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Comment on the Born approximation in Aharonov-Bohm scattering

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The cylindrically symmetric partial-wave amplitude for scattering of an electron on an impenetrable infinite solenoid of finite radius is studied for small magnetic-flux values and an electron wavelength much larger than the solenoid radius. The result of a recent study for the Born series expression of this amplitude is discussed.

I. INTRODUCTION

As was first noted by Feinberg,[1] the Born approximation fails to give the angular behavior of the Aharonov-Bohm amplitude[2] for scattering of an electron on an infinite magnetic-flux line (flux Φ, we put $\alpha=-e\Phi/hc$). Instead of the correct behavior, $e^{-i\theta/2}/\sin\frac{1}{2}\theta=\cot\frac{1}{2}\theta-i$, the Born approximation gives in order α the result $\cot\frac{1}{2}\theta$. θ is the scattering angle. The reason for this discrepancy is known[3,4] to be the fact that the isotropic $m=0$ cylindrical-wave Born contribution, with perturbation α^2/r^2, vanishes in order α, and is hence absent from the first-order Born approximation. At the same time this perturbation gives a divergent matrix element. The true $m=0$ contribution goes as $|\alpha|$ for small α, and is hence nonanalytic at zero.

In a recent paper Aharonov, Au, Lerner, and Liang[5] studied the Born series for the case of a flux cylinder (solenoid) with an infinite barrier of radius R. They show that although the lowest-order $m=0$ Born approximation, which converges for $R\neq 0$, is of order α^2, as expected, the Born series is essentially an expansion in powers of $\alpha\ln(2/kR)$, where k is the momentum of the electron. Summation of the Born series and going to the limit $R=0$ would then give rise to a leading contribution proportional to $|\alpha|$.

Although Ref. 5 qualitatively explains the behavior of the $m=0$ amplitude, the quantitative result is incomplete as a consequence of some unallowed approximations in the study of the scattering integral equation for the $m=0$ wave.

The purpose of this Comment is to study the behavior of the $m=0$ partial-wave scattering amplitude for small α and kR and in particular how the α^2 behavior for $kR\neq 0$ goes over into the $|\alpha|$ behavior for $kR=0$. This is done in Sec. II. In Sec. III we complement the Born series treatment of Ref. 5.

First, we remark on the symmetry of the scattering solution: Although the scattering equation is invariant under rotations around the z axis (the axis of the flux cylinder), it is not invariant under reflection in a plane through this axis, unless one combines the reflection with a change of sign of α.[6] This implies for the scattering phase shifts $\delta_m(\alpha)=\delta_{-m}(-\alpha)$. In particular, $\delta_0(\alpha)=\delta_0(-\alpha)$, the $m=0$ phase shift is an even function of α. In the $R=0$ limit we have the well-known result (assuming $|\alpha|<1$, see below)

$$\delta_m(\alpha)=-\frac{\pi}{2}\alpha\,\mathrm{sgn}m \text{ for } m\neq 0, \text{ and } \delta_0(\alpha)=-\frac{\pi}{2}|\alpha| .$$

A further remark is that since the scattering cross section is periodic in α with period one—more precisely one can show for the scattering solution that $\psi(r,\theta,\alpha+1)=e^{i(\pi-\theta)}\psi(r,\theta,\alpha)$—one can restrict the study to values of α such that $|\alpha|\leq\frac{1}{2}$.

II. BEHAVIOR OF THE AMPLITUDE FOR SMALL α AND kR

The expression for the $m=0$ amplitude, or rather the corresponding S-matrix element, is (see e.g., Ref. 4)

$$\sqrt{2\pi ik}\,f_0(\alpha,k,R)+1=S_0(\alpha,kR)=e^{i2\delta_0(\alpha,kR)}=-e^{-i\pi|\alpha|}\frac{H^{(2)}_{|\alpha|}(kR)}{H^{(1)}_{|\alpha|}(kR)} . \quad (2.1)$$

Using standard relations for Hankel functions one can write

$$S_0(\alpha,kR)=-\frac{H^{(2)}_{-\alpha}(kR)}{H^{(1)}_{\alpha}(kR)}=-\frac{H^{(2)}_{\alpha}(kR)}{H^{(1)}_{-\alpha}(kR)} . \quad (2.2)$$

Since $H^{(i)}_\alpha(kR)$ is, for $kR\neq 0$, an entire function of α, (2.2) shows that $S_0(\alpha,kR)$ is for $kR\neq 0$ an even function of α with a power-series expansion convergent up to the first zero of $H^{(1)}_\alpha(kR)$ as function of α. Assuming $\ln(2/kR)>>1$ one can derive from (2.5) below a radius of convergence approximately equal to $\pi/\{[\ln(2/\gamma kR)]^2+\pi^2/4\}^{1/2}$ (for the constant γ see below). From the small-kR expression for $H^{(i)}_\alpha(kR)$ we get, ignoring terms of order $(kR)^2$,

$$S_0(\alpha,kR)=\frac{\Gamma(1+\alpha)\exp\{\alpha[\ln(2/kR)-i\pi/2]\}-\Gamma(1-\alpha)\exp\{-\alpha[\ln(2/kR)-i\pi/2]\}}{\Gamma(1+\alpha)\exp\{\alpha[\ln(2/kR)+i\pi/2]\}-\Gamma(1-\alpha)\exp\{-\alpha[\ln(2/kR)+i\pi/2]\}} . \quad (2.3)$$

The standard infinite product representation

$$\Gamma(1+\alpha)=e^{-C\alpha} \Big/ \prod_{n=1}^{\infty} [(1+\alpha/n)e^{-\alpha/n}]$$

easily leads to

$$\Gamma(1+\alpha)=e^{-C\alpha}(1+c_2\alpha^2+c_4\alpha^4)[1-c_3\alpha^3+O(\alpha^5)] \quad , \quad (2.4)$$

where $C=\ln\gamma=\ln 1.7811\ldots=0.5772\ldots$ and

$$c_3=\frac{1}{3}\sum_{n=1}^{\infty} 1/n^3=0.4007\ldots .$$

Introducing (2.4) into (2.3) one obtains, ignoring order α^5,

$$S_0(\alpha,kR)=\frac{\sinh\alpha z^*-c_3\alpha^3\cosh\alpha z^*}{\sinh\alpha z-c_3\alpha^3\cosh\alpha z} \quad , \quad (2.5)$$

where $z=\ln(2/\gamma kR)+i\pi/2$.

Expressing the hyperbolic functions of complex arguments in hyperbolic and trigonometric functions of real variables, and going to the phase shift [see (2.1)] we have, still up to terms of order α^5,

$$\delta_0(\alpha,kR)=-\arctan\{\tan(\alpha\pi/2)\coth[\alpha\ln(2/\gamma kR)-c_3\alpha^3]\} \quad . \quad (2.6)$$

Recall that in (2.5) and (2.6) we have ignored terms of order $(kR)^2$ and α^5. It should be observed that although $(kR)^2 << 1$, we can well have $\ln(2/\gamma kR)$ not much larger than 1; e.g., if $kR=10^{-2}$ we have $\ln(2/\gamma kR)=4.72$.

Expanding (2.5) to relative order α^2, assuming $kR\neq 0$, we have

$$S_0(\alpha,kR)=\frac{\ln(2/\gamma kR)-i\pi/2}{\ln(2/\gamma kR)+i\pi/2}\left[1-i\alpha^2\frac{\pi}{3}\left(\ln(2/\gamma kR)+\frac{3c_3}{[\ln(2/\gamma kR)]^2+\pi^2/4}\right)+O(\alpha^4)\right] \quad . \quad (2.7)$$

In the limit $kR\to 0$, i.e., $\ln(2/\gamma kR)\to\infty$, we get from (2.3)

$$S_0(\alpha,0)=e^{-i\pi|\alpha|}, \quad \text{or } \delta_0(\alpha,0)=-\frac{\pi}{2}|\alpha| \quad . \quad (2.8)$$

If $\alpha\to 0$ in (2.3) or (2.7) we get

$$S_0(0,kR)=\frac{\ln(2/\gamma kR)-i\pi/2}{\ln(2/\gamma kR)+i\pi/2} \quad . \quad (2.9)$$

Ignoring order α^3 in (2.6) we obtain

$$\delta_0(\alpha,kR)=-\arctan\{(\pi/2)|\alpha|\coth[|\alpha|\ln(2/\gamma kR)]\} \quad . \quad (2.10)$$

Equation (2.10) shows the transition between the two limiting cases (2.8) and (2.9) as $|\alpha|\ln(2\gamma kR)$ goes from large values ($>>1$) to small values ($<<1$). Quantitatively, (2.10) shows that for (2.8) to be a good approximation we must have $|\alpha|\ln(2/\gamma kR)>c$, where c is at least 2 or 3. This means $kR<e^{-c/|\alpha|}$; thus for small α the flux line limit $kR=0$ is reached extremely slowly.

III. COMPARISON WITH AND DISCUSSION OF RESULT IN REFERENCE 5

The expression (39) in Ref. 5 for the Aharonov-Bohm contribution to the scattering can be written

$$\sqrt{2\pi ik}\,[f_0(\alpha,k,R)-f_0(0,k,R)]$$
$$=S_0(\alpha,kR)-S_0(0,kR)=-i\pi\left(\alpha-\alpha\frac{(kR/2)^\alpha}{\Gamma(1+\alpha)}\right) \quad . \quad (3.1)$$

[We have introduced factors $\sqrt{2\pi ik}$ and $-i\pi$ omitted in (39) of Ref. 5; also, $\alpha>0$ is assumed in (3.1), in the general case $|\alpha|$ should be substituted for α.]

Equation (3.1) gives the correct limit $-i\pi\alpha$ as $kR\to 0$, and also qualitatively the α^2 behavior as $\alpha\to 0$ for $kR\neq 0$, but does not address higher corrections.

For small α we can use $\Gamma(1+\alpha)=e^{-C\alpha}$. Introducing for short $\beta=|\alpha|\ln(2/\gamma kR)$ we can then write (3.1) as

$$-i\pi|\alpha|(1-e^{-\beta}) \quad . \quad (3.2)$$

On the other hand, from (2.5) we get, ignoring order α^2,

$$S_0(\alpha,kR)-S_0(0,kR)=\frac{\sinh\alpha z^*}{\sinh\alpha z}-\frac{z^*}{z}$$
$$=-i\pi|\alpha|\frac{1}{\beta}(\beta\coth\beta-1) \quad . \quad (3.3)$$

Here besides $\alpha<<1$ we have assumed $\ln(2/\gamma kR)>>1$, so that $i\pi/2$ is ignored in comparison; $\beta=|\alpha|\ln(2/\gamma kR)$ can take any (non-negative) value.

For small β, (3.2) gives $-i\alpha^2\pi\ln(2/\gamma kR)$ compared to the correct value $-i\alpha^2(\pi/3)\ln(2/\gamma kR)$ obtained from (3.3) [cf. also (2.7)]. Equation (3.2) fails also for large β: (3.3) gives for $\beta>>1$ the expression

$$-i\pi|\alpha|(1+2e^{-2\beta}-1/\beta) \quad ,$$

to be compared with (3.2). This means that the approach to the line flux limit is much slower as $kR\to 0$ than suggested by (3.2). The difference goes inversely as β instead of decaying exponentially.

To trace the origin of the quantitative discrepancy between the results of the present paper and of Ref. 5, we look at the integral equation for the $m=0$ partial-wave function used in Ref. 5, Eqs. (21) and (22),

$$\phi_0(r)=a_0[N_0(kR)J_0(kr)-J_0(kR)N_0(kr)]+\alpha^2\frac{\pi}{2}\int_R^r [N_0(kr)J_0(kr')-J_0(kr)N_0(kr')]\frac{1}{r'}\phi_0(r')dr' \quad . \quad (3.4)$$

The constant a_0 should be determined in such a way that asymptotically for large r

$$\phi_0(r)\approx J_0(kr)+f_0e^{ikr}/\sqrt{r} \quad . \quad (3.5)$$

This leads to

$$a_0=\frac{i}{H_0^{(1)}(kR)}\left[1-i\alpha^2\frac{\pi}{2}\int_R^\infty H_0^{(1)}(kr)\frac{1}{r}\phi_0(r)dr\right] ,$$

and

$$\sqrt{2\pi ik}f_0(\alpha,k,R)+1=S_0(\alpha,kR)=-\frac{H_0^{(2)}(kR)}{H_0^{(1)}(kR)}+\alpha^2\pi\frac{1}{H_0^{(1)}(kR)}\int_R^\infty [N_0(kR)J_0(kr)-J_0(kR)N_0(kr)]\frac{1}{r}\phi_0(r)dr . \tag{3.6}$$

The second term on the right-hand side of (3.6) is just $S_0(\alpha,kR)-S_0(0,kR)$. Using the exact $m=0$ partial-wave function [cf. (2) and (4) of Ref. 5],

$$\phi_0(r)=\frac{ie^{-i\pi|\alpha|/2}}{H_{|\alpha|}^{(1)}(kR)}[N_{|\alpha|}(kR)J_{|\alpha|}(kr)-J_{|\alpha|}(kR)N_{|\alpha|}(kr)] , \tag{3.7}$$

we get

$$S_0(\alpha,kR)-S_0(0,kR)=i\alpha^2\pi\frac{e^{-i\pi|\alpha|/2}}{H_0^{(1)}(kR)H_{|\alpha|}^{(1)}(kR)}\int_R^\infty [N_0(kR)J_0(kr)-J_0(kR)N_0(kr)] \times[N_{|\alpha|}(kR)J_{|\alpha|}(kr)-J_{|\alpha|}(kR)N_{|\alpha|}(kr)]\frac{1}{r}dr . \tag{3.8}$$

The lowest-order, i.e., order α^2, Born approximation corresponds to using the unperturbed solution $\phi_0(r)_{\alpha=0}$ from (3.7) in (3.6). This gives

$$S_0(\alpha,kR)-S_0(0,kR)\approx i\alpha^2\pi\frac{1}{[H_0^{(1)}(kR)]^2}\int_R^\infty [N_0(kR)J_0(kr)-J_0(kR)N_0(kr)]^2\frac{1}{r}dr . \tag{3.9}$$

Now the argument in Ref. 5 is that for $kR \ll 1$ we can ignore the functions $J_0(kR)$ and $J_{|\alpha|}(kR)$ compared to $N_0(kR)$ and $N_{|\alpha|}(kR)$ in (3.8). Though this is alright for the factors $H_0^{(1)}(kR)$ and $H_{|\alpha|}^{(1)}(kR)$, it is not allowed in the integrand. The reason for this is somewhat subtle: For a fixed $\alpha\neq 0$ it is correct that of the two terms multiplying $N_0(kR)$ in the expansion of the integral in (3.8) the term

$$J_{|\alpha|}(kR)\int_R^\infty J_0(kr)N_{|\alpha|}(kr)\frac{1}{r}dr ,$$

ignored in Ref. 5, vanishes as $kR\to 0$ compared to the term retained,

$$N_{|\alpha|}(kR)\int_R^\infty J_0(kr)J_{|\alpha|}(kr)\frac{1}{r}dr .$$

The same conclusion is true also for the remaining two terms, and corresponds to the fact that the result (3.1) gives the correct limiting value when $kR=0$. However, the conclusion above is not valid uniformly in α as α also goes to zero. We shall see below that for $\alpha=0$ all the terms in the expansion of the integral in (3.9) give contributions of the same order in $\ln(2/\gamma kR)$.

Using the indefinite-integral formula valid for any two Bessel functions $W_0(x)$ and $w_\alpha(x)$,[7]

$$\alpha^2\int W_0(x)w_\alpha(x)\frac{dx}{x}=x[W_0(x)w_\alpha'(x)-W_0'(x)w_\alpha(x)] , \tag{3.10}$$

we can derive for (3.8) the expression

$$S_0(\alpha,kR)-S_0(0,kR)=-e^{-i\pi|\alpha|}\frac{H_{|\alpha|}^{(2)}(kR)}{H_{|\alpha|}^{(1)}(kR)}+\frac{H_0^{(2)}(kR)}{H_0^{(1)}(kR)} , \tag{3.11}$$

as should be [cf. (2.1)].

The integral in the lowest-order Born approximation (3.9) gives, using the notation $kR=\epsilon$ for short,

$$N_0(\epsilon)^2\int_\epsilon^\infty J_0(x)^2\frac{dx}{x}-2N_0(\epsilon)J_0(\epsilon) \times\int_\epsilon^\infty J_0(x)N_0(x)\frac{dx}{x}+J_0(\epsilon)^2\int_\epsilon^\infty N_0(x)^2\frac{dx}{x} . \tag{3.12}$$

Here the leading $\ln(2/\gamma\epsilon)$ contributions to the integrals are

$$\int_\epsilon^\infty J_0(x)^2\frac{dx}{x}\approx\ln(2/\gamma\epsilon) ,$$

$$\int_\epsilon^\infty J_0(x)N_0(x)\frac{dx}{x}\approx-\frac{1}{\pi}[\ln(2/\gamma\epsilon)]^2 ,$$

$$\int_\epsilon^\infty N_0(x)^2\frac{dx}{x}\approx\frac{4}{\pi^2}\frac{1}{3}[\ln(2/\gamma\epsilon)]^3 .$$

Since $N_0(\epsilon)\approx-(2/\pi)\ln(2/\gamma\epsilon)J_0(\epsilon)$ this means that the first and second contributions in (3.12) cancel, and the whole contribution in leading $\ln(2/\gamma\epsilon)$ order comes from the third integral. This gives the α^2 contribution in (2.7). The approximation of Aharonov *et al.* consists in retaining only the first integral in (3.12).

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[5]Y. Aharonov, C. K. Au, E. C. Lerner, and J. Q. Liang, Phys. Rev. D **29**, 2396 (1984).

[6]Compare the discussion in W. C. Henneberger, Phys. Rev. A **22**, 1383 (1980), p. 1386 bottom and p. 1387 top; the z component of the angular momentum is a good quantum number as a consequence of the SO(2) symmetry, whereas the relation $\psi_{k,m}(r,\theta)=\psi_{k,-m}(r,-\theta)$ should be changed to $\psi_{k,m}(r,\theta,\alpha)=\psi_{k,-m}(r,-\theta,-\alpha)$.

[7]See, e.g., *Higher Transcendental Functions,* edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. II, p. 90, formula (8).