Nonrelativistic potential scattering through the shifted large-dimension expansion

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Low-energy scattering by spherically symmetric potentials is studied in the light of the shifted large-dimension expansion technique. The present approach yields fairly accurate results for the scattering lengths and reasonable ones for the phase shifts for a class of potentials which can support up to one bound state. Remarks are made stressing the advantages of using the shifted large-N technique as compared to the unshifted formalism.

The large-N (N being the spatial dimensionality or degrees of freedom) expansion method with a built-in natural generalization of the quantum theories in which $1/N$ is a new expansion parameter has already generated immense activities by proving its efficacy in widely different fields.[1-10] One of the advantages of using this method is that the large-N technique is not an expansion in powers of the potentials and, hence, it can be used for problems which do not manifestly involve a small coupling or an expansion parameter for doing the perturbation theory. At a mundane level, the method provides surprising simplicity in obtaining an approximate solution of the Schrödinger equation. However, most of the previous works were restricted to bound-state problems only. Quite recently, one of us (R.S.G.) with his co-workers suggested an intuitive approach[10] to obtain estimates for the scattering lengths for spherically symmetric potentials. In this approach, one needs to replace the actual potential by an effective δ-function pseudopotential in the limit of large dimension and the strength and the position of the equivalent δ potential are determined in terms of the parameters of the true potential.

It has been observed that although the results obtained from this simple-minded prescription exhibit the broad qualitative features of the variation of the scattering length as a function of the strength (depth) of the potential, numerical agreement of the predicted values with the exact ones is rather poor, particularly when the strength of the interaction approaches the optimal strength necessary to produce a zero-energy bound state. In fact, for all the potentials considered by these authors, the scattering length becomes infinitely large negative at a strength much lower than the value which is actually required for the appearance of the first bound state. Moreover, the phase shifts computed according to their prescription show significant deviation from the exact ones.

We suspect that the fact that the position of the effective δ-function potential which was obtained to be completely independent of the strength of the actual potential[10] may play a crucial role in yielding these serious discrepancies. The purpose of this Brief Report is to demonstrate that the required strength dependence can be generated through the use of the shifted $1/N$ expansion proposed recently by Sukhatme and Imbo.[7] Needless to mention, the shifted large-N technique has been shown so far to be more effective and accurate than the unshifted large-N expansion method for the bound-state properties of a number of potentials.[7-9] The present extension of the shifted $1/N$ formalism to the scattering domain indicates that it is equally effective in yielding more accurate estimates of the scattering lengths and the phase shifts in comparison to those obtained within the framework of an unshifted $1/N$ expansion scheme.[10] For the sake of illustrating this point, we prefer to discuss scattering by the Gaussian potential for which a maximum discrepancy was observed [see Fig. 1(a) of Ref. 10]. The improved results for other short-range potentials are then briefly mentioned.

The radial Schrödinger equation in N spatial dimensions for an arbitrary spherically symmetric potential $V(r)$ is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{(K-1)(K-3)\hbar^2}{8mr^2}+V(r)\right]u_l(r)=Eu_l(r)\ , \tag{1}$$

where $K=N+2l$. Following the prescription of Sukhatme and Imbo,[7] Eq. (1) can be recast to the form

$$\frac{d^2u_l}{dr^2}+\left[k^2-\frac{L(L+1)}{r^2}-\frac{2m}{\hbar^2}V(r)\right]u_l(r)=0\ , \tag{2}$$

in which

$$\begin{aligned} k^2&=2mE/\hbar^2\ ,\\ L&=(\bar{K}+s-3)/2\ , \end{aligned} \tag{3}$$

where $\bar{K}=K-s$, s being the shift parameter. For potentials less singular than $1/r^2$, the radial function $u_l(r)$ behaves as r^{L+1} for small r. Accordingly, for short-range potentials, the product $V(r)u_l(r)$ occurring in Eq. (2) will have a maximum which in the limit of large N with the appropriate scaling can be replaced by the approximate potential[10]

$$V_{\text{eff}}(r)\approx-\lambda\delta(r-a)\ , \tag{4}$$

in which the effective strength parameter is given by

$$\lambda=e^{f(a)}[2\pi/f''(a)]^{1/2}\ , \tag{5}$$

and the location of the δ potential is determined from the condition

$$\left.\frac{df(r)}{dr}\right|_{r=a}=0\ , \tag{6}$$

where

$$f(r)=\ln V(r)+(L+1)\ln r\ . \tag{7}$$

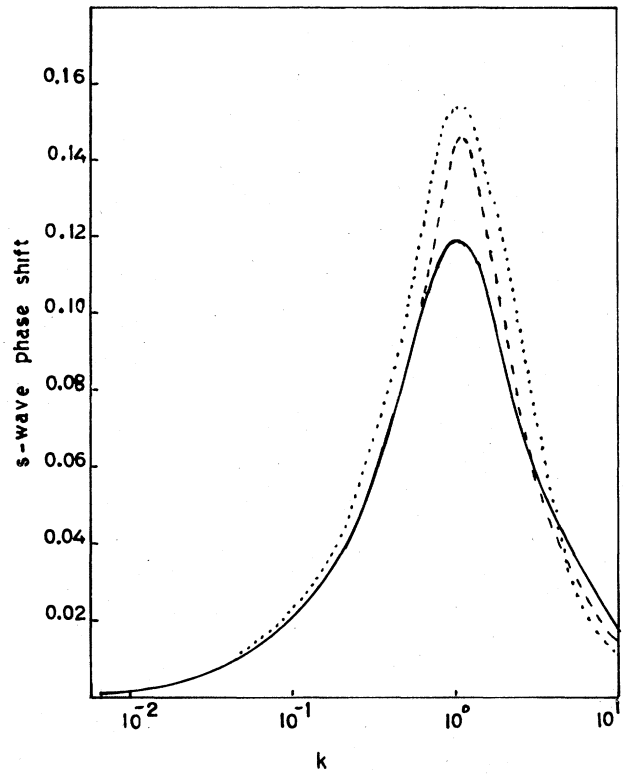

FIG. 2. s-wave scattering phase shift as a function of k (in logarithmic scale) for the Gaussian potential with depth $V_0=0.4$. Exact values are displayed by the full curve while the shifted and the unshifted large-N approximates are shown by the broken and dotted curves, respectively.

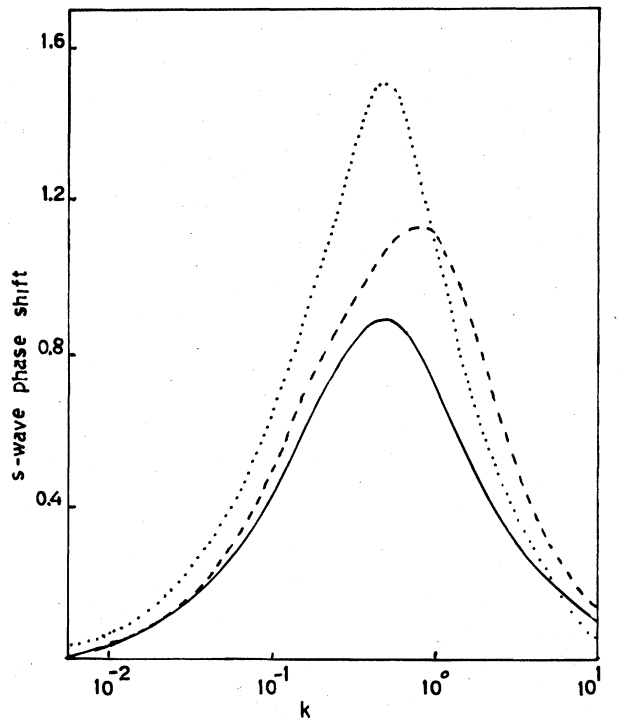

FIG. 3. s-wave scattering phase shift as a function of k (in logarithmic scale) for the Gaussian potential with depth $V_0=2.2$. Exact values are displayed by the full curve while the shifted and the unshifted large-N approximates are shown by the broken and dotted curves, respectively.

and (14). The phase shifts calculated in the energy range $k=0.001$ to 10 for $V_0=0.4$ and $V_0=2.2$ are shown in Figs. 2 and 3 along with the results obtained numerically. The agreement of the shifted $1/N$ results with the exact ones is satisfactory at least in the low-energy region. We find that the phase shifts predicted by the simple approximate formula (10) are accurate by about 5% for k up to 0.5 for $V_0=0.4$. The results become gradually inaccurate for a higher strength of the potential. Furthermore, as the energy increases, the disagreement becomes more pronounced. Similar features have been noticed in cases of other short-range potentials. We guess that this discrepancy may be reduced by incorporating the appropriate energy dependence of the shift parameter in addition to its dependence on the strength coefficient V_0. It is rather interesting to note that although the numerical accuracy decreases in the high-energy region, our pseudopotential yields the same asymptotic behavior, $\delta_0(k)\sim 1/k$ as $k\rightarrow\infty$, as one obtains in the Born approximation.

We conclude that the shifted large-N technique, which has proved its efficacy in dealing with the nonrelativistic bound-state problems, seems to play a vital role in providing a consistent description of the scattering phenomena. To our knowledge, this observation has not been made earlier. Our work extends the scope of applicability of the idea behind the shifted large-N method proposed by Sukhatme and his co-workers. One of the merits of our approach is that it brings significant simplicity in the computations of scattering lengths and phase shifts for a number of useful potentials without sacrificing the elegance of an analytic treatment. Furthermore, the present method, although far from being rigorous, exhibits certain promising trends which require careful examination. It is certainly worthwhile to investigate the phase shifts for the Coulomb plus short-range potentials.[14] In recent times, nice results have been obtained for the energy-dependent effective range function which predicts the phase shifts correctly even at high energies.[15] The present method can be extended to obtain an equivalent representation for the effective range function for any arbitrary angular momentum state.

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For the Gaussian potential $V(r) = -V_0 e^{-\mu r^2}$ with the appropriate scaling $\mu = \tilde{\mu}\bar{K}$, we obtain from Eqs. (4) to (7) the effective pseudopotential in the limit of large K (i.e., large N),

$$-\left(\frac{2m}{\hbar^2}\right) V_0 e^{-\bar{K}/4} \left(\frac{\pi}{2\tilde{\mu}\bar{K}}\right)^{1/2} \delta(r - \tfrac{1}{2}(1/\tilde{\mu})^{1/2}) \ ,$$

and next substituting the value $N = 3$, the approximate potential in three dimensions is obtained to be

$$V_{\text{eff}}^{\text{Gaussian}} \approx -\lambda \delta(r - a) \ ,$$

with

$$\lambda = V_0(\pi/2\mu)^{1/2} \exp[-(3 + 2l - s)/4] \ ,$$

$$a = \frac{1}{2}\left(\frac{3 + 2l - s}{\mu}\right)^{1/2} \ . \tag{8}$$

The partial-wave phase shift for the δ potential is given by[11]

$$\tan\delta_l = k\lambda a^2 [j_l(ka)]^2 / [1 + k\lambda a^2 j_l(ka)\dot{\eta}_l(ka)] \ , \tag{9}$$

which for the s wave takes the form

$$\tan\delta_0 = \lambda \sin^2(ka)/[k - \lambda \sin(2ka)/2] \ , \tag{10}$$

and the corresponding scattering length is given by

$$a_{\text{sc}} = -\lambda a^2/(1 - \lambda a) \ . \tag{11}$$

It may be seen from Eq. (8) that although the effective strength λ of the pseudopotential depends on both the depth (V_0) and the range (μ) parameters of the actual potential, the position a of the approximate Dirac-δ potential remains independent of V_0. This feature seems to be unrealistic in a sense that it is quite natural to expect that the location of the effective potential should also alter as V_0 changes. Unlike in the case of the unshifted $1/N$ expansion calculation,[10] we find that due to the appearance of the shift parameter s in a, there is a scope to adjust this parameter to obtain the desired strength dependence of the position of the effective δ potential. For bound-state problems, prescriptions have been suggested[7,8] to fix up the shift parameter in terms of the potential parameters. For the scattering states, it is true that apparently there is no first principle to ascertain the functional dependence of s on V_0. However, it is found that an intuitive choice

$$s = \ln(1 + \alpha V_0) \tag{12}$$

yields a dramatic improvement of the values of the scattering lengths in comparison to those obtained by the unshifted $1/N$ technique.[10] Furthermore, it leads to estimating the phase shifts over a wide range of energies with improved accuracy.

To determine the parameter α in (12), we employ the criterion that the scattering length becomes infinitely large negative at $V_0 = V_b$, where V_b is the minimal strength of the attractive potential necessary to support at least a zero-energy bound state. From Eq. (11), we then require

$$\lambda(V_b) a(V_b) = 1 \ . \tag{13}$$

For the Gaussian potential, a stringent analytic lower bound $V_b = 2.684$ was obtained by Glaser, Martin, Grosse, and Thirring.[12] Using this value in Eqs. (8), (12), and (13), we finally get the shift parameter for the Gaussian potential (with $\mu = 1$),

$$s = \ln(1 + 4.4439 V_0) \ . \tag{14}$$

As we have already mentioned that the functional dependence of the shift parameter on the potential strength has to be made in an *ad hoc* manner, Eq. (12) is not adequately motivated and so one should try with alternative choices. The shift $s = s_b$ at $V_0 = V_b$ has been chosen in such a way that $s = 0$ at $V_0 = 0$. In fact, there may be many interpolations between these two points—Eq. (12) being one of them. Another simpler choice is the linear form, $s = \alpha V_0$, which in conjunction with Eqs. (8) and (13) gives

$$s = 0.9299 V_0 \ . \tag{15}$$

Using either of the two forms (14) or (15) in Eqs. (8) and (11), we compute the scattering length (in units of $2m/\hbar^2 = 1$) for different strengths of the potential and the results are compared with the exact values graphically in Fig. 1. For the sake of assessing the improved accuracy of the present approach, the scattering lengths obtained as a function of V_0 in the framework of the unshifted $1/N$ expansion method are also depicted. The surprising improvement of the present approximation with logarithmic dependence of s on V_0 is quite apparent from the diagram. As a specific example, we like to mention that for $V_0 \approx 2$, while the unshifted calculation[10] predicts $a_{\text{sc}} = -\infty$, the shifted $1/N$ technique with s given by (14) yields $a_{\text{sc}} = -3.30$ which is quite close to the exact value $a_{\text{sc}} = -3.48$.

For other short-range potentials considered by Sinha-Roy *et al.*, a similar procedure to that delineated above has been followed. For the solvable potentials such as those of Hulthén, Bargmann, and Pöschl and Teller, the threshold depth V_b necessary to support a bound state is known analytically. For the exponential and Yukawa potentials, we take the appropriate values of V_b obtained by us[13] using an iterative trace method. In all cases, we obtain much improved results for the scattering lengths, particularly in the neighborhood of the first bound state.

The s-wave scattering phase shifts for different strengths of the Gaussian potential are computed from Eqs. (8), (10),

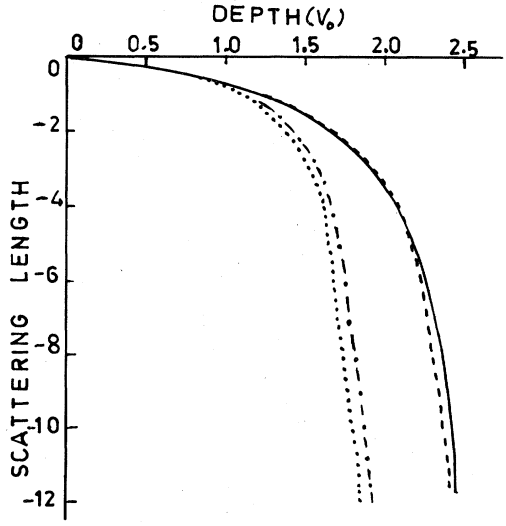

FIG. 1. Scattering length (in units of $2m/\hbar^2 = 1$) as a function of the potential depth for the Gaussian potential $V(r) = -V_0 e^{-r^2}$: exact (solid curve), unshifted (dotted curve), shifted with the shift parameter (14) (dashed curve), shifted with the shift parameter (15) (dot-dashed curve).

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