with the results of the usual formulation of QM. Moreover, it is shown how one can obtain information on the position and momentum of the detector from measurement of the position and momentum of the meter.

In Sec. V the problem of the estimation of the gravitational force is considered. First it is shown how this problem can be handled in the general case; then two particular schemes for the measuring apparatus are treated in detail and the uncertainties in the estimate of the force are calculated.

## II. CONTINUOUS OBSERVATIONS IN QUANTUM MECHANICS

In this section we give an account of the formalism of continuous observations in QM. We shall try to pay more attention to physical ideas than to mathematical rigor; however, because notions from different fields, such as open system theory, measurement theory in QM, and generalized stochastic processes, are involved, the exposition of the mathematical formalism is lengthy. A full presentation of continuous measurements can be found in Ref. 15.

The most general framework for treating continuously measured quantities is that of *generalized stochastic processes* (GSP's),[22] which turns out to be essential for introducing continuous measurements in QM. Let us denote by

$$\mathbf{z}(t)\equiv(z_1(t),z_2(t),\ldots,z_n(t))$$

the set of observables that are continuously measured. As any measuring apparatus has a certain "inertia" in its response, in general, its output will give only "time averages" of $\mathbf{z}(t)$ of the form

$$z_h=\sum_{j=1}^{n}\int dt\, h_j(t)z_j(t)\ , \tag{2.1}$$

where the *test functions*

$$\mathbf{h}(t)\equiv(h_1(t),\ldots,h_n(t))$$

belong to a suitable function space $E$.

In this context, a physically meaningful question is the following: given a set of test functions $\mathbf{h}^{(1)}(t),\ldots,\mathbf{h}^{(s)}(t)$, what is the probability that the $s$-tuple of values of the corresponding stochastic variables $z_{h^{(1)}},\ldots,z_{h^{(s)}}$ belongs to a Borel subset $B$ of $\mathbb{R}^s$? More precisely, we take for $E$ the nuclear space of $n$-component, real, $C^\infty$ functions on $\mathbf{R}$ with compact support and denote by $E'$ the topological dual space of $E$. The subsets of $E'$ of the form

$$C(\mathbf{h}^{(1)},\ldots,\mathbf{h}^{(s)};B)=\{\mathbf{z}\in E':(z_{h^{(1)}},\ldots,z_{h^{(s)}})\in B\} \tag{2.2}$$

($B$ is a Borel set in $\mathbf{R}^s$) are called cylinder sets. Let $\Sigma$ be the $\sigma$ algebra generated by the cylinder sets. Then the continuous observation is described by a probability measure $P(N\mid\rho)$, $N\in\Sigma$ [for instance, $N$ can be the set (2.2)], which gives the probability of the result $\mathbf{z}\in N$; $\rho$ is a symbol which denotes how the system is prepared.

A GSP is uniquely determined by its *characteristic functional* $L([\boldsymbol{\varphi}(t)]\mid\rho)$,[22] which is defined as the mean value of $\exp(iz_\varphi)$ with respect to the probability measure $P(\cdot\mid\rho)$. Vice versa, probabilities can be obtained from the characteristic functional by taking the Fourier transform

$$P(C(\mathbf{h}^{(1)},\ldots,\mathbf{h}^{(s)};B)\mid\rho)$$
$$=\int_B d_s\mathbf{z}\frac{1}{(2\pi)^s}\int d_s\mathbf{k}\exp\left[-i\sum_{r=1}^{s}k_r z_r\right]\times L\left[\left[\sum_{r=1}^{s}k_r\mathbf{h}^{(r)}(t)\right]\middle|\rho\right]. \tag{2.3}$$

Moreover, the moments of the variables (2.1) can be obtained directly from the characteristic functional by functional differentiation,

$$\langle z_{h^{(1)}}z_{h^{(2)}}\cdots z_{h^{(s)}}\rangle$$
$$=\sum_{j_1,\ldots,j_s=1}^{n}\int dt_1\cdots dt_s h_{j_1}^{(1)}(t_1)\cdots h_{j_s}^{(s)}(t_s)\times\langle z_{j_1}(t_1)\cdots z_{j_s}(t_s)\rangle\ , \tag{2.4a}$$

$$\langle z_{j_1}(t_1)\cdots z_{j_s}(t_s)\rangle$$
$$=(-i)^s\frac{\delta^s}{\delta\varphi_{j_1}(t_1)\cdots\delta\varphi_{j_s}(t_s)}L([\boldsymbol{\varphi}]\mid\rho)\bigg|_{\boldsymbol{\varphi}=0}\ . \tag{2.4b}$$

The prototype of a GSP is white noise, whose characteristic functional is

$$L_{\mathrm{wn}}[\varphi]=\exp\left[-\tfrac{1}{2}\Gamma\int dt\,\varphi^2(t)\right],\quad \Gamma>0\ ; \tag{2.5}$$

it is a zero-mean Gaussian GSP with a two-time correlation function given by

$$\langle[z(t)-\langle z(t)\rangle][z(t')-\langle z(t')\rangle]\rangle=\Gamma\delta(t-t')\ . \tag{2.6}$$

Many manipulations, such as linear transformations on stochastic variables, are easily made at the level of the characteristic functional. For instance, consider a GSP described by the characteristic functional $L([\varphi]\mid\rho)$; $z(t)$ is the continuously observed quantity. Suppose we are interested in the probabilities for the derivative $\dot z(t)$; we denote by $\eta(t)$ the generic test function associated with $\dot z(t)$. Recalling that test functions vanish with all their derivatives at the ends of the interval of measurement, we can write

$$\int dt\,\eta(t)\dot z(t)=-\int dt\,\dot\eta(t)z(t)\ . \tag{2.7}$$

By taking $\varphi(t)=-\dot\eta(t)$ we obtain the characteristic functional $\tilde L(\cdots)$ for $\dot z(t)$, i.e.,

$$\tilde L([\eta(t)]\mid\rho)\equiv L([-\dot\eta(t)]\mid\rho)\ ; \tag{2.8}$$

then probabilities can be obtained from $\tilde L(\cdots)$ by Eq. (2.3). Transformations of this kind will be used in the following sections.

Now let us consider quantum mechanics. In the most general formulation,[8-12] measurements are described by *operation-valued measures*. Let $T(\mathscr{h})$ be the Banach space of trace class operators on the Hilbert space $\mathscr{h}$ of the system; $T(\mathscr{h})$ is the space spanned by statistical operators. Let $\Sigma$ be a $\sigma$ algebra on a space $\Omega$ (for instance, for $s$ real-valued observables we take $\Omega=\mathbb{R}^s$ and $\Sigma$ is the $\sigma$ algebra of Borel sets). An operation-valued measure $\mathscr{F}(N)$, $N\in\Sigma$, is a set of linear operators in $T(\mathscr{h})$, with the following properties: (i) $\mathscr{F}(N)$ is $\sigma$ additive on $\Sigma$; (ii) $\mathscr{F}(N)$ is completely positive; and (iii) $\mathscr{F}(\Omega)$ is trace preserving (normalization), i.e.,

$$\mathrm{Tr}[\mathscr{F}(\Omega)\hat X]=\mathrm{Tr}(\hat X),\quad \forall\hat X\in T(\mathscr{h})\ . \tag{2.9}$$

Then, for the system prepared in the state $\hat\rho$ (statistical operator), the probability of obtaining the result $\mathbf{z}\in N$ ($\mathbf{z}$ denotes the measured quantities) is given by

$$P(N\mid\rho)=\mathrm{Tr}[\mathscr{F}(N)\hat\rho]\ . \tag{2.10}$$

Moreover, operation-valued measures give also the change of the state due to the measurement: indeed

$$\hat\rho'\equiv\mathscr{F}(N)\hat\rho/P(N\mid\rho) \tag{2.11}$$

is the state of the system after the measurement, when the result was $\mathbf{z}\in N$. In this respect operations give a generalization of the Von Neumann reduction postulate. The usual scheme of QM (observables represented by self-adjoint operators , . . .) can be restated in the language of operations and represents only a particular kind of measurement in QM.

Properties (i)—(iii) guarantee that Eq. (2.10) defines a probability measure on $\Sigma$ (a positive, $\sigma$ additive, normalized measure) and that $\hat{\rho}'$ [Eq. (2.11)] is a statistical operator (a self-adjoint, positive operator with trace one). In particular, property (ii) implies the positivity of quantities (2.10) and (2.11); however, complete positivity is mathematically more restrictive than positivity of (2.10) and (2.11). This condition is now largely used in axiomatic open system theory[23] and measurement theory[11,12] and can be justified by physical considerations.[11,23] As in the following, I have no need of using explicitly complete positivity, I refer to the existing literature for its definition (see, for instance, Ref. 23).

Using operations and the language of GSP's, continuous observations can be introduced in QM, avoiding problems such as the "Zeno paradox."[24] For any time interval $(t_1,t_2)$ of measurement, we assume there exists an operation-valued measure $\mathscr{F}(t_2,t_1;N)$, $N\in\Sigma_{(t_1,t_2)}$, where $\Sigma_{(t_1,t_2)}$ is the sub-$\sigma$-algebra in $E'$ generated by the cylinder sets (2.2) with the restriction that the test functions $\mathbf{h}^{(j)}(t)$ have support only in the interval $(t_1,t_2)$. When at time $t_1$ the system is prepared in the state $\hat{\rho}$, the quantity

$$P(N\mid\rho,t_1)=\mathrm{Tr}[\mathscr{F}(t_2,t_1;N)\hat{\rho}],\quad N\in\Sigma_{(t_1,t_2)}\ , \tag{2.12}$$

is the probability of finding the result $\mathbf{z}\in N$ in the measurement interval $(t_1,t_2)$ and

$$\hat{\rho}'=\mathscr{F}(t_2,t_1;N)\hat{\rho}/P(N\mid\rho,t_1) \tag{2.13}$$

is the state at time $t_2$, conditioned upon the result $\mathbf{z}\in N$.

In order to have a consistent description when different measurement intervals are considered, we must also require the following property:

$$\mathscr{F}(t_3,t_2;M)\mathscr{F}(t_2,t_1;N)=\mathscr{F}(t_3,t_1;M\cap N)\ ,$$

$$\forall N\in\Sigma_{(t_1,t_2)},\quad \forall M\in\Sigma_{(t_2,t_3)},\quad t_1<t_2<t_3\ . \tag{2.14}$$

Let us explain the meaning of this condition. If we introduce the conditional probability

$$P(M\mid N;\rho,t_1)\equiv\frac{P(M\cap N\mid\rho,t_1)}{P(N\mid\rho,t_1)}$$

$$\equiv\frac{\mathrm{Tr}[\mathscr{F}(t_3,t_1;M\cap N)\hat{\rho}]}{\mathrm{Tr}[\mathscr{F}(t_2,t_1;N)\hat{\rho}]}\ , \tag{2.15}$$

the condition (2.14) allows us to write

$$P(M\mid N;\rho,t_1)=P(M\mid\rho',t_1)\equiv\mathrm{Tr}[\mathscr{F}(t_3,t_2;M)\hat{\rho}']\ , \tag{2.16}$$

where the state $\hat{\rho}'$ is given by Eq. (2.13). Roughly speaking, Eq. (2.14) means that the description of measurements referring to the time interval $(t_1,t_3)$ is the same as the description obtained by considering measurements in the interval $(t_1,t_2)$ followed by measurements in the interval $(t_2,t_3)$.

In this setup measurement and dynamics are intimately connected. Indeed, consider the operator

$$\mathscr{U}(t_2,t_1)\equiv\mathscr{F}(t_2,t_1;E')\ . \tag{2.17}$$

To take in $\mathscr{F}(\cdots)$ the total set $E'$ means that no registration of the result of the measurement is made in the interval $(t_1,t_2)$; here $E'$, the dual of the function space $E$, plays the same role as $\Omega$ in the general case. By property (iii) we have

$$P(E'\mid\rho,t_1)=1\ , \tag{2.18}$$

and, so, by Eq. (2.13),

$$\hat{\rho}(t_2)\equiv\mathscr{U}(t_2,t_1)\hat{\rho} \tag{2.19}$$

is the state at time $t_2$, when no selection is made in the interval $(t_1,t_2)$. Therefore, Eq. (2.17) gives the dynamics of the system. We assume $\mathscr{U}(t,t_0)$ to satisfy an evolution equation

$$\frac{\partial}{\partial t}\mathscr{U}(t,t_0)=\mathscr{L}(t)\mathscr{U}(t,t_0)\ , \tag{2.20a}$$

$$\mathscr{U}(t_0,t_0)=1\ , \tag{2.20b}$$

where the generator $\mathscr{L}(t)$ is a linear operator in $T(\mathscr{H})$ with the structure

$$\mathscr{L}(t)=-\frac{i}{\hbar}[\hat{H}(t),\cdot]-\frac{1}{4\hbar^2}\sum_{r,r'=1}^{m}D_{rr'}[\hat{Q}_r,[\hat{Q}_{r'},\cdot]]$$

$$-\frac{i}{4\hbar}\sum_{r,r'=1}^{m}E_{rr'}[\hat{Q}_r,\{\hat{Q}_{r'},\cdot\}]\ , \tag{2.21}$$

$$\hat{H}(t)=\hat{H}(t)^\dagger,\quad \hat{Q}_r=\hat{Q}_r^\dagger\ , \tag{2.22}$$

$$D_{rr'}=D_{r'r}\in\mathbb{R},\quad E_{rr'}=-E_{r'r}\in\mathbb{R}\ , \tag{2.23}$$

$$\mathbf{D}+i\hbar\mathbf{E}\geq\mathbf{0}\ . \tag{2.24}$$

Here, $[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$, $\{\hat{A},\hat{B}\}=\hat{A}\hat{B}+\hat{B}\hat{A}$ and we have introduced the matrices

$$\mathbf{D}=(D_{rr'})_{m\times m},\quad \mathbf{E}=(E_{rr'})_{m\times m}\ . \tag{2.25}$$

The argument $t$ in $\hat{H}(t)$ means an explicit time dependence (we are working in the Schrödinger picture).

Equation (2.20) is the operator form of a generic quantum master equation. The expression (2.21) for $\mathscr{L}(t)$ is equivalent to the structure studied by Lindblad,[23] which is the most general one in the case of bounded operators; moreover, all known master equations can be rewritten in this form. In principle, one could consider a unitary dynamics for the system and the external world; then, by suitable approximations, one obtains the master equation (2.20a) for the reduced dynamics of the system; the generator $\mathscr{L}(t)$ turns out to have the structure (2.21). In particular, the explicit expressions for the operators $\hat{Q}_r$ and the matrices $\mathbf{D}$ and $\mathbf{E}$ depend on the interaction between the system and the external world. Conditions (2.22)—(2.24) guarantee that $\mathscr{U}(t,t_0)$ transforms statistical operators into statistical operators; in particular, Eq. (2.24) guarantees complete positivity of the dynamics.

Roughly speaking, the first term on the right-hand side of Eq. (2.21) gives the reversible (Hamiltonian) part of the dynamics, while the second and the third terms give the dissipative part (they can describe diffusion, damping, etc.); however, note that the splitting between Hamiltonian and dissipative parts is, in general, not unique.[23] In our case the dissipative part is due to the interaction with some reservoir and with the measuring apparatus itself; however, it is not possible to distinguish within the formalism the contributions of these two sources of noise. In the following we shall see that if dissipation is not present, no continuous measurement is possible.

Given the dynamics, we have to introduce explicitly the measurement. Technically, it is useful to work with the *characteristic operator* $\mathcal{G}(t_2,t_1;[\boldsymbol{\varphi}])$ (the analog of the characteristic functional of a GSP) defined as the mean value of $\exp(iz_\varphi)$ with respect to the operation-valued measure $\mathcal{F}(t_2,t_1;\cdot)$, i.e.,

$$\mathcal{G}(t_2,t_1;[\boldsymbol{\varphi}])\equiv\int_{E'}\exp\left[i\int_{t_1}^{t_2}dt\,\mathbf{z}(t)\cdot\boldsymbol{\varphi}(t)\right]\mathcal{F}(t_2,t_1;d\mathbf{z}) , \tag{2.26}$$

where $d\mathbf{z}$ means an infinitesimal set in $E'$. The characteristic operator is the functional Fourier transform of the measure $\mathcal{F}(t_2,t_1;\cdot)$.

First, for $\boldsymbol{\varphi}=\mathbf{0}$ the integral in Eq. (2.26) gives $\mathcal{F}(t_2,t_1;E')$, so that, by Eq. (2.17), we obtain

$$\mathcal{U}(t,t_0)=\mathcal{G}(t,t_0;[\mathbf{0}]) . \tag{2.27}$$

Moreover, just as the trace of an operation-valued measure applied to a state $\hat{\rho}$ gives probabilities, the trace of $\mathcal{G}(\cdots)$ applied to $\hat{\rho}$ gives the characteristic functional of a GSP; in particular, the quantity

$$L([\boldsymbol{\varphi}]\,|\,\rho)\equiv\lim_{t\to+\infty}\mathrm{Tr}\{\mathcal{G}(t,0;[\boldsymbol{\varphi}])\hat{\rho}\} \tag{2.28}$$

is the characteristic functional corresponding to a continuous measurement in the interval $(0,+\infty)$. Probabilities (2.12), which are the quantities containing all the physical information, are obtained from $L([\boldsymbol{\varphi}]\,|\,\rho)$ via Eq. (2.3); moments are directly given by Eqs. (2.4).

For a large class of continuous observations the characteristic operator is determined by the differential equation[13–15]

$$\frac{\partial}{\partial t}\mathcal{G}(t,t_0;[\boldsymbol{\varphi}])=\mathcal{K}(t;\boldsymbol{\varphi}(t))\mathcal{G}(t,t_0;[\boldsymbol{\varphi}]) \tag{2.29}$$

with the initial condition

$$\mathcal{G}(t_0,t_0;[\boldsymbol{\varphi}])=1 . \tag{2.30}$$

The generator $\mathcal{K}(t;\boldsymbol{\varphi}(t))$ is a linear operator in $T(\mathscr{h})$ and must be such that the probabilities constructed via Eqs. (2.28) and (2.3) are actually positive (and normalized). The structure of this generator determines the type of the measurement. Essentially, it can contain a "Poisson" and a "Gaussian" contribution;[21] the pure Poisson case corresponds to the counting processes studied by Davies and Srinivas,[9,25] while the Gaussian case has been studied in Refs. 13 and 15 and is suitable for observables with continuous spectra. As in the following we shall treat continuous measurements of position and momentum of an oscillator, here we consider only the Gaussian case. The generator for this case has been found in Refs. 13–15, starting from repeated imprecise measurements and then taking the limit to a continuous sequence of measurements in a suitable way. We have

$$\mathcal{K}(t;\boldsymbol{\varphi}(t))=\mathcal{L}(t)+\frac{i}{2}\sum_{j=1}^{n}\{\hat{C}_j,\cdot\}\varphi_j(t)+\tfrac{1}{2}\sum_{j=1}^{n}[\hat{M}_j,\cdot]\varphi_j(t)-\frac{\hbar^2}{4}\sum_{i,j=1}^{n}\varphi_i(t)\Gamma_{ij}\varphi_j(t) , \tag{2.31}$$

where $\mathcal{L}(t)$ is the generator of the dynamics introduced above [it must appear in $\mathcal{K}(t,\boldsymbol{\varphi})$ because of Eq. (2.27)] and

$$\hat{C}_j=\sum_{r=1}^{m}\hat{Q}_rC_{rj},\quad j=1,\ldots,n , \tag{2.32}$$

$$\hat{M}_j=\sum_{r=1}^{m}\hat{Q}_rM_{rj},\quad j=1,\ldots,n , \tag{2.33}$$

$$\Gamma_{ij}=\Gamma_{ji}\in\mathbf{R},\quad C_{rj}\in\mathbf{R},\quad M_{rj}\in\mathbf{R} , \tag{2.34}$$

$$\boldsymbol{\Gamma}\geq 0,\quad \det\boldsymbol{\Gamma}>0 , \tag{2.35}$$

$$\mathbf{D}+i\hbar\mathbf{E}-(\mathbf{C}+i\mathbf{M})\boldsymbol{\Gamma}^{-1}(\mathbf{C}^T-i\mathbf{M}^T)\geq\mathbf{0} . \tag{2.36}$$

Here we have introduced the matrices

$$\boldsymbol{\Gamma}=(\Gamma_{ij})_{n\times n} , \tag{2.37a}$$

$$\mathbf{C}=(C_{rj})_{m\times n},\quad \mathbf{M}=(M_{rj})_{m\times n} ; \tag{2.37b}$$

the superscript $T$ denotes the transpose.

As we shall see, the operators $\hat{C}_j$ are associated with the continuously measured quantities so that Eq. (2.32) says that these quantities must be linear combinations of the operators $\hat{Q}_r$ appearing in the dissipative part of $\mathcal{L}(t)$. Mathematically, there is no relation between $n$ (number of measured quantities) and $m$ (number of operators appearing in the dissipative part of the dynamics). However, in the physically interesting cases we expect to have $m\geq n$ ($m=n$ when dissipation is wholly due to the measurement). Another link between the measurement procedure and dissipation is given by Eq. (2.36), which is the mathematical condition that guarantees the positivity of the probabilities one obtains by Eqs. (2.28) and (2.3), for any initial state $\hat{\rho}$. If this relation is violated, we do not have true probabilities and, therefore, a meaningful theory.

Now, the characteristic operator is only a mathematical tool without a direct physical meaning; in order to clarify these considerations and to gain insight into the role of the various terms introduced above, we must consider physically meaningful quantities such as mean values and correlations. Moments can be easily calculated by using Eqs. (2.28) and (2.4) and the formal solution of Eq. (2.29),

$$\mathcal{G}(t_1,t_0;[\boldsymbol{\varphi}])=T\exp\left[\int_{t_0}^{t_1}dt\,\mathcal{K}(t;\boldsymbol{\varphi}(t))\right] ; \tag{2.38}$$

here $T$ denotes the time-ordered product.

For mean values we obtain

$$\langle z_j(t)\rangle=\mathrm{Tr}[\hat{C}_j\hat{\rho}(t)] , \tag{2.39}$$

where

$$\hat{\rho}(t)=\mathcal{U}(t,0)\hat{\rho}\ . \tag{2.40}$$

We can interpret Eq. (2.39) by saying that the measured observables are associated with the self-adjoint operators $\hat{C}_j$; this equation is the standard quantum formula for the mean value of $\hat{C}_j$ at time $t$ when the dynamics is given by $\mathcal{U}(t,0)$. However, recall that $\mathcal{U}(t,0)$ contains also a perturbation due to the measuring apparatus. Note that formally $\hat{C}_j$ can be any self-adjoint operator; however, it is not true that for any observable $\hat{C}_j$ the Gaussian generator (2.31) is a reasonable choice. For instance, for an observable such as the number of particles in a certain region a generator of the "Poisson" type must be used.

For two-time correlation functions we have

$$\begin{aligned}\langle[z_{j_1}(t_1)-\langle z_{j_1}(t_1)\rangle][z_{j_2}(t_2)-\langle z_{j_2}(t_2)\rangle]\rangle &= \frac{\hbar^2}{2}\Gamma_{j_1j_2}\delta(t_1-t_2)\\ &+\tfrac{1}{2}\theta(t_1-t_2)[\mathrm{Tr}([\hat{C}_{j_1}-\langle z_{j_1}(t_1)\rangle]\mathcal{U}(t_1,t_2)\{\hat{C}_{j_2}-\langle z_{j_2}(t_2)\rangle,\hat{\rho}(t_2)\})-i\,\mathrm{Tr}(\hat{C}_{j_1}\mathcal{U}(t_1,t_2)[\hat{M}_{j_2},\hat{\rho}(t_2)])]\\ &+\tfrac{1}{2}\theta(t_2-t_1)[\text{as above with } t_1\rightleftarrows t_2, j_1\rightleftarrows j_2]\ ,\end{aligned} \tag{2.41}$$

where

$$\theta(t)=\begin{cases}0 & \text{for } t<0\ ,\\ \frac{1}{2} & \text{for } t=0\ ,\\ 1 & \text{for } t>0\ .\end{cases} \tag{2.42}$$

The first term in Eq. (2.41) is a white-noise contribution [cf. Eq. (2.6)]; it can be interpreted as an internal noise of the measuring apparatus affecting the accuracy of the measurement. This contribution is essentially due to condition (2.14); we can say that this condition implies the internal noise of the measuring apparatus to be white. When this is not a reasonable approximation for the internal noise of the measuring apparatus, Eq. (2.14) cannot hold and some "memory" of the past history of the system must be introduced. In this case an alternative way to proceed is to retain Eq. (2.14), but to include in the "system" the part of the measuring apparatus responsible for the "non-Markovian" behavior.

The terms multiplying the step function also require some discussion. The first is a kind of "quantum symmetrized correlation function" sometimes introduced in the literature at a phenomenological level (see for instance Ref. 26) and contains all the internal fluctuations of the measured system. The second term gives a correction to these fluctuations due to the particular measuring procedure adopted. To change $\hat{M}_j$ does not change the dynamics or the observed quantities $\hat{C}_j$; what changes is the way in which the $\hat{C}_j$'s are measured. The operators $\hat{M}_j$ can be taken to vanish; however, the "best" measuring procedure is in some sense the one characterized by the equality sign in Eq. (2.36). In this case, if $\mathbf{E}\neq 0$, then $\mathbf{M}\neq 0$ and the operators $\hat{M}_j$ do not vanish.

Finally, let us consider the variances of the time smoothed variables (2.1):

$$\begin{aligned}(\Delta z_h)^2 &= \sum_{j_1,j_2=1}^{n}\int_0^{+\infty}dt_1dt_2h_{j_1}(t_1)\langle[z_{j_1}(t_1)-\langle z_{j_1}(t_1)\rangle][z_{j_2}(t_2)-\langle z_{j_2}(t_2)\rangle]\rangle h_{j_2}(t_2)\\ &= \frac{\hbar^2}{2}\int_0^{+\infty}dt\,\mathbf{h}(t)^T\mathbf{\Gamma}\mathbf{h}(t)\\ &+\sum_{j_1,j_2=1}^{n}\int_0^{+\infty}dt_1\int_0^{t_1}dt_2h_{j_1}(t_1)h_{j_2}(t_2)[\,\mathrm{Tr}([\hat{C}_{j_1}-\langle z_{j_1}(t_1)\rangle]\mathcal{U}(t_1,t_2)\{\hat{C}_{j_2}-\langle z_{j_2}(t_2)\rangle,\hat{\rho}(t_2)\}\\ &\qquad -i\,\mathrm{Tr}(\hat{C}_{j_1}\mathcal{U}(t_1,t_2)[\hat{M}_{j_2},\hat{\rho}(t_2)])]\ .\end{aligned} \tag{2.43}$$

If we wanted to have the variances of the nonsmoothed variables, we would let the function $\mathbf{h}(t)$ go to a $\delta$ function, but in this limit the first term goes to infinity. The stochastic process is a true generalized one, and only the time smoothed variables have finite variances. If we consider the time averages

$$\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,z_i(t)\ ,$$

we must take

$$h_j(t)=\delta_{ji}\chi_{(\bar{t},\bar{t}+\tau)}(t)/\tau\ ,$$

where $\chi_{(a,b)}(t)$ is the characteristic function of the interval $(a,b)$. Then, for small $\tau$, we obtain

$$\left[\Delta\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,z_i(t)\right]^2 \underset{\tau\to 0}{\simeq} \frac{\hbar^2}{2\tau}\Gamma_{ii}+\mathrm{Tr}\{[\hat{C}_i-\langle z_i(\bar{t})\rangle]^2\rho(\bar{t})\} -\frac{i}{2}\mathrm{Tr}\{[\hat{C}_i,\hat{M}_i]\hat{\rho}(\bar{t})\}\ . \tag{2.44}$$

The first term is due to the white noise, the second is the usual quantum-mechanical variance, and the third is a further contribution (not necessarily positive) due to the measuring procedure, as seen above. Note that for very small $\tau$ the first term in Eq. (2.44) diverges; this means that we always need a finite time of measurement in order to collect sufficient information about the measured quantity.

About the first term in Eq. (2.43), note also that if we have no dissipation ($\mathbf{D}=\mathbf{0},\mathbf{E}=\mathbf{0}$), by Eq. (2.36) this term grows to infinity and we have no sensible measurement. In other words we can say that the coupling to the measuring apparatus gives a dissipative contribution to the dynamics: the more precise the continuous measurement, the more dissipative the dynamics.

## III. THE MODEL FOR THE GRAVITATIONAL-WAVE DETECTOR

A gravitational-wave detector can be schematized as a harmonic oscillator (for instance, the fundamental acoustic mode of a massive bar) interacting with the gravitational field. The dynamics of such systems, when dissipation is included, was studied in a coherent quantum-mechanical way by Lindblad,[27] and in Ref. 14 I applied to these linear models the theory of continuous measurements. A preliminary analysis of continuous measurements (including measurement of QND observables) on a gravitational-wave detector was given in Ref. 18.

As stressed in Ref. 2, in order to have a more complete view of the problem, it is useful to include in the quantum analysis also the first stage of the measuring apparatus (the meter[2,4]). Our aim is to apply the formalism of continuous observations to some observables of the meter and to deduce from this measurement information about observables of the detector and ultimately about the gravitational force.

The meter is essentially an electrical circuit. This is a very complex system from a quantum point of view. However, in some physically interesting cases (see, for instance, Ref. 2), it can be schematized as a generic quantum one-dimensional linear system. Even in more complicated cases, this schematization can give a qualitative idea of the effects induced by the meter into the results of the measurement.

Hence, we have two coupled linear systems: system 1 is the detector, system 2 is the meter. The generators of the dynamics for these two systems, when no coupling is present, are given by[27,14,18]

$$\mathscr{L}^{(k)}=-\frac{i}{\hbar}[\hat{H}^{(k)},\cdot]-\frac{1}{4\hbar^2}D_{ij}^{(k)}[\hat{Q}_i^{(k)},[\hat{Q}_j^{(k)},\cdot]] -\frac{i\gamma_k}{4\hbar}\epsilon_{ij}[\hat{Q}_i^{(k)},\{\hat{Q}_j^{(k)},\cdot\}],\quad k=1,2\ , \tag{3.1}$$

where

$$\hat{Q}_1^{(k)}=\hat{x}^{(k)},\quad \hat{Q}_2^{(k)}=\hat{p}^{(k)} \tag{3.2}$$

($\hat{x}^{(k)}$ and $\hat{p}^{(k)}$ are the canonical position and momentum operators for the two systems),

$$\hat{H}^{(k)}=\tfrac{1}{2}\hat{Q}_i^{(k)}B_{ij}^{(k)}\hat{Q}_j^{(k)}\ , \tag{3.3}$$

$$B_{ij}^{(k)}=B_{ji}^{(k)}\in\mathbf{R},\quad D_{ij}^{(k)}=D_{ji}^{(k)}\in\mathbf{R},\quad \gamma_k\in\mathbf{R}\ , \tag{3.4}$$

$$\epsilon=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{3.5}$$

$$\mathbf{D}^{(k)}+i\hbar\gamma_k\epsilon\geq\mathbf{0}\ . \tag{3.6}$$

In these equations summation (from one to two) over repeated lower indices is understood (elsewhere we shall use also matrix notation). No summation convention is applied to the upper indices, which distinguish the detector and the meter.

The dynamics of a system $k$ generated by an operator $\mathscr{L}^{(k)}$ of the form (3.1) is extensively studied in Ref. 27. Let us recall some results. The first term in Eq. (3.1) (Hamiltonian part) gives the main features of the motion of the system. We set

$$\omega_k=(\det\mathbf{B}^{(k)})^{1/2}\ . \tag{3.7}$$

Then, if $\det\mathbf{B}^{(k)}>0$, $\omega_k$ is a positive real number and, in this case, the system $k$ has an oscillatory behavior with angular frequency $\omega_k$. A simple choice for the matrices $\mathbf{B}^{(k)}$ is, for instance,

$$\mathbf{B}^{(k)}=\begin{bmatrix}m_k\omega_k^2 & 0\\ 0 & 1/m_k\end{bmatrix},\quad k=1,2\ . \tag{3.8}$$

The second term in Eq. (3.1) gives diffusion (both for position and momentum); the matrix $\mathbf{D}^{(k)}$ has the same role as the diffusion matrix in a classical Fokker-Planck equation. This does not mean that the diffusion here has a classical origin; it can be due to purely quantum phenomena. The third term gives damping (in the physical case $\gamma_k\geq 0$). Condition (3.6), which is equivalent to

$$D_{11}^{(k)}\geq 0,\quad D_{22}^{(k)}\geq 0,\quad \det\mathbf{D}^{(k)}\geq\hbar^2\gamma_k^{\,2}\ , \tag{3.9}$$

is needed to ensure that the dynamics transforms statistical operators into statistical operators. It is a mathematical formulation of the well-known fact that in QM we cannot have damping without diffusion.

Now we introduce the interaction of system 1 with the gravitational wave (treated as a classical external force). This interaction is described by the term

$$\mathscr{L}_{\rm gw}(t)=\frac{i}{\hbar}f_i(t)[\hat{Q}_i^{(1)},\cdot]\ , \tag{3.10a}$$

where

$$\mathbf{f}(t)=f(t)\mathbf{u}\ ; \tag{3.10b}$$

$f(t)$ is the unknown gravitational force and $\mathbf{u}$ describes the coupling to canonical position and momentum. If the gravitational wave couples only to position (as it does when the detector is a mechanical oscillator), we have $u_1=1$, $u_2=0$.

Let us introduce now a generic quadratic, possibly time-dependent interaction between the two systems

$$\mathscr{L}_I(t)=-\frac{i}{\hbar}\lambda[\hat{Q}_i^{(1)}A_{ij}(t)\hat{Q}_j^{(2)},\cdot]\ , \tag{3.11}$$

where $\mathbf{A}(t)$ is a real $2\times 2$ matrix.

Interesting examples of interactions are given by the following choices for the interaction matrix $\mathbf{A}(t)$:

$$\mathbf{A}(t)=\mathbf{A}\ , \tag{3.12}$$

$$\mathbf{A}(t)=\exp(\mathbf{B}^{(1)}\boldsymbol{\epsilon}t)\mathbf{A}\ , \tag{3.13}$$

$$\mathbf{A}(t)=g(t)\mathbf{A}\ . \tag{3.14}$$

These interactions have been introduced in the literature in connection with gravitational-wave detection; the structure of the matrix $\mathbf{A}(t)$ determines the essential features of the measurement. Matrix (3.12) describes a time-independent coupling; this is the standard way of operating existing gravitational-wave detectors. The measurement scheme with interaction (3.12) is called "amplitude and phase" measurement.[2,6] Matrix (3.13), when $\det\mathbf{A}=0$, describes the so-called "back action evading" coupling.[2] This name comes from the fact that the behavior of the observable of system 1 entering the coupling is not affected by the action of system 2. This coupling, in principle, allows a precise estimation of the force; however, it requires unrealizable (today) components. The model of Eq. (3.16) of Ref. 2 is of this kind; the matrices $\mathbf{B}^{(k)}$ are given by Eq. (3.8) with $\omega_2=0$ and the interaction matrix by Eq. (3.13) with

$$\mathbf{A}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}. \tag{3.15}$$

The interaction (3.14), with a suitable $g(t)$, can give better results than amplitude and phase measurements; moreover, it is experimentally realizable and a prototype has been constructed.[7] For the model proposed in Appendix D of Ref. 2, $\mathbf{A}(t)$ is given by Eq. (3.14) with

$$g(t)=\cos\omega_1 t\ ; \tag{3.16}$$

$\mathbf{B}^{(1)},\mathbf{B}^{(2)}$ are given by Eq. (3.8) with $\omega_2=0$ and $\mathbf{A}$ by Eq. (3.15). For the model of Eq. (3) of Ref. 3 $\mathbf{A}(t)$ is given by Eq. (3.14) with $\mathbf{A}$ given by Eq. (3.15) and

$$g(t)=\cos\omega_1 t\cos\omega_2 t\ , \tag{3.17}$$

the matrices $\mathbf{B}^{(k)}$ are given by Eq. (3.8). Also the models proposed in Ref. 4 could be translated into our notation with some more complicated interaction matrix $\mathbf{A}(t)$.

Dissipation is considered in Ref. 3, where an external source of white noise is added to the equations of motion, and in Ref. 4, where the same treatment as ours is used with

$$\mathbf{D}^{(i)}=\hbar\frac{\gamma_i}{\omega_i}\mathbf{B}^{(i)}\ . \tag{3.18}$$

Summing up, the generator for the dynamics of the two coupled systems (detector plus meter) is

$$\mathscr{L}(t)=\mathscr{L}^{(1)}+\mathscr{L}^{(2)}+\mathscr{L}_{\rm gw}(t)+\mathscr{L}_I(t)\ , \tag{3.19}$$

where the various terms are given by Eqs. (3.1)–(3.6), (3.10), and (3.11).

Now we have to introduce the continuous observation. Here we assume that the measured observables are the position and momentum of system 2 (to choose some linear combination of them gives no more generality). From this measurement we want to deduce information about the behavior of system 1 and ultimately about the gravitational force. From the results of the previous section we have that the generator for the characteristic operator is given by

$$\mathscr{K}(t;\boldsymbol{\varphi}(t))=\mathscr{L}(t)+\frac{i}{2}\{\hat{Q}_j^{(2)},\cdot\}\varphi_j(t)+\tfrac{1}{2}[\hat{Q}_i^{(2)},\cdot]M_{ij}\varphi_j(t) -\frac{\hbar^2}{4}\varphi_i(t)\Gamma_{ij}\varphi_j(t)\ . \tag{3.20}$$

The $2\times2$ matrix $\boldsymbol{\Gamma}$ is real, symmetric, and invertible and $\mathbf{M}$ is a real $2\times2$ matrix. Moreover, condition (2.36) now becomes

$$\mathbf{D}^{(2)}+i\hbar\gamma_2\boldsymbol{\epsilon}-(1+i\mathbf{M})\boldsymbol{\Gamma}^{-1}(1-i\mathbf{M}^T)\geq\mathbf{0}\ . \tag{3.21}$$

The matrix $\mathbf{C}$ appearing in Eqs. (2.32) and (2.36) here is taken to be the identity, so that the measured quantities are the $\hat{Q}_i^{(2)}$'s as wanted [cf. Eq. (2.39) and the following discussion].

In Appendix A we show how to calculate the characteristic functional (2.28) (which is sufficient for reconstructing probabilities) for the model we have introduced. Here we give only the final results. The characteristic functional can be written as

$$L([\boldsymbol{\varphi}]\,|\,\rho)=e^{\beta}\mathrm{Tr}\{\hat{W}(\boldsymbol{\alpha}(0))\hat{\rho}\}\ , \tag{3.22}$$

where $\hat{\rho}$ is the initial state at time $t=0$, $\hat{W}(\boldsymbol{\alpha})$ is the Weyl operator

$$\hat{W}(\boldsymbol{\alpha})=\exp(i\alpha_j^{(1)}\hat{Q}_j^{(1)}+i\alpha_j^{(2)}\hat{Q}_j^{(2)})\ , \tag{3.23}$$

the quantity $\beta$ is given by

$$\beta=\int_0^{+\infty}dt\left[i\mathbf{f}(t)^T\boldsymbol{\epsilon}\boldsymbol{\alpha}^{(1)}(t)-\tfrac{1}{4}\sum_{k=1}^{2}\boldsymbol{\alpha}^{(k)}(t)^T\boldsymbol{\epsilon}\mathbf{D}^{(k)}\boldsymbol{\epsilon}^T\boldsymbol{\alpha}^{(k)}(t)-\frac{\hbar}{2}\boldsymbol{\alpha}^{(2)}(t)^T\boldsymbol{\epsilon}\mathbf{M}\boldsymbol{\varphi}(t)-\frac{\hbar^2}{4}\boldsymbol{\varphi}(t)^T\boldsymbol{\Gamma}\boldsymbol{\varphi}(t)\right], \tag{3.24}$$

and the functions $\boldsymbol{\alpha}^{(k)}(t)$ are the solutions of the differential equations

$$\frac{d}{dt}\boldsymbol{\alpha}^{(1)}(t)=\left[\frac{\gamma_1}{2}+\mathbf{B}^{(1)}\boldsymbol{\epsilon}\right]\boldsymbol{\alpha}^{(1)}(t)+\lambda\mathbf{A}(t)\boldsymbol{\epsilon}\boldsymbol{\alpha}^{(2)}(t)\ , \tag{3.25}$$

$$\frac{d}{dt}\boldsymbol{\alpha}^{(2)}(t)=\left[\frac{\gamma_2}{2}+\mathbf{B}^{(2)}\boldsymbol{\epsilon}\right]\boldsymbol{\alpha}^{(2)}(t)+\lambda\mathbf{A}(t)^T\boldsymbol{\epsilon}\boldsymbol{\alpha}^{(1)}(t)-\boldsymbol{\varphi}(t) \tag{3.26}$$

with the final conditions

$$\boldsymbol{\alpha}^{(k)}(+\infty)=\mathbf{0}\ . \tag{3.27}$$

In the Weyl operator in Eq. (3.22) the functions $\boldsymbol{\alpha}^{(k)}(t)$ are evaluated at $t=0$.

The trace appearing in Eq. (3.22) can be explicitly calculated, for instance, in the case of "Gaussian" states.[10,14] Coherent states and squeezed states are in this class.

## IV. MEAN VALUES AND CORRELATIONS

In this section we want to see what kind of physical picture emerges from the continuous-observation formalism. Here we are interested primarily in testing the theory we have developed and not in problems connected with gravitational-wave detection. For this purpose we do not need all aspects of the general model outlined in Sec. III; therefore, we specialize to the situation in which no explicitly time-dependent term is present, and we take

$$\mathbf{f}(t)=\mathbf{0}, \quad \mathbf{A}(t)=\mathbf{A} \ . \tag{4.1}$$

In order to compare our results with the usual formulation of QM, we introduce the quantities

$$Q_i^{(k)}(t)\equiv \mathrm{Tr}[\hat{Q}_i^{(k)}\hat{\rho}(t)] \ , \tag{4.2}$$

$$\sigma_{ij}^{(ks)}(t)\equiv \tfrac{1}{2}\,\mathrm{Tr}[\{\hat{Q}_i^{(k)}-Q_i^{(k)}(t),\hat{Q}_j^{(s)}-Q_j^{(s)}(t)\}\hat{\rho}(t)] \ . \tag{4.3}$$

Recalling the definitions of the previous section, i.e.,

$$\hat{\rho}(t)=\mathcal{U}(t,0)\hat{\rho}=\exp[(\mathcal{L}^{(1)}+\mathcal{L}^{(2)}+\mathcal{L}_I)t]\hat{\rho} \ , \tag{4.4}$$

$$\hat{Q}_1^{(k)}=\hat{x}^{(k)}, \quad \hat{Q}_2^{(k)}=\hat{p}^{(k)} \ , \tag{4.5}$$

we see that Eq. (4.2) gives mean values for position and momentum of the two systems, while Eq. (4.3) defines what can be called a covariance matrix. Note that $\sigma_{11}^{(22)}(t)$, for instance, gives the usual variance for an instantaneous measurement of $\hat{x}^{(2)}$ and $\sigma_{22}^{(22)}(t)$ the usual variance for $\hat{p}^{(2)}$. Obviously, the matrices (4.3) satisfy the Heisenberg principle

$$\det\boldsymbol{\sigma}^{(kk)}(t)\geq \hbar^2/4 \ . \tag{4.6}$$

However, these variances refer to distinct measurements of position and momentum, not to a joint measurement; joint measurements of noncommuting observables can be introduced in QM only by using the general framework from which we developed the continuous-observation formalism. A full treatment of joint instantaneous measurements of position and momentum is given in Ref. 10.

The quantities (4.2) and (4.3) can be explicitly calculated (see Appendix B). The mean values (4.2) satisfy the differential equations

$$\frac{d}{dt}\mathbf{Q}^{(1)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]\mathbf{Q}^{(1)}(t)+\lambda\boldsymbol{\epsilon}\mathbf{A}\mathbf{Q}^{(2)}(t) \ , \tag{4.7a}$$

$$\frac{d}{dt}\mathbf{Q}^{(2)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{Q}^{(2)}(t)+\lambda\boldsymbol{\epsilon}\mathbf{A}^T\mathbf{Q}^{(1)}(t) \ . \tag{4.7b}$$

We introduce the $4\times 4$ matrices

$$\mathbf{G}=\begin{bmatrix}\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\dfrac{\gamma_1}{2} & \lambda\boldsymbol{\epsilon}\mathbf{A}\\ \lambda\boldsymbol{\epsilon}\mathbf{A}^T & \boldsymbol{\epsilon}\mathbf{B}^{(2)}-\dfrac{\gamma_2}{2}\end{bmatrix} , \tag{4.8a}$$

$$\mathbf{R}(t)=\exp(\mathbf{G}t) \ ; \tag{4.8b}$$

then, using the $2\times 2$ blocks $\mathbf{R}^{(ks)}(t)$ of $\mathbf{R}(t)$, the solution of Eqs. (4.7) can be written as

$$\mathbf{Q}^{(k)}(t)=\sum_{s=1}^{2}\mathbf{R}^{(ks)}(t)\mathbf{Q}^{(s)}(t) \ . \tag{4.9}$$

With the same notation the covariance matrix (4.3) becomes

$$\boldsymbol{\sigma}^{(ks)}(t)=\sum_{r,l=1}^{2}\mathbf{R}^{(kr)}(t)\boldsymbol{\sigma}^{(rl)}(0)\mathbf{R}^{(sl)}(t)^T + \tfrac{1}{2}\int_0^t dt'\sum_{r=1}^{2}\mathbf{R}^{(kr)}(t-t')\boldsymbol{\epsilon}\mathbf{D}^{(r)}\boldsymbol{\epsilon}^T\mathbf{R}^{(sr)}(t-t')^T \ . \tag{4.10}$$

When all the eigenvalues of the matrix (4.8a) have a negative real part, the system exhibits approach to equilibrium: for any initial state

$$\lim_{t\to+\infty}\hat{\rho}(t)=\hat{\rho}_{\mathrm{eq}} \ , \tag{4.11}$$

where $\hat{\rho}_{\mathrm{eq}}$ is the Gaussian state defined by mean values

$$\mathbf{Q}_{\mathrm{eq}}^{(k)}=\lim_{t\to+\infty}\mathbf{Q}^{(k)}(t)=\mathbf{0} \tag{4.12}$$

and covariance matrix

$$\boldsymbol{\sigma}_{\mathrm{eq}}^{(ks)}=\lim_{t\to+\infty}\boldsymbol{\sigma}^{(ks)}(t) =\sum_{r=1}^{2}\tfrac{1}{2}\int_0^{+\infty}dt\,\mathbf{R}^{(kr)}(t)\boldsymbol{\epsilon}\mathbf{D}^{(r)}\boldsymbol{\epsilon}^T\mathbf{R}^{(sr)}(t)^T \ . \tag{4.13}$$

The matrices $\boldsymbol{\sigma}_{\mathrm{eq}}^{(ks)}$ are equivalently given by the equation

$$\mathbf{G}\boldsymbol{\sigma}_{\mathrm{eq}}+\boldsymbol{\sigma}_{\mathrm{eq}}\mathbf{G}^T=-\tfrac{1}{2}\begin{bmatrix}\boldsymbol{\epsilon}\mathbf{D}^{(1)}\boldsymbol{\epsilon}^T & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\epsilon}\mathbf{D}^{(2)}\boldsymbol{\epsilon}^T\end{bmatrix} , \tag{4.14a}$$

where $\mathbf{G}$ is given by Eq. (4.8a) and

$$\boldsymbol{\sigma}_{\mathrm{eq}}=\begin{bmatrix}\boldsymbol{\sigma}_{\mathrm{eq}}^{(11)} & \boldsymbol{\sigma}_{\mathrm{eq}}^{(12)}\\ \boldsymbol{\sigma}_{\mathrm{eq}}^{(21)} & \boldsymbol{\sigma}_{\mathrm{eq}}^{(22)}\end{bmatrix} . \tag{4.14b}$$

Consider now the continuous observation, first in the simple case of a single oscillator. This situation is described by the characteristic functional (3.22)–(3.27) with $\lambda=0$ (no interaction between the two systems). The continuously measured observables are the position and momentum of system 2:

$$z_1(t)=x^{(2)}(t), \quad z_2(t)=p^{(2)}(t) \ . \tag{4.15}$$

The solution of Eqs. (3.25) and (3.26) (for $\lambda=0$) is given by [cf. Eqs. (B13)]

$$\boldsymbol{\alpha}^{(2)}(t)=\int_t^{+\infty}dt'\mathbf{R}^{(22)}(t'-t)^T\boldsymbol{\varphi}(t'), \quad \boldsymbol{\alpha}^{(1)}(t)=\mathbf{0} \ , \tag{4.16}$$

with

$$\mathbf{R}^{(22)}(t)=\exp\left[t\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\right] =e^{-\gamma_2 t/2}\left[\cos\omega_2 t+\boldsymbol{\epsilon}\mathbf{B}^{(2)}\frac{\sin\omega_2 t}{\omega_2}\right] , \tag{4.17a}$$

$$\omega_2{}^2=\det\mathbf{B}^{(2)} \ . \tag{4.17b}$$

Then, by Eqs. (2.4) we obtain

$$\langle \mathbf{z}(t)\rangle=\mathbf{Q}^{(2)}(t)=\mathbf{R}^{(22)}(t)\mathbf{Q}^{(2)}(0)\,, \tag{4.18a}$$

$$\langle[\mathbf{z}(t_1)-\langle\mathbf{z}(t_1)\rangle][\mathbf{z}(t_2)-\langle\mathbf{z}(t_2)\rangle]^T\rangle=\frac{\hbar^2}{2}\mathbf{\Gamma}\delta(t_1-t_2)+\theta(t_1-t_2)\mathbf{R}^{(22)}(t_1-t_2)\left[\boldsymbol{\sigma}^{(22)}(t_2)+\frac{\hbar}{2}\boldsymbol{\epsilon}\mathbf{M}\right]$$
$$+\theta(t_2-t_1)\left[\boldsymbol{\sigma}^{(22)}(t_1)+\frac{\hbar}{2}\mathbf{M}^T\boldsymbol{\epsilon}^T\right]\mathbf{R}^{(22)}(t_2-t_1)^T\,, \tag{4.18b}$$

where now we have

$$\boldsymbol{\sigma}^{(22)}(t)=\mathbf{R}^{(22)}(t)\boldsymbol{\sigma}^{(22)}(0)\mathbf{R}^{(22)}(t)^T+\tfrac{1}{2}\int_0^t dt'\mathbf{R}^{(22)}(t')\boldsymbol{\epsilon}\mathbf{D}^{(2)}\boldsymbol{\epsilon}^T\mathbf{R}^{(22)}(t')^T\,. \tag{4.19}$$

The mean values (4.18a) coincide with the quantities $\mathbf{Q}^{(2)}(t)$ [Eqs. (4.2) and (4.9) for $\lambda=0$] as stated by the general result (2.39). In the correlations (4.18b) we can identify the following contributions:

(1) the term containing the $\delta$ function, which gives the unavoidable internal white noise of the measuring apparatus;

(2) the terms containing the matrix $\boldsymbol{\sigma}^{(22)}(t)$, which give the usual internal fluctuations of the system;

(3) the terms containing the matrix $\mathbf{M}$, which can be interpreted as a correction to the covariance matrix due to the way the measurement is performed.

The variances (2.43) for the time smoothed variables become now

$$(\Delta z_h)^2=\frac{\hbar^2}{2}\int_0^{+\infty}dt\,\mathbf{h}(t)^T\mathbf{\Gamma}\mathbf{h}(t)+2\int_0^{+\infty}dt_1\int_0^{t_1}dt_2\mathbf{h}(t_1)^T\mathbf{R}^{(22)}(t_1-t_2)\left[\boldsymbol{\sigma}^{(22)}(t_2)+\frac{\hbar}{2}\boldsymbol{\epsilon}\mathbf{M}\right]\mathbf{h}(t_2)\,. \tag{4.20}$$

To gain insight into the meaning of this equation, consider the time average of position in the interval $(\bar{t},\bar{t}+\tau)$ for the case of a free particle, i.e., choose

$$h_1(t)=\chi_{(\bar{t},\bar{t}+\tau)}(t)/\tau,\quad h_2(t)=0\,, \tag{4.21}$$

$$B_{11}^{(2)}=B_{12}^{(2)}=0,\quad B_{22}^{(2)}=1/m,\quad \gamma_2=0\,. \tag{4.22}$$

With this choice Eq. (4.20) gives

$$\left[\Delta\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,x^{(2)}(t)\right]^2=\Delta_1{}^2(\bar{t})+\Delta_2{}^2\,, \tag{4.23a}$$

where

$$\Delta_1{}^2(\bar{t})=\sigma_{11}^{(22)}(\bar{t})+\frac{\tau}{m}\sigma_{12}^{(22)}(\bar{t})+\frac{\tau^2}{4m^2}\sigma_{22}^{(22)}(\bar{t})\,, \tag{4.23b}$$

$$\Delta_2{}^2=\frac{\hbar^2}{2\tau}\Gamma_{11}+\frac{\hbar}{2}M_{21}-\frac{\tau\hbar}{6m}M_{11}+\frac{\tau}{6}D_{22}-\frac{\tau^2}{8m}D_{12}+\frac{\tau^3}{10m^2}D_{11}\,. \tag{4.23c}$$

We have written the variance (4.23a) as a sum of two terms: the first is due to the usual covariance matrix $\boldsymbol{\sigma}^{(22)}$ at time $\bar{t}$, the second is due to diffusion and to the characteristics of the measuring apparatus. Once the averaging interval $(\bar{t},\bar{t}+\tau)$ has been fixed, one can ask what is the best measuring procedure. It is possible to minimize the variance (4.23a) under the constraints (4.6) and (3.21); for $\bar{t}=0$ the minimum is reached for

$$\Gamma_{11}=\frac{6}{\sqrt{155}}\frac{\tau^2}{m\hbar},\quad \Gamma_{12}=0,\quad 1/\Gamma_{22}=0\,, \tag{4.24a}$$

$$M_{11}=(\tfrac{5}{31})^{1/2},\quad M_{12}=M_{21}=0,\quad M_{22}=0\,, \tag{4.24b}$$

$$D_{11}=6\left[\frac{5}{31}\right]^{1/2}\frac{m\hbar}{\tau^2},\quad D_{12}=0,\quad D_{22}=0\,, \tag{4.24c}$$

$$\sigma_{11}^{(22)}(0)=\frac{\tau\hbar}{4m},\quad \sigma_{12}^{(22)}(0)=0,\quad \sigma_{22}^{(22)}(0)=\frac{\hbar m}{\tau}\,. \tag{4.24d}$$

In this case we obtain

$$\Delta_1{}^2(0)=\frac{\tau\hbar}{2m},\quad \Delta_2{}^2=\frac{1}{3}\left[\frac{31}{5}\right]^{1/2}\frac{\hbar\tau}{2m}\simeq 0.83\frac{\hbar\tau}{2m}\,. \tag{4.25}$$

Therefore, the two contributions are of the same order of magnitude and the variance can be made as small as one wants by taking $\tau$ sufficiently small. However, if for a small $\tau$ we make the choice (4.24) for the measuring procedure and the initial conditions, then $\Delta_1{}^2(\bar{t})$ grows rapidly in time; in fact, from Eq. (4.19) we have

$$\sigma_{11}^{(22)}(t)=\sigma_{11}^{(22)}(0)+\sigma_{12}^{(22)}(0)t+\sigma_{22}^{(22)}(0)\frac{t^2}{m^2}+D_{11}^{(2)}\frac{t^3}{6m^2}\,, \tag{4.26a}$$

$$\sigma_{12}^{(22)}(t)=\sigma_{12}^{(22)}(0)+\sigma_{22}^{(22)}(0)\frac{t}{m}+D_{11}^{(2)}\frac{t^2}{4m}\,, \tag{4.26b}$$

$$\sigma_{22}^{(22)}(t)=\sigma_{22}^{(22)}(0)+\tfrac{1}{2}D_{11}^{(2)}t\,, \tag{4.26c}$$

and from Eqs. (4.24c), (4.24d), and (4.23b) we obtain

$$\Delta_1{}^2(\bar{t})=\frac{\tau\hbar}{m}\left\{\tfrac{1}{2}+(1+\tfrac{3}{4}\sqrt{\tfrac{5}{31}})\frac{t}{\tau}+\left[1+\frac{3}{2}\left[\frac{5}{31}\right]^{1/2}\right]\frac{t^2}{\tau^2}+\left[\frac{5}{31}\right]^{1/2}\frac{t^3}{\tau^3}\right\}. \tag{4.27}$$

Another interesting example is that of a harmonic oscillator decaying to its ground state; it is described by the master equation

$$\frac{d}{dt}\hat{\rho}(t)=-i\omega_2[\hat{a}^\dagger\hat{a},\hat{\rho}(t)]+\gamma_2\hat{a}\hat{\rho}(t)\hat{a}^\dagger-\frac{\gamma_2}{2}\{\hat{a}^\dagger\hat{a},\hat{\rho}(t)\}\,, \tag{4.28a}$$

$$\omega_2>0,\quad \gamma_2>0\,. \tag{4.28b}$$

If we introduce position and momentum by the equations

$$\hat{x}^{(2)}=\left[\frac{\hbar}{2m_2\omega_2}\right]^{1/2}(\hat{a}^\dagger+\hat{a}) ,$$
$$\hat{p}^{(2)}=i\left[\frac{\hbar m_2\omega_2}{2}\right]^{1/2}(\hat{a}^\dagger-\hat{a}),\quad m_2>0 , \tag{4.29}$$

we can write the generator of the evolution operator in the form (3.1)—(3.6), with

$$\mathbf{B}^{(2)}=\begin{bmatrix} m_2\omega_2^2 & 0 \\ 0 & 1/m_2 \end{bmatrix},\quad \mathbf{D}^{(2)}=\frac{\hbar\gamma_2}{\omega_2}\mathbf{B}^{(2)} . \tag{4.30}$$

As $\det(\mathbf{D}^{(2)}+i\hbar\gamma_2\epsilon)=0$, Eq. (3.21) cannot be satisfied: this means that we cannot have a joint measurement of position and momentum when, under measurement, the dynamics is given by Eq. (4.28a). Vice versa, we can say that the interaction with the measuring apparatus for a joint measurement of $x^{(2)}(t)$ and $p^{(2)}(t)$ modifies the dynamics, and we cannot have decay to a pure state but only to a mixed state.

However, if we go back to the general formalism of Sec. II, we see that we can introduce continuous measurement of $x^{(2)}(t)$ alone. Using the results we have obtained up to now, we modify the diffusion matrix by taking

$$\mathbf{D}^{(2)}=\begin{bmatrix} \hbar\gamma_2 m_2\omega_2^2 & 0 \\ 0 & \dfrac{\hbar\gamma_2}{m_2\omega_2}+\epsilon \end{bmatrix} \tag{4.31}$$

and choose

$$\boldsymbol{\Gamma}=\begin{bmatrix} 1/\hbar\gamma_2 m_2\omega_2 & 0 \\ 0 & 1/\epsilon \end{bmatrix} , \tag{4.32}$$

$$\mathbf{M}=\begin{bmatrix} 0 & 0 \\ -1/m_2\omega_2 & 0 \end{bmatrix} . \tag{4.33}$$

Then, to recover the dynamics (4.28a), we take into account only the results for the measurement of $x^{(2)}(t)$ and let $\epsilon$ go to zero. Note that the matrices $\mathbf{D}^{(2)}$, $\boldsymbol{\Gamma}$, and $\mathbf{M}$ satisfy Eq. (3.21) with the equality sign; this means that the noise is the minimum compatible with the given dynamics.

For mean values and correlations we obtain

$$\langle x^{(2)}(t)\rangle=e^{-\gamma_2 t/2}\left[\cos\omega_2 t\; Q_1^{(2)}(0)+\frac{\sin\omega_2 t}{m_2\omega_2}Q_2^{(2)}(0)\right] , \tag{4.34}$$

$$\begin{aligned}\langle[x^{(2)}(t_1)-\langle x^{(2)}(t_1)\rangle][x^{(2)}(t_2)-\langle x^{(2)}(t_2)\rangle]\rangle \\ =\frac{\hbar}{2m_2\omega_2\gamma_2}\delta(t_1-t_2) \\ +\theta(t_1-t_2)\exp\left[-\frac{\gamma_2}{2}(t_1-t_2)\right]\left\{\cos\omega_2(t_1-t_2)\left[\sigma_{11}^{(22)}(t_2)-\frac{\hbar}{2m_2\omega_2}\right]+\frac{1}{m_2\omega_2}\sin\omega_2(t_1-t_2)\,\sigma_{12}^{(22)}(t_2)\right\} \\ +\theta(t_2-t_1)\exp\left[-\frac{\gamma_2}{2}(t_2-t_1)\right](\text{as above with } t_1\rightleftarrows t_2) ,\end{aligned} \tag{4.35}$$

where

$$\sigma_{11}^{(22)}(t)=e^{-\gamma_2 t}\left[\sigma_{11}^{(22)}(0)\cos^2\omega_2 t+\sigma_{22}^{(22)}(0)\frac{\sin^2\omega_2 t}{m_2^2\omega_2^2}+\sigma_{12}^{(22)}(0)\frac{\sin\omega_2 t}{m_2\omega_2}\cos\omega_2 t\right]+\frac{\hbar}{2m_2\omega_2}(1-e^{-\gamma_2 t}) , \tag{4.36a}$$

$$\sigma_{12}^{(22)}(t)=e^{-\gamma_2 t}\left\{\left[\frac{\sigma_{22}^{(22)}(0)}{m_2\omega_2}-m_2\omega_2\sigma_{11}^{(22)}(0)\right]\sin\omega_2 t\cos\omega_2 t+\sigma_{12}^{(22)}(0)(\cos^2\omega_2 t-\sin^2\omega_2 t)\right\} . \tag{4.36b}$$

At equilibrium we have

$$\sigma_{11}^{(\mathrm{eq})}=\lim_{t\to+\infty}\sigma_{11}^{(22)}(t)=\frac{\hbar}{2m_2\omega_2} ,$$
$$\sigma_{12}^{(\mathrm{eq})}=\lim_{t\to+\infty}\sigma_{12}^{(22)}(t)=0 , \tag{4.37}$$

and Eqs. (4.34) and (4.35) become

$$\langle x^{(2)}(t)\rangle_{\mathrm{eq}}=0 , \tag{4.38}$$

$$\langle x^{(2)}(t_1)x^{(2)}(t_2)\rangle_{\mathrm{eq}}=\frac{\hbar}{2m_2\omega_2\gamma_2}\delta(t_1-t_2)$$
$$=\sigma_{11}^{(\mathrm{eq})}\frac{1}{\gamma_2}\delta(t_1-t_2) . \tag{4.39}$$

Note that the contribution $\frac{1}{2}\hbar M_{21}=-\hbar/2m_2\omega_2$ cancels exactly the contribution of $\sigma_{11}^{(\mathrm{eq})}$ in the terms containing the step function so that only the term containing the $\delta$ function survives.

If we consider the time average of $x^{(2)}(t)$, we obtain

$$\left[\Delta^{(\mathrm{eq})}\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,x^{(2)}(t)\right]^2=\frac{\hbar}{2m_2\omega_2}\frac{1}{\gamma_2\tau}\ . \tag{4.40}$$

This equation gives the uncertainty in our knowledge of the random variable

$$\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,x^{(2)}(t)$$

as obtained through the considered measuring procedure. For $\tau \ll 1/\gamma_2$ we have a very big variance [cf. Eq. (2.44)]: the measuring apparatus has not had enough time to collect information about the measured quantity. For an averaging time $\tau$ of the order of the decay time $1/\gamma_2$ we obtain the usual variance $\sigma_{11}^{(\mathrm{eq})}$; however, note that $\sigma_{11}^{(\mathrm{eq})}$ is the variance for an instantaneous measurement of position, while Eq. (4.40) gives the variance for the time average of position in an interval of duration $\tau$ obtained via continuous measurement. For $\tau \gg 1/\gamma_2$ we have a very precise estimation of

$$\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\,x^{(2)}(t)\ ,$$

but a time average of $x^{(2)}(t)$ on a large interval gives very little information about $x^{(2)}(t)$ itself: the time average of the position of a damped oscillator in a large interval is zero with practically no uncertainty.

Now, let us go back to the case of nonvanishing coupling ($\lambda \neq 0$). The solution of Eqs. (3.25) and (3.26) [for $\mathbf{A}(t)=\mathbf{A}$] can be written as [cf. Eqs. (B13)]

$$\boldsymbol{\alpha}^{(k)}(t)=\int_t^{+\infty}dt'\,\mathbf{R}^{(2k)}(t'-t)^T\boldsymbol{\varphi}(t')\ . \tag{4.41}$$

Then, by Eqs. (2.4), we obtain

$$\langle \mathbf{z}(t)\rangle=\mathbf{Q}^{(2)}(t)\ , \tag{4.42}$$

$$\begin{aligned}\langle[\mathbf{z}(t_1)-\langle\mathbf{z}(t_1)\rangle][\mathbf{z}(t_2)-\langle\mathbf{z}(t_2)\rangle]^T\rangle=&\frac{\hbar^2}{2}\boldsymbol{\Gamma}\delta(t_1-t_2)+\theta(t_1-t_2)\left[\sum_{k=1}^{2}\mathbf{R}^{(2k)}(t_1-t_2)\boldsymbol{\sigma}^{(k2)}(t_2)+\frac{\hbar}{2}\mathbf{R}^{(22)}(t_1-t_2)\boldsymbol{\epsilon}\mathbf{M}\right]\\&+\theta(t_2-t_1)\left[\sum_{k=1}^{2}\boldsymbol{\sigma}^{(2k)}(t_1)\mathbf{R}^{(2k)}(t_2-t_1)^T+\frac{\hbar}{2}\mathbf{M}^T\boldsymbol{\epsilon}^T\mathbf{R}^{(22)}(t_2-t_1)^T\right]\ ,\end{aligned} \tag{4.43}$$

where $\mathbf{Q}^{(2)}(t)$ and $\boldsymbol{\sigma}^{(ks)}(t)$ are given by Eqs. (4.9) and (4.10). Equation (4.43) has the same structure as Eq. (4.18b), but now we have a more complicated dynamics.

Up to now we have considered mean values and correlations for the variables of system 2. However, system 2 is only a stage of the measuring apparatus (the meter), while we are interested in the behavior of system 1 (the measured system). If we assume

$$\det\mathbf{A}\neq 0\ , \tag{4.44}$$

Eq. (4.7b) can be written as

$$\mathbf{Q}^{(1)}(t)=\frac{1}{\lambda}(\mathbf{A}^T)^{-1}\boldsymbol{\epsilon}^T\left[\frac{d}{dt}\mathbf{Q}^{(2)}(t)-\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{Q}^{(2)}(t)\right]; \tag{4.45}$$

this equation enables us to obtain the position and momentum of system 1 by starting from the position and momentum of system 2. Indeed, we can introduce the new variables

$$\mathbf{y}(t)=\frac{1}{\lambda}(\mathbf{A}^T)^{-1}\boldsymbol{\epsilon}^T\left[\frac{d}{dt}\mathbf{z}(t)-\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{z}(t)\right]\ , \tag{4.46}$$

which represent the position and momentum of system 1 $[y_1(t)=x^{(1)}(t),\ y_2(t)=p^{(1)}(t)]$ as continuously measured through the interaction with the meter. In Sec. II [Eqs. (2.7) and (2.8)] we have seen how the derivative process can be introduced; here we proceed in an analogous way. Denote by $\boldsymbol{\eta}(t)$ the generic test function associated with $\mathbf{y}(t)$. Recalling that test functions must vanish at the ends of the measuring interval with all their derivatives, we have, by Eq. (4.46)

$$\begin{aligned}\int_0^{+\infty}dt\,\boldsymbol{\eta}(t)^T\mathbf{y}(t)&\\=-\frac{1}{\lambda}\int_0^{+\infty}dt&\left[\frac{d\boldsymbol{\eta}(t)^T}{dt}(\mathbf{A}^T)^{-1}\boldsymbol{\epsilon}^T+\boldsymbol{\eta}(t)^T(\mathbf{A}^T)^{-1}\times\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}^T\right]\right]\mathbf{z}(t)\ .\end{aligned} \tag{4.47}$$

Then, if in the characteristic functional for $\mathbf{z}(t)$ we take

$$\boldsymbol{\varphi}(t)=-\frac{1}{\lambda}\left[\boldsymbol{\epsilon}\mathbf{A}^{-1}\frac{d}{dt}\boldsymbol{\eta}(t)+\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]\mathbf{A}^{-1}\boldsymbol{\eta}(t)\right]\ , \tag{4.48}$$

we obtain the characteristic functional for $\mathbf{y}(t)$. The result is

$$L_y([\boldsymbol{\eta}]\,|\,\rho)=e^{\beta}\mathrm{Tr}\{\hat{W}(\boldsymbol{\alpha}(0))\hat{\rho}\}\ , \tag{4.49}$$

$$\boldsymbol{\alpha}^{(k)}(t)=\frac{1}{\lambda}\delta_{2k}\boldsymbol{\epsilon}\mathbf{A}^{-1}\boldsymbol{\eta}(t)+\int_t^{+\infty}dt'\mathbf{R}^{(1k)}(t'-t)^T\boldsymbol{\eta}(t')\ , \tag{4.50}$$

$$\begin{aligned}\beta=-\tfrac{1}{2}\int_0^{+\infty}dt\Bigg\{&\tfrac{1}{2}\sum_{k=1}^{2}\boldsymbol{\alpha}^{(k)}(t)^T\boldsymbol{\epsilon}\mathbf{D}^{(k)}\boldsymbol{\epsilon}^T\boldsymbol{\alpha}^{(k)}(t)-\frac{\hbar}{\lambda}\boldsymbol{\alpha}^{(2)}(t)^T\boldsymbol{\epsilon}\mathbf{M}\left[\boldsymbol{\epsilon}\mathbf{A}^{-1}\frac{d}{dt}\boldsymbol{\eta}(t)+\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]\mathbf{A}^{-1}\boldsymbol{\eta}(t)\right]\\&+\frac{\hbar^2}{2\lambda}\left[\frac{d\boldsymbol{\eta}(t)^T}{dt}(\mathbf{A}^{\mathrm{T}})^{-1}\boldsymbol{\epsilon}^{\mathrm{T}}+\boldsymbol{\eta}(t)^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{-1}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}^{\mathrm{T}}\right]\right]\boldsymbol{\Gamma}\\&\times\left[\boldsymbol{\epsilon}\mathbf{A}^{-1}\frac{d}{dt}\boldsymbol{\eta}(t)+\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]\mathbf{A}^{-1}\boldsymbol{\eta}(t)\right]\Bigg\}\ .\end{aligned} \tag{4.51}$$

From this characteristic functional we obtain for the mean values

$$\langle\mathbf{y}(t)\rangle=\mathbf{Q}^{(1)}(t)\ , \tag{4.52}$$

as it must be, and for the two-time correlation functions

$$\begin{aligned}&\langle[\mathbf{y}(t_1)-\langle\mathbf{y}(t_1)\rangle][\mathbf{y}(t_2)-\langle\mathbf{y}(t_2)\rangle]^T\rangle\\&=-\frac{\hbar^2}{2\lambda^2}(\mathbf{A}^T)^{-1}\boldsymbol{\epsilon}^T\boldsymbol{\Gamma}\boldsymbol{\epsilon}\mathbf{A}^{-1}\ddot{\delta}(t_1-t_2)+\frac{\hbar}{2\lambda^2}(\mathbf{A}^T)^{-1}(\hbar\mathbf{B}^{(2)}\boldsymbol{\Gamma}\boldsymbol{\epsilon}-\hbar\boldsymbol{\epsilon}^T\boldsymbol{\Gamma}\mathbf{B}^{(2)}+\boldsymbol{\epsilon}^T\mathbf{M}^T-\mathbf{M}\boldsymbol{\epsilon})\mathbf{A}^{-1}\dot{\delta}(t_1-t_2)\\&\quad+\frac{1}{2\lambda^2}(\mathbf{A}^T)^{-1}\left[\hbar^2\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}^T\right]\boldsymbol{\Gamma}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]-\hbar\mathbf{M}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]-\hbar\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}^T\right]\mathbf{M}^T+\mathbf{D}^{(2)}\right]\mathbf{A}^{-1}\delta(t_1-t_2)\\&\quad+\vartheta(t_1-t_2)\left\{\sum_{k=1}^{2}\mathbf{R}^{(1k)}(t_1-t_2)\boldsymbol{\sigma}^{(k1)}(t_2)+\frac{1}{2\lambda}\mathbf{R}^{(12)}(t_1-t_2)\boldsymbol{\epsilon}\left[\mathbf{D}^{(2)}-\hbar\mathbf{M}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}\right]\right]\mathbf{A}^{-1}\right\}\\&\quad+\vartheta(t_2-t_1)\left\{\sum_{k=1}^{2}\boldsymbol{\sigma}^{(1k)}(t_1)\mathbf{R}^{(1k)}(t_2-t_1)^T+\frac{1}{2\lambda}(\mathbf{A}^T)^{-1}\left[\mathbf{D}^{(2)}-\hbar\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\boldsymbol{\epsilon}^T\right]\mathbf{M}^T\right]\boldsymbol{\epsilon}^T\mathbf{R}^{(12)}(t_2-t_1)^T\right\}\ .\end{aligned} \tag{4.53}$$

Equation (4.53) is much more complicated than Eq. (4.43): in it appear not only the $\delta$ function, but also the first and second derivatives of the $\delta$ function. This fact is simply due to the "propagation of errors" from $\mathbf{z}(t)$ to $\mathbf{y}(t)$: indeed $\mathbf{y}(t)$ is obtained from $\mathbf{z}(t)$ by means of Eq. (4.46). Moreover, note the close analogy between the nonsingular terms in Eqs. (4.43) and (4.53).

A much simpler situation appears when the characteristic times for the evolution of system 2 are much shorter than the characteristic times of system 1. Mathematically, this situation can be realized[20] by making the substitutions

$$\begin{aligned}&\mathbf{B}^{(2)}\to\mathbf{B}^{(2)}/\lambda^2,\quad\mathbf{D}^{(2)}\to\mathbf{D}^{(2)}/\lambda^2,\quad\gamma_2\to\gamma_2/\lambda^2\ ,\\&\mathbf{A}\to\mathbf{A}/\lambda^2,\quad\boldsymbol{\Gamma}\to\lambda^2\boldsymbol{\Gamma}\end{aligned} \tag{4.54}$$

and taking the limit $\lambda\to 0$; this is known in the literature as the *weak-coupling limit* (see, for instance, Ref. 28). If the limit is to exist, the real part of the eigenvalues of $(\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\gamma_2/2)$ must be negative; more explicitly, we must have

$$\gamma_2>0,\quad\text{if }\det\mathbf{B}^{(2)}\geq 0\ , \tag{4.55a}$$

or

$$\gamma_2>2(\,|\det\mathbf{B}^{(2)}|\,)^{1/2},\quad\text{if }\det\mathbf{B}^{(2)}<0\ . \tag{4.55b}$$

Using the results of Appendix B, Eqs. (B14)–(B25), we obtain in the weak-coupling limit

$$\langle\mathbf{y}(t)\rangle=\mathbf{Q}^{(1)}(t)=\widetilde{\mathbf{R}}^{(11)}(t)\mathbf{Q}^{(1)}(0)\ , \tag{4.56}$$

$$\begin{aligned}&\langle[\mathbf{y}(t_1)-\langle\mathbf{y}(t_1)\rangle][\mathbf{y}(t_2)-\langle\mathbf{y}(t_2)\rangle]^T\rangle\\&=\frac{\hbar^2}{2}\widetilde{\boldsymbol{\Gamma}}\delta(t_1-t_2)+\theta(t_1-t_2)\widetilde{\mathbf{R}}^{(11)}(t_1-t_2)\left[\widetilde{\boldsymbol{\sigma}}^{(11)}(t_2)+\frac{\hbar}{2}\boldsymbol{\epsilon}\widetilde{\mathbf{M}}\right]+\theta(t_2-t_1)\left[\widetilde{\boldsymbol{\sigma}}^{(11)}(t_1)+\frac{\hbar}{2}\widetilde{\mathbf{M}}^T\boldsymbol{\epsilon}^T\right]\widetilde{\mathbf{R}}^{(1)}(t_2-t_1)^T,\end{aligned} \tag{4.57}$$

where

$$\widetilde{\mathbf{R}}^{(11)}(t)=\exp\left[\left[\epsilon\widetilde{\mathbf{B}}-\frac{\widetilde{\gamma}}{2}\right]t\right], \tag{4.58a}$$

$$\widetilde{\mathbf{B}}=\mathbf{B}^{(1)}-\left[\frac{\gamma_2^2}{2}+2\omega_2^2\right]^{-1}\mathbf{A}\epsilon\mathbf{B}^{(2)}\epsilon^T\mathbf{A}^T, \tag{4.58b}$$

$$\widetilde{\gamma}=\gamma_1+\gamma_2\left[\frac{\gamma_2^2}{4}+\omega_2^2\right]^{-1}\det\mathbf{A}, \tag{4.58c}$$

$$\widetilde{\mathbf{\Gamma}}=(\mathbf{A}^T)^{-1}\left[\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon^T\right]\mathbf{\Gamma}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon\right]-\frac{1}{\hbar}\mathbf{M}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon\right]-\frac{1}{\hbar}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon^T\right]\mathbf{M}^T+\frac{1}{\hbar^2}\mathbf{D}^{(2)}\right]\mathbf{A}^{-1}, \tag{4.59}$$

$$\widetilde{\sigma}^{(11)}(t)=\widetilde{\mathbf{R}}^{(11)}(t)\sigma^{(11)}(0)\widetilde{\mathbf{R}}^{(11)}(t)^T+\tfrac{1}{2}\int_0^t dt'\widetilde{\mathbf{R}}^{(11)}(t')\epsilon\widetilde{\mathbf{D}}\epsilon^T\widetilde{\mathbf{R}}^{(11)}(t')^T, \tag{4.60a}$$

$$\widetilde{\mathbf{D}}=\mathbf{D}^{(1)}+\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\times\epsilon\mathbf{D}^{(2)}\epsilon^T\left[\frac{\gamma_2}{2}-\mathbf{B}^{(2)}\epsilon^T\right]^{-1}\mathbf{A}^T, \tag{4.60b}$$

$$\widetilde{\mathbf{M}}=\mathbf{A}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon^T\right]^{-1}\times\left[\frac{1}{\hbar}\mathbf{D}^{(2)}-\mathbf{M}\left[\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\epsilon\right]\right]\mathbf{A}^{-1}. \tag{4.61}$$

With some algebraic manipulation and using Eq. (3.21), one can prove that

$$\widetilde{\mathbf{\Gamma}}\geq 0,\quad \widetilde{\mathbf{D}}+i\hbar\widetilde{\gamma}\epsilon\geq\mathbf{0}, \tag{4.62a}$$

$$\widetilde{\mathbf{D}}+i\hbar\widetilde{\gamma}\epsilon-(1+i\widetilde{\mathbf{M}})\widetilde{\mathbf{\Gamma}}^{-1}(1-i\widetilde{\mathbf{M}}^T)\geq\mathbf{0}. \tag{4.62b}$$

These results are clearly the same as for a direct continuous measurement of the position and momentum of system 1 alone. This is a general result of the formalism of Sec. II.[20] The interaction with the measuring apparatus has changed the dynamics of the measured system (both the Hamiltonian and the dissipative parts) and has given rise to the terms $\widetilde{\mathbf{\Gamma}}$ and $\widetilde{\mathbf{M}}$, directly connected with the way $x^{(1)}(t)$ and $p^{(1)}(t)$ are measured. Note that, even if we started with $\mathbf{M}=0$, we obtain $\widetilde{\mathbf{M}}\neq 0$. When $\mathbf{D}^{(1)}=\mathbf{0}$ and $\gamma_1=0$, then the dissipation in system 1 is wholly due to the interaction with the measuring apparatus. Moreover, in this case, if we start from the "best" measuring procedure on 2, i.e.,

$$\mathbf{D}^{(2)}+i\hbar\gamma_2\epsilon=(1+i\mathbf{M})\mathbf{\Gamma}^{-1}(1-i\mathbf{M}^T), \tag{4.63}$$

we obtain the "best" measuring procedure on 1, i.e.,

$$\widetilde{\mathbf{D}}+i\hbar\widetilde{\gamma}\epsilon=(1+i\widetilde{\mathbf{M}})\widetilde{\mathbf{\Gamma}}^{-1}(1-i\widetilde{\mathbf{M}}). \tag{4.64}$$

However, for a physically realizable meter (a transducer from mechanical to electrical quantities) the approximation of very short characteristic times is probably not a good one, so we cannot take the weak-coupling limit. On the contrary, short characteristic times are typical of an amplification stage. Therefore, the model of Sec. III gives a sensible description of the measuring chain: detector + meter + amplifier.

## V. ESTIMATION OF THE GRAVITATIONAL FORCE

In this section we want to discuss the problem of estimation of the gravitational force, using the general results of Sec. III. Let us denote by $\mathbf{R}^{(ks)}(t,t_0)$ the formal solution of Eqs. (B9)–(B10), when the interaction matrix is time dependent; then Eqs. (3.24)–(3.27) give

$$\alpha^{(k)}(t)=\int_t^{+\infty}dt'\mathbf{R}^{(2k)}(t',t)^T\varphi(t'). \tag{5.1}$$

Inserting this result into Eqs. (3.22) and (3.24), we obtain the characteristic functional for the continuously measured quantities $\mathbf{z}(t)$ $[z_1(t)=x^{(2)}(t), z_2(t)=p^{(2)}(t)]$. In particular, for the mean values we have

$$\langle\mathbf{z}(t)\rangle=\sum_{k=1}^{2}\mathbf{R}^{(2k)}(t,0)\mathbf{Q}^{(k)}(0)+\int_0^t dt'\mathbf{R}^{(21)}(t,t')\epsilon^T\mathbf{f}(t'). \tag{5.2}$$

By inspection of Eq. (3.24), we see that only the mean value of $\mathbf{z}(t)$ depends on $\mathbf{f}(t)$; therefore, we can introduce a new stochastic process $\mathbf{y}(t)$, independent of the gravitational force, by setting

$$\mathbf{z}(t)=\int_0^t dt'\mathbf{R}^{(21)}(t,t')\epsilon^T\mathbf{f}(t')+\mathbf{y}(t). \tag{5.3}$$

Thus the output $\mathbf{z}(t)$ is composed of a deterministic signal (the term containing the force) plus a noise $\mathbf{y}(t)$. This is the typical situation considered in classical estimation theory.[29] Note that the noise in our case has a quantum origin, but at this stage we have a purely classical problem: the estimation of the signal in the given noise.

The solution of the problem depends on what is *a priori* known about the force. In Ref. 30, Holevo uses our theory of continuous measurements in the simpler case of a single harmonic oscillator (treated in Ref. 14) and assumes the force to be

$$\mathbf{f}(t)=(\theta_1 g_1(t),\theta_2 g_2(t))$$

with $g_1(t)$ and $g_2(t)$ known; then he treats the two-parameter estimation problem for $\theta_1$ and $\theta_2$. Instead, in this paper we treat the case of two harmonic oscillators (detector and meter) and consider the force to be

$$\mathbf{f}(t)=f(t)\mathbf{u} \tag{5.4}$$

with $\mathbf{u}$ known (i.e., it is known how the gravitational field couples to the detector), but $f(t)$ unknown. We have a "waveform" estimation problem.

Following the terminology of estimation theory, we define an unbiased estimator for $f(t)$ to be any functional

$$F(t)=F([\mathbf{z}];t) \tag{5.5}$$

of the measured quantities $\mathbf{z}(t)$, whose mean values gives the force

$$\langle F(t)\rangle = f(t), \quad \forall f(t)\;; \tag{5.6}$$

if $F(t)$ depends linearly on $\mathbf{z}$ we speak of a linear unbiased estimator. If the estimator $F(t)$ is not unique, one can determine the "best" choice using some optimality criterion. For the most used criteria, when the process is Gaussian (as it is in our case when the initial state is Gaussian), the best unbiased estimator is linear.

Given a certain estimator $F(t)$, one can obtain its characteristic functional $L_F[\eta]$ from the characteristic functional $L([\boldsymbol{\varphi}]\,|\,\rho)$ of $\mathbf{z}$ by using the dependence (5.5) of $F(t)$ on $\mathbf{z}$. In Appendix C we show that Eq. (5.6) implies that

$$\boldsymbol{\alpha}^{(1)}(0)=\boldsymbol{\alpha}^{(2)}(0)=\mathbf{0}\;, \tag{5.7}$$

so that

$$\mathrm{Tr}[\hat{W}(\boldsymbol{\alpha}(0))\hat{\rho}]=1$$

and $L_F[\eta]$ does not depend on the initial state $\hat{\rho}$. Moreover, we have

$$\ln L_F[\eta]\equiv\beta=i\int_0^{+\infty}dt\,f(t)\eta(t)-\tfrac{1}{2}\int_0^{+\infty}dt\left[\frac{\hbar^2}{2}\boldsymbol{\varphi}(t)^T\boldsymbol{\Gamma}\boldsymbol{\varphi}(t)+\tfrac{1}{2}\sum_{k=1}^{2}\boldsymbol{\alpha}^{(k)}(t)^T\boldsymbol{\epsilon}\mathbf{D}^{(k)}\boldsymbol{\epsilon}^T\boldsymbol{\alpha}^{(k)}(t)+\hbar\boldsymbol{\alpha}^{(2)}(t)^T\boldsymbol{\epsilon}\mathbf{M}\boldsymbol{\varphi}(t)\right]\;, \tag{5.8}$$

where one component of $\boldsymbol{\varphi}$, say $\varphi_2$, is an arbitrary functional of $\eta$, and $\boldsymbol{\alpha}^{(k)}(t)$ and $\varphi_1(t)$ are linear combinations of $\eta(t)$ and $\varphi_2(t)$ and their derivatives (they can be explicitly calculated as sketched in Appendix C). The dependence of $\varphi_2$ on $\eta$ can be determined by imposing some optimality condition.

From Eq. (5.8) one can calculate the variances of the time smoothed variables

$$\int_0^{+\infty}dt\,F(t)h(t)$$

for an arbitrary test function $h(t)$, so obtaining the "precision" of the estimate of

$$\int_0^{+\infty}dt\,f(t)h(t)$$

(for instance one can consider the Fourier components of the force). The first result is that this precision does not depend on the initial state $\hat{\rho}$ of detector and meter [because of Eq. (5.7)], but only on the dynamics of the system and on the chosen measuring procedure. This is not a strange result: a classical model for the detecting apparatus would give the same qualitative result. Indeed, if we want to estimate a force acting on a classical particle by measuring the position $x(t)$ of the particle, we find that the estimate does not depend on the initial distribution in phase space, essentially because the force is linked to the acceleration which does not depend on initial conditions.

All the variants of the model introduced in Sec. III can be worked out. However, we limit ourselves to discussion of two simple cases: the "back action evading" coupling (3.13) and the time-independent coupling (3.12). Moreover, we take for the matrices $\mathbf{B}^{(k)}$ the expressions (3.8) and consider the case in which the force couples only to $\hat{x}^{(1)}$, i.e.,

$$u_1=1,\quad u_2=0\;. \tag{5.9}$$

First we study the case of the coupling (3.13), with $\mathbf{A}$ given by Eq. (3.15); explicitly, the interaction Hamiltonian is

$$\hat{H}_I(t)=\lambda\hat{x}^{(2)}\hat{X}_1(t)\;, \tag{5.10a}$$

$$\hat{X}_1(t)=\hat{x}^{(1)}\cos\omega_1 t-\hat{p}^{(1)}\frac{\sin\omega_1 t}{m_1\omega_1}\;. \tag{5.10b}$$

The quantities $\boldsymbol{\alpha}^{(k)}(t)$ and $\varphi_1(t)$ are calculated in Appendix C [Eqs. (C2) and (C16)–(C19)]. The quantity $\varphi_2(t)$ is an arbitrary linear functional of $\eta(t)$, which we write as

$$\varphi_2(t)=\int_0^{+\infty}dt'\eta(t')K(t',t)\;. \tag{5.11}$$

Then, using Eqs. (5.11), (C17), and (C19), we can write

$$\int_0^{+\infty}dt\,\boldsymbol{\varphi}(t)^T\mathbf{z}(t)=\int_0^{+\infty}dt\,\eta(t)F(t)\;, \tag{5.12a}$$

where

$$F(t)=\int_0^{+\infty}dt'K(t,t')\left\{p^{(2)}(t')-m_2\left[\frac{d}{dt'}x^{(2)}(t')+\frac{\gamma_2}{2}x^{(2)}(t')\right]\right\}$$
$$+\frac{m_1m_2\omega_1}{\lambda\sin\omega_1 t}\left[\frac{d^3}{dt^3}x^{(2)}(t)+\left[\frac{\gamma_1}{2}+\gamma_2\right]\frac{d^2}{dt^2}x^{(2)}(t)+\left[\omega_2{}^2+\frac{\gamma_2{}^2}{4}+\frac{\gamma_1\gamma_2}{2}\right]\frac{d}{dt}x^{(2)}(t)+\frac{\gamma_1}{2}\left[\omega_2{}^2+\frac{\gamma_2{}^2}{4}\right]x^{(2)}(t)\right]\;. \tag{5.12b}$$

Equation (5.12b) gives the dependence of the estimator $F(t)$ on the measured variables [$x^{(2)}(t)$ and $p^{(2)}(t)$ are position and momentum of system 2].

By inserting Eqs. (C2) and (C16)–(C19) into Eq. (5.8) we obtain the characteristic functional for the estimator $F(t)$. Here we consider only the case of a very large coupling between detector and meter ($\lambda\to\infty$); the existence of this limit is due to the structure (3.13) with $\det\mathbf{A}=0$ (back-action-evading interaction). Moreover, in this case, the best choice is to take $\varphi_2(t)=0$, because the terms containing $\varphi_2$ give a positive contribution to all variances. From the results of Appendix C, we see that in the strong-coupling limit we have

$$\boldsymbol{\alpha}^{(2)}(t)=\mathbf{0},\quad \boldsymbol{\varphi}(t)=\mathbf{0}\ , \tag{5.13}$$

$$\alpha_1^{(1)}(t)=-m_1\omega_1\frac{\cos\omega_1 t}{\sin\omega_1 t}\eta(t),\quad \alpha_2^{(1)}(t)=\eta(t)\ . \tag{5.14}$$

Therefore, the characteristic functional (5.8) becomes

$$\ln L_F[\eta]=i\int_0^{+\infty}dt\, f(t)\eta(t)-\tfrac{1}{4}\int_0^{+\infty}dt\,\eta^2(t)\left[D_{11}^{(1)}+2D_{12}^{(1)}m_1\omega_1\frac{\cos\omega_1 t}{\sin\omega_1 t}+D_{22}^{(1)}m_1{}^2\omega_1{}^2\frac{\cos^2\omega_1 t}{\sin^2\omega_1 t}\right]\ . \tag{5.15}$$

From Eq. (5.15) all the moments of $F(t)$ can be obtained via Eqs. (2.4). Note that if $\mathbf{D}^{(1)}=0$, all correlations vanish and the force is estimated without uncertainty. Thus we agree with the conclusion that a "back-action-evading coupling" allows a precise estimation of the force (cf. Ref. 2).

The simplest case of nonvanishing dissipation is when the matrix $\mathbf{D}^{(1)}$ is given by Eq. (3.18) (for $i=1$). With this choice, when the interaction is off, system 1 decays to its ground state [cf. Eqs. (4.28) and (4.30)]; recall that in this equilibrium state

$$\sigma_{22}^{(\mathrm{eq})}\equiv\mathrm{Tr}\{(\hat{p}^{(1)}-\langle p^{(1)}\rangle_{\mathrm{eq}})^2\hat{\rho}_{\mathrm{eq}}\}=\tfrac{1}{2}\hbar m_1\omega_1\ . \tag{5.16}$$

If we denote by $\Delta_h{}^2$ the variance for the estimate of

$$\int_0^{+\infty}dt\, f(t)h(t)\ ,$$

we obtain from Eq. (5.15)

$$\Delta_h{}^2=\tfrac{1}{2}\hbar m_1\omega_1\gamma_1\int_0^{+\infty}dt\frac{h^2(t)}{\sin^2\omega_1 t}\ . \tag{5.17}$$

For instance, if we consider the time average of $f(t)$ in the time interval $(\bar{t},\bar{t}+\tau)$ [so that $h(t)$ is the characteristic function of the considered time interval divided by $\tau$], we obtain

$$\left[\Delta\frac{1}{\tau}\int_{\bar{t}}^{\bar{t}+\tau}dt\, f(t)\right]^2=\tfrac{1}{2}\hbar m_1\omega_1\frac{\gamma_1}{\tau}\times\frac{\sin\omega_1\tau/\omega_1\tau}{\sin\omega_1\bar{t}\,\sin\omega_1(\bar{t}+\tau)}\ . \tag{5.18}$$

Note in Eqs. (5.17) and (5.18) how the uncertainty in the estimation of the force is linked to the equilibrium variance $\sigma_{22}^{(\mathrm{eq})}$ for the momentum. This does not mean that we are considering an equilibrium situation: when the time-dependent interaction is on we have no equilibrium state. Rather both of the quantities (5.16) and (5.17) depend on the dynamical features of the system and, so, they must be connected.

It is interesting to note that for the case $\omega_1\tau\ll 1$, Eq. (5.18) can be obtained from a very simple argument. During the time $\tau$ the force changes the value of the measured quantity

$$X_1=x^{(1)}\cos\omega_1 t-(p^{(1)}/m_1\omega_1)\sin\omega_1 t$$

[cf. Eq. (5.10b)] by an amount

$$\delta X_1\simeq-(f\tau/m_1\omega_1)\sin\omega_1\bar{t}\ . \tag{5.19}$$

Consider now the variance of $X_1$ defined as usual as

$$\mathrm{Tr}\{[\hat{X}_1(t)-\langle X_1(t)\rangle]^2\hat{\rho}(t)\}\ ;$$

it can be easily calculated. During the same time the internal fluctuations increase this variance by an amount

$$(\Delta X_1)^2\simeq\frac{\hbar\gamma_1\tau}{2m_1\omega_1}\ . \tag{5.20}$$

Setting Eqs. (5.19) and (5.20) together yields Eq. (5.18).

Let us study now an example of a standard detecting apparatus; we consider the time-independent coupling (3.12) with $\mathbf{A}$ given by Eq. (3.15). The interaction Hamiltonian in this case is simply

$$\hat{H}_I=\lambda\hat{x}^{(1)}\hat{x}^{(2)}\ . \tag{5.21}$$

Moreover, we take the simple diffusion matrices given by Eq. (3.18). This is the situation discussed in Sec. IV [Eqs. (4.28)–(4.33); now with interaction between the two systems] and corresponds to the zero-temperature case. As shown in Sec. IV, $p^{(2)}(t)$ is not measured, so that $\varphi_2(t)=0$. Then we take

$$\Gamma_{11}=1/\hbar\gamma_2 m_2\omega_2,\quad M_{21}=-1/m_2\omega_2\ ,\qquad M_{11}=M_{12}=M_{22}=0 \tag{5.22}$$

[cf. Eqs. (4.32) and (4.33)]. This choice corresponds to the minimum noise compatible with the given dynamics.

Following the procedure given in Appendix C, we obtain

$$\alpha_1^{(1)}(t)=m_1\left[-\frac{d}{dt}\eta(t)+\frac{\gamma_1}{2}\eta(t)\right]\ ,\qquad \alpha_2^{(1)}(t)=\eta(t)\ , \tag{5.23}$$

$$\alpha_1^{(2)}(t)=\frac{1}{\lambda}m_1m_2\left[\frac{d}{dt}\zeta(t)-\frac{\gamma_2}{2}\zeta(t)\right]\ ,\qquad \alpha_2^{(2)}(t)=-\frac{m_1}{\lambda}\zeta(t)\ , \tag{5.24}$$

$$\varphi_1(t)=\lambda\eta(t)-\frac{m_1m_2}{\lambda}\left[\frac{d^2}{dt^2}\zeta(t)-\gamma_2\frac{d}{dt}\zeta(t)+\left[\omega_2^2+\frac{\gamma_2^2}{4}\right]\zeta(t)\right], \quad (5.25)$$

where

$$\zeta(t)\equiv\frac{d^2}{dt^2}\eta(t)-\gamma_1\frac{d}{dt}\eta(t)+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]\eta(t) . \quad (5.26)$$

First, by proceeding as for Eqs. (5.12), we obtain the estimator $F(t)$ for the force

$$F(t)=\lambda x^{(2)}(t)-\frac{1}{\lambda}m_1m_2\left[\frac{d^4}{dt^4}x^{(2)}(t)+(\gamma_1+\gamma_2)\frac{d^3}{dt^3}x^{(2)}(t)+\left[\frac{\gamma_1^2+\gamma_2^2}{4}+\gamma_1\gamma_2+\omega_1^2+\omega_2^2\right]\frac{d^2}{dt^2}x^{(2)}(t)\right.$$
$$\left.+\left[\frac{\gamma_1+\gamma_2}{4}\gamma_1\gamma_2+\gamma_2\omega_1^2+\gamma_1\omega_2^2\right]\frac{d}{dt}x^{(2)}(t)+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]\left[\frac{\gamma_2^2}{4}+\omega_2^2\right]x^{(2)}(t)\right] . \quad (5.27)$$

Then, by inserting Eqs. (5.22)–(5.26), into Eq. (5.8), we obtain the characteristic functional for $F(t)$

$$\ln L_F[\eta]=i\int_0^{+\infty}dt\, f(t)\eta(t)$$
$$-\frac{\hbar}{2}\int_0^{+\infty}dt\left\{\frac{\gamma_1m_1}{2\omega_1}\left[\left[\frac{d\eta(t)}{dt}\right]^2+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]\eta^2(t)\right]\right.$$
$$-\frac{m_1}{\gamma_2\omega_2}\left[\left[\frac{d^2\eta(t)}{dt^2}\right]^2-\left[\frac{\gamma_1^2-\gamma_2^2}{4}+\omega_1^2+\omega_2^2\right]\left[\frac{d\eta(t)}{dt}\right]^2\right.$$
$$\left.+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]\left[\omega_2^2-\frac{\gamma_2^2}{4}\right]\eta^2(t)\right]+\frac{\lambda^2\eta^2(t)}{2\gamma_2\omega_2m_2}$$
$$\left.+\frac{m_1^2m_2}{2\lambda^2\gamma_2\omega_2}\left[\left[\frac{d^2\zeta(t)}{dt^2}\right]^2+2\left[\frac{\gamma_2^2}{4}-\omega_2^2\right]\left[\frac{d\zeta(t)}{dt}\right]^2+\left[\frac{\gamma_2^2}{4}+\omega_2^2\right]\zeta^2(t)\right]\right\} . \quad (5.28)$$

The simplest way to study this characteristic functional is to use Fourier expansion. Consider a finite interval measurement $(0,T)$ and set

$$\eta(t)=\sum_{n=1}^{\infty}\eta_n\left[\frac{2}{T}\right]^{1/2}\sin\Omega_nt, \quad \Omega_n=\frac{n\pi}{T} . \quad (5.29)$$

Inserting this expression into Eq. (5.28), we obtain

$$L_F(\eta_1,\eta_2,\ldots)=\prod_{n=1}^{\infty}\exp[if_n\eta_n-\tfrac{1}{2}\Delta^2(\Omega_n)\eta_n^2] , \quad (5.30)$$

where

$$f_n=\left[\frac{2}{T}\right]^{1/2}\int_0^T dt\, f(t)\sin\Omega_nt , \quad (5.31)$$

$$\Delta^2(\Omega)=\frac{\hbar\gamma_1m_1}{2\omega_1}\left[\Omega^2+\omega_1^2+\frac{\gamma_1^2}{4}\right]-\frac{\hbar m_1}{\gamma_2\omega_2}\left[\Omega^4-\Omega^2\left[\frac{\gamma_1^2-\gamma_2^2}{4}+\omega_1^2+\omega_2^2\right]+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]\left[\omega_2^2-\frac{\gamma_2^2}{4}\right]\right]$$
$$+\frac{\lambda^2\hbar}{2\gamma_2m_2\omega_2}+\frac{\hbar m_1^2m_2}{2\lambda^2\gamma_2\omega_2}\left[\Omega^4+2\left[\frac{\gamma_2^2}{4}-\omega_2^2\right]\Omega^2+\left[\frac{\gamma_2^2}{4}+\omega_2^2\right]^2\right]\left[\Omega^4+2\left[\frac{\gamma_1^2}{4}-\omega_1^2\right]\Omega^2+\left[\frac{\gamma_1^2}{4}+\omega_1^2\right]^2\right] . \quad (5.32)$$

Equation (5.30) gives the characteristic function of the joint probability for the Fourier components of the estimator; these Fourier components are statistically independent Gaussian variables with variances $\Delta^2(\Omega_n)$. Equation (5.32) gives the so-called *power spectrum* of the noise[29] and can be used for computing such quantities as the *signal-to-noise ratio* and in general for studying the sensitivity characteristics of the detecting apparatus. The temperature-dependent case [when Eq. (3.18) does not hold] can be discussed in a similar way and a temperature-dependent power spectrum can be obtained.

## APPENDIX A

The characteristic functional (2.28) for the model of Sec. III can be calculated by using the Weyl operator technique.[14,27] The Weyl operators are defined by Eq. (3.23) and enjoy the following properties:

$$\frac{\partial \hat{W}(\boldsymbol{\alpha})}{\partial \alpha_j^{(k)}} = \frac{i}{2}\{\hat{Q}_j^{(k)}, \hat{W}(\boldsymbol{\alpha})\} , \tag{A1}$$

$$[\hat{Q}_i^{(k)}, \hat{W}(\boldsymbol{\alpha})] = \hbar \alpha_j^{(k)} \epsilon_{ji} \hat{W}(\boldsymbol{\alpha}) . \tag{A2}$$

The strategy is to use Weyl operators for converting the operatorial equation (2.29) into a set of classical differential equations.

For any operator $\mathscr{A}$ in $T(\mathscr{h})$ we can define its transpose in $B(\mathscr{h})$ (bounded operators in $\mathscr{h}$) by the equation

$$\begin{aligned} &\mathrm{Tr}[\hat{Y}(\mathscr{A}\hat{X})] = \mathrm{Tr}[(\mathscr{A}^T\hat{Y})\hat{X}] , \\ &\forall \hat{X} \in T(\mathscr{h}), \quad \forall \hat{Y} \in B(\mathscr{h}) . \end{aligned} \tag{A3}$$

Using this notation, the characteristic functional can be rewritten as

$$\begin{aligned} L([\boldsymbol{\varphi}]\,|\,\rho) &= \lim_{\bar{t}\to+\infty} \mathrm{Tr}[\hat{I}(\mathscr{G}(\bar{t},0;[\boldsymbol{\varphi}])\hat{\rho})] \\ &= \lim_{\bar{t}\to\infty} \mathrm{Tr}[(\mathscr{G}^T(\bar{t},0;[\boldsymbol{\varphi}])\hat{I})\hat{\rho}] \\ &= \lim_{\bar{t}\to+\infty} \mathrm{Tr}[(\mathscr{G}^T(\bar{t},0;[\boldsymbol{\varphi}])\hat{W}(\mathbf{0}))\hat{\rho}] . \end{aligned} \tag{A4}$$

Taking the transpose of Eq. (2.38) and differentiating with respect to $t_0$, we obtain the equation

$$\frac{\partial}{\partial t}\mathscr{G}^T(\bar{t},t;[\boldsymbol{\varphi}]) = -\mathscr{K}^T(t;\boldsymbol{\varphi}(t))\mathscr{G}^T(\bar{t},t;[\boldsymbol{\varphi}]) , \tag{A5a}$$

$$\mathscr{G}^T(\bar{t},\bar{t};[\boldsymbol{\varphi}]) = 1 . \tag{A5b}$$

The transpose of the generator (3.20) can be easily calculated by using definition (A3), and it can be written as

$$\mathscr{K}^T(t;\boldsymbol{\varphi}(t)) = \mathscr{L}^T(t) + \frac{i}{2}\{\hat{Q}_j^{(2)},\cdot\}\varphi_j(t) - \tfrac{1}{2}[\hat{Q}_i^{(2)},\cdot]M_{ij}\varphi_j(t) - \frac{\hbar^2}{4}\varphi_i(t)\Gamma_{ij}\varphi_j(t) , \tag{A6a}$$

$$\mathscr{L}^T(t) = \mathscr{L}^{(1)T} + \mathscr{L}^{(2)T} - \frac{i}{\hbar}f_i(t)[\hat{Q}_i^{(1)},\cdot] + \frac{i\lambda}{2\hbar}A_{ij}(t)(\{\hat{Q}_i^{(1)},[\hat{Q}_j^{(2)},\cdot]\} + \{\hat{Q}_j^{(2)},[\hat{Q}_i^{(1)},\cdot]\}) , \tag{A6b}$$

$$\mathscr{L}^{(k)T} = \frac{i}{2\hbar}B_{ij}^{(k)}\{\hat{Q}_i^{(k)},[\hat{Q}_j^{k},\cdot]\} - \frac{1}{4\hbar^2}D_{ij}^{(k)}[\hat{Q}_i^{(k)},[\hat{Q}_j^{(k)},\cdot]] - \frac{i\gamma_k}{4\hbar}\epsilon_{ij}\{\hat{Q}_i^{(k)},[\hat{Q}_j^{(k)},\cdot]\} . \tag{A6c}$$

In the case of a quadratic generator such as (A6), we have

$$\mathscr{G}^T(\bar{t},t;[\boldsymbol{\varphi}])\hat{W}(\boldsymbol{\alpha}(\bar{t})) = \hat{W}(\boldsymbol{\alpha}(t))\exp\beta(t) . \tag{A7}$$

From Eqs. (A7) and (A4), by taking

$$\boldsymbol{\alpha}^{(k)}(+\infty) = \mathbf{0} , \tag{A8}$$

we see that the characteristic functional can be written in the form (3.22) [where $\beta \equiv \beta(0)$].

To calculate the functions $\boldsymbol{\alpha}^{(k)}(t)$ and $\beta(t)$ we differentiate Eq. (A7) with respect to $t$; using Eqs. (A1), (A2), (A5a), and (A6) and identifying the coefficients of the various operatorial terms, we obtain the coupled differential equations (3.25), (3.26) for $\boldsymbol{\alpha}^{(k)}(t)$ and a differential equation for $\beta(t)$ which can be immediately integrated; Eq. (3.24) gives the value of $\beta(t)$ at $t=0$.

## APPENDIX B

In this appendix we calculate the quantities (4.2) and (4.3). By differentiating Eq. (4.2), we obtain

$$\frac{d}{dt}Q_i^{(k)}(t) = \mathrm{Tr}[(\mathscr{L}^T\hat{Q}_i^{(k)})\hat{\rho}(t)] , \tag{B1}$$

and computing $\mathscr{L}^T$ applied to $\hat{Q}_i^{(k)}$, we find Eqs. (4.7). Using the same procedure for the matrices $\boldsymbol{\sigma}^{(ks)}(t)$ we obtain the equations

$$\frac{d}{dt}\boldsymbol{\sigma}^{(11)}(t) = \left[\boldsymbol{\epsilon}\mathbf{B}^{(1)} - \frac{\gamma_1}{2}\right]\boldsymbol{\sigma}^{(11)}(t) + \lambda\boldsymbol{\epsilon}\mathbf{A}\boldsymbol{\sigma}^{(21)}(t) + \boldsymbol{\sigma}^{(11)}(t)\left[\mathbf{B}^{(1)}\boldsymbol{\epsilon}^T - \frac{\gamma_1}{2}\right] + \lambda\boldsymbol{\sigma}^{(12)}(t)\mathbf{A}^T\boldsymbol{\epsilon}^T + \tfrac{1}{2}\boldsymbol{\epsilon}\mathbf{D}^{(1)}\boldsymbol{\epsilon}^T , \tag{B2a}$$

$$\frac{d}{dt}\boldsymbol{\sigma}^{(22)}(t) = \left[\boldsymbol{\epsilon}\mathbf{B}^{(2)} - \frac{\gamma_2}{2}\right]\boldsymbol{\sigma}^{(22)}(t) + \lambda\boldsymbol{\epsilon}\mathbf{A}^T\boldsymbol{\sigma}^{(12)}(t) + \boldsymbol{\sigma}^{(22)}(t)\left[\mathbf{B}^{(2)}\boldsymbol{\epsilon}^T - \frac{\gamma_2}{2}\right] + \lambda\boldsymbol{\sigma}^{(21)}(t)\mathbf{A}\boldsymbol{\epsilon}^T + \tfrac{1}{2}\boldsymbol{\epsilon}\mathbf{D}^{(2)}\boldsymbol{\epsilon}^T , \tag{B2b}$$

$$\frac{d}{dt}\boldsymbol{\sigma}^{(12)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]\boldsymbol{\sigma}^{(12)}(t)+\lambda\boldsymbol{\epsilon}\mathbf{A}\boldsymbol{\sigma}^{(22)}(t)+\boldsymbol{\sigma}^{(12)}(t)\left[\mathbf{B}^{(2)}\boldsymbol{\epsilon}^T-\frac{\gamma_2}{2}\right]+\lambda\boldsymbol{\sigma}^{(11)}(t)\mathbf{A}\boldsymbol{\epsilon}^T\ ; \tag{B2c}$$

moreover, by definition, we have

$$\boldsymbol{\sigma}^{(21)}(t)=\boldsymbol{\sigma}^{(12)}(t)^T\ . \tag{B3}$$

By introducing the four-component vector

$$\mathbf{Q}(t)=\begin{bmatrix}\mathbf{Q}^{(1)}(t)\\ \mathbf{Q}^{(2)}(t)\end{bmatrix} \tag{B4}$$

and the $4\times4$ matrices

$$\boldsymbol{\sigma}(t)=\begin{bmatrix}\boldsymbol{\sigma}^{(11)}(t) & \boldsymbol{\sigma}^{(12)}(t)\\ \boldsymbol{\sigma}^{(21)}(t) & \boldsymbol{\sigma}^{(22)}(t)\end{bmatrix}, \tag{B5}$$

$$\mathbf{G}=\begin{bmatrix}\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2} & \lambda\boldsymbol{\epsilon}\mathbf{A}\\ \lambda\boldsymbol{\epsilon}\mathbf{A}^T & \boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\end{bmatrix}, \tag{B6a}$$

$$\mathbf{R}(t)=\exp(\mathbf{G}t)\ , \tag{B6b}$$

we have that the solutions of Eqs. (4.7) and (B2) can be written as

$$\mathbf{Q}(t)=\mathbf{R}(t)\mathbf{Q}(0)\ , \tag{B7}$$

$$\boldsymbol{\sigma}(t)=\mathbf{R}(t)\boldsymbol{\sigma}(0)\mathbf{R}(t)^T+\tfrac{1}{2}\int_0^t dt'\mathbf{R}(t-t')\begin{bmatrix}\boldsymbol{\epsilon}\mathbf{D}^{(1)}\boldsymbol{\epsilon}^T & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\epsilon}\mathbf{D}^{(2)}\boldsymbol{\epsilon}^T\end{bmatrix}\times\mathbf{R}(t-t')^T\ . \tag{B8}$$

By using the $2\times2$ blocks $\mathbf{R}^{(ks)}(t)$ of $\mathbf{R}(t)$, we obtain Eqs. (4.9) and (4.10).

The $2\times2$ matrices $\mathbf{R}^{(ks)}(t)$ can be defined equivalently as the solutions of the differential equations

$$\frac{d}{dt}\mathbf{R}^{(11)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]\mathbf{R}^{(11)}(t)+\lambda\boldsymbol{\epsilon}\mathbf{A}\mathbf{R}^{(21)}(t)\ , \tag{B9a}$$

$$\frac{d}{dt}\mathbf{R}^{(21)}(t)=\lambda\boldsymbol{\epsilon}\mathbf{A}^T\mathbf{R}^{(11)}(t)+\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{R}^{(21)}(t)\ , \tag{B9b}$$

$$\frac{d}{dt}\mathbf{R}^{(22)}(t)=\lambda\boldsymbol{\epsilon}\mathbf{A}^T\mathbf{R}^{(12)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{R}^{(22)}(t)\ , \tag{B10a}$$

$$\frac{d}{dt}\mathbf{R}^{(12)}(t)=\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]\mathbf{R}^{(12)}(t)+\lambda\boldsymbol{\epsilon}\mathbf{A}\mathbf{R}^{(22)}(t)\ , \tag{B10b}$$

with the initial conditions

$$\begin{aligned}&\mathbf{R}^{(11)}(0)=\mathbf{R}^{(22)}(0)=1\ ,\\ &\mathbf{R}^{(12)}(0)=\mathbf{R}^{(21)}(0)=0\ .\end{aligned} \tag{B11}$$

An equivalent set of differential equations is given by

$$\frac{d}{dt}\mathbf{R}^{(11)}(t)=\mathbf{R}^{(11)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]+\lambda\mathbf{R}^{(12)}(t)\boldsymbol{\epsilon}\mathbf{A}^T\ , \tag{B12a}$$

$$\frac{d}{dt}\mathbf{R}^{(12)}(t)=\lambda\mathbf{R}^{(11)}(t)\boldsymbol{\epsilon}\mathbf{A}+\mathbf{R}^{(12)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right], \tag{B12b}$$

$$\frac{d}{dt}\mathbf{R}^{(22)}(t)=\lambda\mathbf{R}^{(21)}(t)\boldsymbol{\epsilon}\mathbf{A}+\mathbf{R}^{(22)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right], \tag{B13a}$$

$$\frac{d}{dt}\mathbf{R}^{(21)}(t)=\mathbf{R}^{(21)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]+\lambda\mathbf{R}^{(22)}(t)\boldsymbol{\epsilon}\mathbf{A}^T\ . \tag{B13b}$$

As a fourth-order algebraic equation can be explicitly solved, the eigenvalues of the matrix $\mathbf{G}$ can be found and $\mathbf{R}(t)$ calculated. However, we treat only the weak-coupling case (4.54) and (4.55). Equations (B12) become

$$\frac{d}{dt}\mathbf{R}^{(11)}(t)=\mathbf{R}^{(11)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(1)}-\frac{\gamma_1}{2}\right]+\frac{1}{\lambda}\mathbf{R}^{(12)}(t)\boldsymbol{\epsilon}\mathbf{A}^T\ , \tag{B14a}$$

$$\frac{d}{dt}\mathbf{R}^{(12)}(t)=\frac{1}{\lambda^2}\mathbf{R}^{(12)}(t)\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]+\frac{1}{\lambda}\mathbf{R}^{(11)}(t)\boldsymbol{\epsilon}\mathbf{A}\ . \tag{B14b}$$

Equation (B14b) gives

$$\begin{aligned}\mathbf{R}^{(12)}(t)&=\frac{1}{\lambda}\int_0^t dt'\mathbf{R}^{(11)}(t')\boldsymbol{\epsilon}\mathbf{A}\exp\left[\frac{1}{\lambda^2}\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right](t-t')\right]\\ &\overset{\lambda\to0}{\simeq}\frac{1}{\lambda}\mathbf{R}^{(11)}(t)\boldsymbol{\epsilon}\mathbf{A}\int_0^t dt'\exp\left[\frac{1}{\lambda^2}\left[\boldsymbol{\epsilon}\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right](t-t')\right]\\ &\overset{\lambda\to0}{\simeq}\lambda\mathbf{R}^{(11)}(t)\boldsymbol{\epsilon}\mathbf{A}\left[\frac{\gamma_2}{2}-\boldsymbol{\epsilon}\mathbf{B}^{(2)}\right]^{-1}\ .\end{aligned} \tag{B15}$$

Using this result in Eq. (B14a), we obtain the equation

$$\frac{d}{dt}\mathbf{R}^{(11)}(t)=\mathbf{R}^{(11)}(t)\left[\epsilon\mathbf{B}^{(1)}-\frac{\gamma_1}{2}+\epsilon\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\epsilon\mathbf{A}^T\right], \tag{B16}$$

whose solution is

$$\mathbf{R}^{(11)}(t)=\exp\left\{\left[\epsilon\mathbf{B}^{(1)}-\frac{\gamma_1}{2}+\epsilon\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\epsilon\mathbf{A}^T\right]t\right\}. \tag{B17}$$

Equation (B10a) becomes

$$\frac{d}{dt}\mathbf{R}^{(22)}(t)=\frac{1}{\lambda}\epsilon\mathbf{A}^T\mathbf{R}^{(12)}(t)+\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{R}^{(22)}(t)$$
$$\overset{\lambda\to 0}{\simeq}\epsilon\mathbf{A}^T\mathbf{R}^{(11)}(t)\epsilon\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}+\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{R}^{(22)}(t), \tag{B18}$$

which gives

$$\mathbf{R}^{(22)}(t)\overset{\lambda\to 0}{\simeq}\exp\left[\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]t\right]+\int_0^t dt'\exp\left[\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right](t-t')\right]\times\epsilon\mathbf{A}^T\mathbf{R}^{(11)}(t')\epsilon\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}$$
$$\overset{\lambda\to 0}{\simeq}\exp\left[\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]t\right]. \tag{B19}$$

Finally, Eq. (B9b) becomes

$$\frac{d}{dt}\mathbf{R}^{(21)}(t)=\frac{1}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\mathbf{R}^{(21)}(t)+\frac{1}{\lambda}\epsilon\mathbf{A}^T\mathbf{R}^{(11)}(t), \tag{B20}$$

which gives

$$\mathbf{R}^{(21)}(t)\overset{\lambda\to 0}{\simeq}\lambda\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\epsilon\mathbf{A}^T\mathbf{R}^{(11)}(t). \tag{B21}$$

Using these results the mean values (4.9) become

$$\mathbf{Q}^{(1)}(t)=\widetilde{\mathbf{R}}^{(11)}(t)\mathbf{Q}^{(1)}(0), \tag{B22a}$$

$$\mathbf{Q}^{(2)}(t)=\exp\left[\frac{t}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\right]\mathbf{Q}^{(2)}(0), \tag{B22b}$$

where

$$\widetilde{\mathbf{R}}^{(11)}(t)=\exp\left[\left[\epsilon\widetilde{\mathbf{B}}-\frac{\widetilde{\gamma}}{2}\right]t\right], \tag{B23a}$$

$$\widetilde{\mathbf{B}}=\widetilde{\mathbf{B}}^{(1)}-\left[\frac{\gamma_2{}^2}{2}+2\omega_2{}^2\right]^{-1}\mathbf{A}\epsilon\mathbf{B}^{(2)}\epsilon^T\mathbf{A}^T, \tag{B23b}$$

$$\widetilde{\gamma}=\gamma_1+\gamma_2\left[\frac{\gamma_2{}^2}{4}+\omega_2{}^2\right]^{-1}\det\mathbf{A}, \tag{B23c}$$

$$\omega_2{}^2=\det\mathbf{B}^{(2)}, \tag{B23d}$$

$$\epsilon\widetilde{\mathbf{B}}-\frac{\widetilde{\gamma}}{2}=\epsilon\mathbf{B}^{(1)}-\frac{\gamma_1}{2}+\epsilon\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\epsilon\mathbf{A}^T. \tag{B23e}$$

For the correlation matrices (4.10) we obtain

$$\boldsymbol{\sigma}^{(11)}(t)\simeq\widetilde{\mathbf{R}}^{(11)}(t)\boldsymbol{\sigma}^{(11)}(0)\widetilde{\mathbf{R}}^{(11)}(t)+\tfrac{1}{2}\int_0^t dt'\widetilde{\mathbf{R}}^{(11)}(t')\epsilon\widetilde{\mathbf{D}}\epsilon^T\widetilde{\mathbf{R}}^{(11)}(t')^T, \tag{B24a}$$

$$\boldsymbol{\sigma}^{(12)}(t)\simeq\widetilde{\mathbf{R}}^{(11)}(t)\boldsymbol{\sigma}^{(12)}(0)\exp\left[\frac{t}{\lambda^2}\left[\mathbf{B}^{(2)}\epsilon^T-\frac{\gamma_2}{2}\right]\right], \tag{B24b}$$

$$\boldsymbol{\sigma}^{(22)}(t)\simeq\exp\left[\frac{t}{\lambda^2}\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\right]\boldsymbol{\sigma}^{(22)}(0)\exp\left[\frac{t}{\lambda^2}\left[\mathbf{B}^{(2)}\epsilon^T-\frac{\gamma_2}{2}\right]\right]+\tfrac{1}{2}\int_0^{t/\lambda^2}d\tau\exp\left[\tau\left[\epsilon\mathbf{B}^{(2)}-\frac{\gamma_2}{2}\right]\right]\epsilon\mathbf{D}^{(2)}\epsilon^T\exp\left[\tau\left[\mathbf{B}^{(2)}\epsilon^T-\frac{\gamma_2}{2}\right]\right], \tag{B24c}$$

where

$$\widetilde{\mathbf{D}}=\mathbf{D}^{(1)}+\mathbf{A}\left[\frac{\gamma_2}{2}-\epsilon\mathbf{B}^{(2)}\right]^{-1}\epsilon\mathbf{D}^{(2)}\epsilon^T\left[\frac{\gamma_2}{2}-\mathbf{B}^{(2)}\epsilon^T\right]^{-1}\mathbf{A}^T. \tag{B25}$$

## APPENDIX C

Consider an unbiased linear estimator $F(t)$ for the force $f(t)$ [Eqs. (5.5) and (5.6)]. The estimator is a stochastic process whose characteristics functional $L_F[\eta]$ can be deduced from the functional for $\mathbf{z}(t)$ using the dependence of $F(t)$ on

z. Therefore, $L_F[\eta]$ is given by Eqs. (3.22)—(3.27) where $\boldsymbol{\varphi}$ is a certain functional of $\eta$. Now, by Eq. (5.6), the force must appear in $\ln L_F[\eta]$ in the form

$$i\int_0^{+\infty} dt\, f(t)\eta(t) ;$$

by comparing this with Eq. (3.24), we obtain

$$\eta(t)=\mathbf{u}^T\boldsymbol{\epsilon}\boldsymbol{\alpha}^{(1)}(t) . \tag{C1}$$

By a canonical transformation on system 1 and by rescaling $f(t)$, we can always obtain $u_1=1$, $u_2=0$; so, Eq. (C1) becomes

$$\alpha_2^{(1)}(t)=\eta(t) . \tag{C2}$$

By using Eq. (C2), Eqs. (3.25) and (3.26) become

$$\frac{d}{dt}\alpha_1^{(1)}(t)=\left[\frac{\gamma_1}{2}-B_{12}^{(1)}\right]\alpha_1^{(1)}(t)+B_{11}^{(1)}\eta(t)-\lambda A_{12}(t)\alpha_1^{(2)}(t)+\lambda A_{11}(t)\alpha_2^{(2)}(t) , \tag{C3}$$

$$\frac{d}{dt}\eta(t)=\left[\frac{\gamma_1}{2}+B_{12}^{(1)}\right]\eta(t)-B_{22}^{(1)}\alpha_1^{(1)}(t)-\lambda A_{22}(t)\alpha_1^{(2)}(t)+\lambda A_{21}(t)\alpha_2^{(2)}(t) , \tag{C4}$$

$$\frac{d}{dt}\alpha_1^{(2)}(t)=\left[\frac{\gamma_2}{2}-B_{12}^{(2)}\right]\alpha_1^{(2)}(t)+B_{11}^{(2)}\alpha_2^{(2)}(t)-\lambda A_{21}(t)\alpha_1^{(1)}(t)+\lambda A_{11}(t)\eta(t)-\varphi_1(t) , \tag{C5}$$

$$\frac{d}{dt}\alpha_2^{(2)}(t)=\left[\frac{\gamma_2}{2}+B_{12}^{(2)}\right]\alpha_2^{(2)}(t)-B_{22}^{(2)}\alpha_1^{(2)}(t)-\lambda A_{22}(t)\alpha_1^{(1)}(t)+\lambda A_{12}(t)\eta(t)-\varphi_2(t) . \tag{C6}$$

From Eq. (C4), by assuming $B_{22}^{(1)}\neq 0$, we obtain

$$\alpha_1^{(1)}(t)=\frac{1}{B_{22}^{(1)}}\left[-\frac{d}{dt}\eta(t)+\left[\frac{\gamma_1}{2}+B_{12}^{(1)}\right]\eta(t)-\lambda A_{22}(t)\alpha_1^{(2)}(t)+\lambda A_{21}(t)\alpha_2^{(2)}(t)\right] \tag{C7}$$

and we insert this result into the other equations.

From Eq. (C3), using Eqs. (C5) and (C6), we have

$$\alpha_2^{(2)}(t)=\zeta(t)+X(t)\alpha_1^{(2)}(t) , \tag{C8}$$

where

$$X(t)\equiv Y(t)^{-1}\left\{\frac{d}{dt}A_{22}(t)-B_{22}^{(1)}A_{12}(t)+A_{21}(t)B_{22}^{(2)}-A_{22}(t)\left[\frac{\gamma_2-\gamma_1}{2}+B_{12}^{(1)}-B_{12}^{(2)}\right]\right\} , \tag{C9}$$

$$\zeta(t)\equiv Y(t)^{-1}\left\{\frac{1}{\lambda}\left[\frac{d^2}{dt^2}\eta(t)-\gamma_1\frac{d}{dt}\eta(t)+\left[\frac{\gamma_1{}^2}{4}+\omega_1{}^2\right]\eta(t)\right]+\lambda[\det\mathbf{A}(t)]\eta(t)-A_{22}(t)\varphi_1(t)+A_{21}(t)\varphi_2(t)\right\} , \tag{C10}$$

$$Y(t)\equiv\frac{d}{dt}A_{21}(t)-A_{22}(t)B_{11}^{(2)}-B_{22}^{(1)}A_{11}(t)+A_{21}(t)\left[\frac{\gamma_2-\gamma_1}{2}+B_{12}^{(1)}+B_{12}^{(2)}\right] . \tag{C11}$$

Inserting Eq. (C8) into Eq. (C6) and using Eq. (C5), we obtain an expression for $\alpha_1^{(2)}(t)$; we do not write the result: $\alpha_1^{(2)}(t)$ is a linear combination of $\eta(t)$, $\varphi_1(t)$, $\varphi_2(t)$ and their derivatives at time $t$. Finally, by putting this result into Eq. (C5), we obtain a link between $\eta,\varphi_1,\varphi_2$; thus, one component of $\boldsymbol{\varphi}$, say $\varphi_2$, is an arbitrary functional of $\eta$, while $\varphi_1$ depends on $\eta$ directly and through $\varphi_2$. In particular, recalling that test functions vanish at $t=0$ with all their derivatives, we obtain $\alpha_1^{(2)}(0)=0$ and, by Eqs. (C8), (C7), and (C2), also $\alpha_2^{(2)}(0)=\alpha_1^{(1)}(0)=\alpha_2^{(1)}(0)=0$.

In order to give an example of these calculations, let us consider the case of Eqs. (3.8), (3.13), (3.15), and (5.9). First of all, Eq. (C2) holds; then Eqs. (C3)—(C6) can be rewritten as

$$\frac{d}{dt}\{e^{-\gamma_1 t/2}[\cos\omega_1 t\,\alpha_1^{(1)}(t)-m_1\omega_1\sin\omega_1 t\,\eta(t)]\}=\lambda e^{-\gamma_1 t/2}\alpha_2^{(2)}(t) , \tag{C12}$$

$$\frac{d}{dt}\left[e^{-\gamma_1 t/2}\left[\frac{\sin\omega_1 t}{m_1\omega_1}\alpha_1^{(1)}(t)+\cos\omega_1 t\,\eta(t)\right]\right]=0 , \tag{C13}$$

$$\frac{d}{dt}\alpha_1^{(2)}(t)=\frac{\gamma_2}{2}\alpha_1^{(2)}(t)+m_2\omega_2{}^2\alpha_2^{(2)}(t)+\lambda\left[\frac{\sin\omega_1 t}{m_1\omega_1}\alpha_1^{(1)}(t)+\cos\omega_1 t\,\eta(t)\right]-\varphi_1(t) , \tag{C14}$$

$$\frac{d}{dt}\alpha_2^{(2)}(t)=\frac{\gamma_2}{2}\alpha_2^{(2)}(t)-\frac{1}{m_2}\alpha_1^{(2)}(t)-\varphi_2(t) . \tag{C15}$$

Equation (C13) with the final condition (3.27) gives

$$\alpha_1^{(1)}(t)=-m_1\omega_1\frac{\cos\omega_1 t}{\sin\omega_1 t}\eta(t) . \tag{C16}$$

Then, from Eq. (C12) we obtain

$$\alpha_2^{(2)}(t)=\frac{m_1\omega_1}{\lambda\sin\omega_1 t}\left[\left[\omega_1\cos\omega_1 t+\frac{\gamma_1}{2}\sin\omega_1 t\right]\frac{\eta(t)}{\sin\omega_1 t}-\frac{d}{dt}\eta(t)\right] . \tag{C17}$$

Finally, from Eqs. (C14) and (C15) we have

$$\alpha_1^{(2)}(t)=m_2\left[\frac{\gamma_2}{2}\alpha_2^{(2)}(t)-\frac{d}{dt}\alpha_2^{(2)}(t)\right]-m_2\varphi_2(t) , \tag{C18}$$

$$\varphi_1(t)=m_2\left[\frac{d^2}{dt^2}\alpha_2^{(2)}(t)-\gamma_2\frac{d}{dt}\alpha_2^{(2)}(t)+\left[\frac{\gamma_2{}^2}{4}+\omega_2{}^2\right]\alpha_2^{(2)}(t)\right]+m_2\left[\frac{d}{dt}\varphi_2(t)-\frac{\gamma_2}{2}\varphi_2(t)\right] . \tag{C19}$$

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