Generalized Klein-Gordon equations in d dimensions from supersymmetry

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The Wess-Zumino model is extended to higher dimensions, leading to a generalized Klein-Gordon equation whose propagator is computed in configuration space.

In general, when extending field theory for an arbitrary number of dimensions, the kinetic part of the evolution equation is kept as the usual second-order Klein-Gordon equation. This is not a unique generalization. There are other prescriptions which emerge naturally when one tries to make a straightforward extension of supersymmetry in higher dimensions.[1] We adopt here the extension which is called "alternative I" in this reference.

This kind of extension may look reasonable, in spite of the ghost which appears, when one thinks about the difficulties that are encountered when not adopting it, as is usually done.[2,3]

It is with this idea that we shall take the simple Wess-Zumino model[4] and extend it to higher dimensions. To avoid unnecessary technicalities, we shall take the number of dimensions as

$$d=4\nu \quad . \tag{1}$$

The number of components of a Weyl spinor[5] is

$$\omega=2^{d/2-1}=2^{2\nu-1} \quad . \tag{2}$$

The generators of simple supersymmetry obey the usual commutation relations. To these generators there correspond ω Grassmann variables θ^α, and the ω conjugates $\bar{\theta}^\alpha$. Just as in four dimensions, we can define superfields and represent the generators as derivative operators acting on them. Also, in the usual way, we can define the covariant derivatives[6] D_α and $D_{\dot{\alpha}}$ leading to the definitions of chiral fields as the solutions of

$$\bar{D}_{\dot{\alpha}}\phi=0 \quad . \tag{3}$$

From this,

$$\phi=\exp\left(\frac{i}{2}\theta^\alpha\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\right)\phi_0(x,\theta) \quad , \tag{4}$$

$$\phi_0=\sum_{s=0}^{\omega}\frac{1}{s!}\theta^{\alpha_1}\cdots\theta^{\alpha_s}\psi_{\alpha_1\cdots\alpha_s}(x) \quad . \tag{5}$$

As usual, we define

$$A(x)=\psi(x) \quad ,$$
$$\psi_{\alpha_1\cdots\alpha_\omega}=\epsilon_{\alpha_1\cdots\alpha_\omega}F(x) \quad . \tag{6}$$

As in four dimensions, the variation of the highest component under a supersymmetric transformation is a divergence, so the Lagrangian for a chiral superfield can be written in the usual way as

$$L=\bar{\phi}\phi|_D+c\phi^2|_F+\text{H.c.}+\text{interaction terms} \quad . \tag{7}$$

As the mass dimension of L is d, from the first term of the Lagrangian we deduce ($[x]$ means the dimension of χ)

$$[\phi]=\frac{d-\omega}{2} \quad , \tag{8}$$

$$[c]=\frac{\omega}{2} \quad . \tag{9}$$

So we can write

$$c=\tfrac{1}{2}m^{\omega/2} \quad . \tag{10}$$

Now we can use (4) and (5) to write (7) explicitly in component form. Noting that

$$\theta^{\alpha_1}\cdots\theta^{\alpha_\omega}=\epsilon^{\alpha_1\cdots\alpha_\omega}\theta^\omega \quad , \tag{11}$$

from

$$L=\bar{\phi}_0\phi_0|_D+\bar{\phi}_0\bar{\theta}\partial_0\theta\phi_0|_D+\bar{\phi}_0\tfrac{1}{2}(i\theta\partial_0\theta)^2\phi_0|_D+\cdots$$
$$+\tfrac{1}{2}m^{\omega/2}\phi_0\phi_0|_F+\tfrac{1}{2}m^{\omega/2}\bar{\phi}_0\bar{\phi}_0+\text{inter. terms} \quad , \tag{12}$$

we deduce

$$L=A^*\Box^{\omega/2}A+\bar{\psi}^{\dot{\alpha}}\Box^{\omega/2-1}i\partial_{\alpha\dot{\alpha}}\psi^\alpha+\tfrac{1}{2}\bar{\psi}^{\dot{\alpha}_1\dot{\alpha}_2}\Box^{\omega/2-2}i\partial_{\alpha_1\dot{\alpha}_1}i\partial_{\alpha_2\dot{\alpha}_2}\psi^{\alpha_1\alpha_2}+\cdots+F^*F$$
$$+m^{\omega/2}\left(AF+\psi_{\alpha_1}\psi_{\alpha_2\cdots\alpha_\omega}\epsilon^{\alpha_1\cdots\alpha_\omega}\frac{1}{(\omega-1)!}+\frac{\psi_{\alpha_1\alpha_2}}{2!(\omega-2)!}\psi_{\alpha_3\cdots\alpha_\omega}\epsilon^{\alpha_1\cdots\alpha_\omega}+\cdots+\text{H.c.}\right) \quad , \tag{13}$$

where we have used

$$\epsilon^{\alpha_1\cdots\alpha_\omega}\partial_{\alpha_1\dot\alpha_1}\cdots\partial_{\alpha_\omega\dot\alpha_\omega}=\epsilon_{\dot\alpha_1\cdots\dot\alpha_\omega}\Box^{\omega/2}\ ,$$
$$\partial_{\alpha\dot\alpha}\partial^{\beta\dot\alpha}=-\Box\delta^\beta_\alpha,\quad \partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}=-\Box\epsilon_{\alpha\beta}\ , \tag{14}$$

from which we deduce

$$\epsilon^{\alpha_1\cdots\alpha_\omega}\epsilon^{\dot\alpha_1\cdots\dot\alpha_\omega}\partial_{\alpha_1\dot\alpha_1}\cdots\partial_{\alpha_\omega\dot\alpha_\omega}=\omega!\Box^{\omega/2}\ . \tag{15}$$

From the Lagrangian (13) we immediately get the wave equations

$$m^{\omega/2}F^*+\Box^{\omega/2}A=0,\quad F^*+m^{\omega/2}A=0\ , \tag{16}$$

$$m^{\omega/2}\bar\psi_{\dot\alpha_2\cdots\dot\alpha_\omega}\epsilon^{\dot\alpha_1\cdots\dot\alpha_\omega}-\Box^{\omega/2-1}i\partial_{\alpha_1}{}^{\dot\alpha_1}\psi^{\alpha_1}=0\ , \tag{17}$$

$$i\partial_{\alpha_1\dot\alpha_1}\bar\psi_{\dot\alpha_2\cdots\dot\alpha_\omega}\epsilon^{\dot\alpha_1\cdots\dot\alpha_\omega}-m^{\omega/2}\psi_{\alpha_1}=0,\ \text{etc.} \tag{18}$$

Equation (16) reduces to the usual ones in four dimensions (see Ref. 6).

For arbitrary ν, we can eliminate F^* from (16), take the derivative $\partial_{\alpha\dot\alpha}$ of (17) and use (18) to eliminate $\phi_{\dot\alpha_2\cdots\dot\alpha_\omega}$, etc. In this way, we obtain for any field component X the free equation of motion:

$$(\Box^{\omega/2}-m^\omega)X=0\ . \tag{19}$$

Except for $d=4(\omega=2)$ the wave equation (19) does not coincide with the Klein-Gordon equation. Expression (19) means also that the free propagators should contain the factor

$$P=\frac{1}{p^\omega-m^\omega}=\frac{1}{(p^2)^{\omega/2}-(m^2)^{\omega/2}}\ . \tag{20}$$

We can treat real gauge superfield V in an analogous way. In $d=4$, and for the Abelian case we have (see Ref. 6)

$$\mathcal{L}=\epsilon^{\alpha\beta}VD_\alpha D^2D_\beta V\ . \tag{21}$$

A natural supersymmetric generalization of (21) is (see Ref. 7)

$$\mathcal{L}=\epsilon^{\alpha_1\cdots\alpha_\omega}VD_{\alpha_1}\cdots D_{\alpha_{\omega/2}}D^\omega D_{\alpha_{\omega/2+1}}\cdots D_{\alpha_\omega}V|_D\ . \tag{22}$$

Using the identity[8]

$$\epsilon^{\alpha_1\cdots\alpha_\omega}\sum_{s=0}^{\omega}\frac{(-1)^s}{s!(\omega-s)!}\times D_{\alpha_1}\cdots D_{\alpha_\beta}D^\omega D_{\alpha_{s+1}}\cdots D_{\alpha_\omega}=\Box^{\omega/2}\ , \tag{23}$$

and noting that the extra terms [as compared with (22)]

$$\epsilon^{\alpha_1\cdots\alpha_\omega}VD_{\alpha_1}\cdots D_{\alpha_s}D^\omega D_{\alpha_{s+1}}\cdots D_{\alpha_\omega}V,\ \text{for}\ s\neq\frac{\omega}{2},$$

can be added, just as in four dimensions (see Ref. 8, p. 343), as gauge-fixing terms, we arrive at

$$\mathcal{L}=V\Box^{\omega/2}V|_D\ , \tag{24}$$

which means that the free equation of motion for the Abelian gauge field V is

$$\Box^{\omega/2}V=0\ , \tag{25}$$

leading to the propagator (20) with $m=0$

$$P=\frac{1}{p^\omega}=\frac{1}{(p^2)^{\omega/2}}\ . \tag{26}$$

The resultant propagators (20) and (26) show then that it is interesting, or convenient, to gain some confidence with their use. The Fourier transform of (26) is easily found (see Ref. 9):

$$\mathcal{F}(P)=2^{4\nu-\omega}\pi^{2\nu}\Gamma\left(2\nu-\frac{\omega}{2}\right)R^{\omega-4\nu}\ , \tag{27}$$

where $R^2=\sum_i x_i^2-x_0^2+i0$.

The behavior of (27) with R depends on the number of dimensions through the exponent

$$\alpha=\omega-4\nu=2^{2\nu-1}-4\nu=2^{d/2-1}-d\ , \tag{28}$$

which is negative for $d=4$ but positive for any integer value of ν greater than one, except $\nu=2(\alpha=0)$.

When taking the finite part of (27) (note the pole in the Γ functions) we get an additional $\ln R$ factor. So we have

$$G\sim\frac{1}{R^2}\ \text{for}\ d=4\ ,$$
$$G\sim\ln R\ \text{for}\ d=8\ , \tag{29}$$
$$G(R)\sim R^\alpha\ln R\ \text{with}\ \alpha>0\ \text{for}\ d=4\nu,\quad \nu>2\ .$$

We see that for $\nu>1$ the Green's function has a confining form. It grows with R.

Let us go now to the massive case. We write the propagator P in the form

$$P=\frac{1}{p^\omega-m^\omega}=\frac{1}{m^\omega}\sum_1^\infty\left(\frac{m}{p}\right)^{\omega l}\ . \tag{30}$$

Using now form (27) we get

$$\mathcal{F}(\rho)=\frac{2^{4\nu}\pi^{2\nu}}{im^\omega R^{4\nu}}\sum_1^\infty\frac{\Gamma(2\nu-(\omega/2)l)}{\Gamma((\omega/2)l)}\left(\frac{mR}{2}\right)^{\omega l}\ . \tag{31}$$

Using $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$ for $x=[2\nu-(\omega/2)l]$, (31) can be written as

$$\mathcal{F}(P)=-\frac{\pi^{2\nu}}{im^\omega}\frac{2^{4\nu}}{R^{4\nu}}\sum_1^\infty\frac{(mR/2)^{\omega l}\pi}{\Gamma((\omega/2)l)\Gamma(1+(\omega/2)l-2\nu)\sin((\omega/2)l\pi)}\ . \tag{32}$$

Each term of the series 3.2 is ill defined near $(\omega/2)l=$ integer. We then write [with $Z=(\omega/2)l$] each term of the sum in (32) as

$$F_l=\frac{(mR/2)^{2z}\pi(-1)^n}{\Gamma(z)\Gamma(1-2\nu+z)\sin\pi(z-n)}\ . \tag{33}$$

Near $Z=n$ we drop the pole term and keep only the finite part

$$PfF_l=\frac{\partial}{\partial z}(z-n)\,F_l(z)\Big|_{z=n}\ . \tag{34}$$

Going back to (32) we obtain for integer ω

$$\mathcal{F}(P)=\frac{(4\pi)^{2\nu}}{im^{\omega}R^{4\nu}}\sum_{l=1}^{\infty}\frac{(mR/2)^{\omega l}}{\Gamma((\omega/2)l)\Gamma((\omega/2)l+1-2\nu)}\left[2\ln\frac{mR}{2}-\psi\left(\frac{\omega}{2}l\right)-\psi\left(\frac{\omega}{2}l+1-2\nu\right)\right] . \tag{35}$$

To check that (35) is the Green's function of (19), we apply the operator $\Box^{\omega/2}$. The first term ($l=1$) is the fundamental solution of (25) (see Ref. 9, p. 276), so that the result of operating with $\Box^{\omega/2}$ is just a δ function. For the rest of the series we use

$$\Box^{\omega/2}R^{\omega+\alpha}=2^{\omega}\frac{\Gamma((\omega+\alpha+2)/2)\Gamma((\omega+\alpha+4\nu)/2)}{\Gamma((\alpha+2)/2)\Gamma((\alpha+4\nu)/2)}R^{\alpha} ,$$

and the equation obtained by taking the derivative of this equation with respect to α. In this way we can show that (35) is indeed the fundamental solution of (19). The sum of the pole parts in (32) is then a solution of the corresponding homogeneous equation. The addition of this sum would then be equivalent to a modification of the boundary behavior of the chosen Green's function. In four dimensions that would be equivalent to a modification of the causality requirement for the usual Feynman function.

If we compare (35) with the Fourier transform of the usual Klein-Gordon equation in dimension d, i.e., $\mathcal{F}(1/(p^2-m^2))$, see Ref. 9, p. 362, we see that it is a subseries where only the $(\omega l/2)$th terms ($l=1,2,\ldots$) are summed up.

This is as far as the configuration variables refer. The propagator $\langle\phi\phi\rangle$ contains the usual $\delta(\theta)=\theta^{\omega}$. Thus the product of two such terms is zero, which leads to the "no-renormalization" theorem.

The presence of tachyons and ghosts is a problem in the application of this regularization scheme but, nevertheless, it is an alternative which should not be abandoned.

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