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Isometries of homogeneous Gödel-type spacetimes

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The isometries of homogeneous Gödel-type Riemannian spacetimes are reexamined. An extension of the Raychaudhuri-Thakurta-Rebouças-Tiomno analysis is made. It is shown that the $m=2\Omega$ case admits a seven-parameter group of motions. Comments on the unique Gödel-type causal model are made. The isometries for the degenerate Gödel-type metrics ($\Omega=0$) are also obtained.

INTRODUCTION

In 1949 Gödel found a solution of Einstein's field equations with a cosmological constant for incoherent matter with rotation.[1] Despite its various striking properties, the cosmological solution discussed by Gödel has a well-recognized historical (and even philosophical) importance, and has given rise to a noticeable stimulus to the investigation of rotating spacetimes. In particular, the search for Gödel-type rotating models has received considerably more attention in recent years, and the literature on those kinds of rotating solutions is fairly large today.[2]

However, apart from Gödel's paper, the problem of the spacetime homogeneity of Gödel-type manifolds was considered only in 1980 by Raychaudhuri and Thakurta.[3] They found the necessary conditions for a Riemannian Gödel-type manifold to be homogeneous in space and time (hereafter called ST homogeneous).

Two years later Rebouças and Tiomno investigated the homogeneity of Riemannian spacetimes of Gödel type, and proved that the Raychaudhuri-Thakurta conditions are also sufficient for a Gödel-type Riemannian manifold to be ST homogeneous.[4] However, not only Raychaudhuri and Thakurta but also Rebouças and Tiomno have assumed explicitly or implicitly that the vector field $u^\alpha=\delta_0^\alpha$ is a left-invariant vector field on the Lie group of isometry, viz.,

$$\mathscr{L}_{\mathbf{K}}\mathbf{u}=[\mathbf{K},\mathbf{u}]=0\ ,$$

where $\mathbf{K}$ stands for a generic Killing vector field. In other words, they have restricted their study to the time-independent Killing vector fields.

Our major aim in this paper is to extend these investigations so as to include the time-dependent isometries. It emerges from our study that among the new Rebouças-Tiomno solutions, the special one with $m=2\Omega$ is peculiar not only as far as its causal properties are concerned, but mainly because it has a seven-parameter maximal group of motions (G_7), while the remaining nondegenerate Gödel-type solutions have a G_5 only. As neither Raychaudhuri and Thakurta nor Rebouças and Tiomno have discussed the limiting case $\Omega=0$, we also examine the isometries for this degenerate Gödel-type manifold, showing that there is a six-parameter isometry group, G_6.

BASIC EQUATIONS AND RESULTS

Consider a four-dimensional Riemannian manifold endowed with a Gödel-type metric:

$$ds^2=[dt+H(r)d\phi]^2-dr^2-D^2(r)d\phi^2-dz^2\ . \tag{1}$$

This may be written as

$$ds^2=\eta_{(A)(B)}\theta^{(A)}\theta^{(B)}=\theta_{(0)}{}^2-\theta_{(1)}{}^2-\theta_{(2)}{}^2-\theta_{(3)}{}^2\ , \tag{2}$$

where $\eta_{(A)(B)}$ is the Minkowski metric and where the one-forms $\theta^{(A)}=e_\alpha^{\ (A)}dx^\alpha$ are clearly

$$\theta^{(0)}=dt+H(r)d\phi,\quad \theta^{(1)}=dr,\quad \theta^{(2)}=D(r)d\phi,\quad \theta^{(3)}=dz\ . \tag{3}$$

Denoting a generic Killing vector field by $K^{(A)}=(T,R,P,Z)$, the ten Killing equations in the Lorentz frame defined by $\theta^{(A)}$ can be written as

$$T_t=0,\quad T_z-Z_t=0\ , \tag{4}$$

$$R_r=0,\quad Z_r-R_z=0\ , \tag{5}$$

$$Z_z=0\ , \tag{6}$$

$$D(T_r-R_t)-H_rP=0\ , \tag{7}$$

$$DP_z+Z_\phi-HZ_t=0\ , \tag{8}$$

$$T_\phi+H_rR-DP_t=0\ , \tag{9}$$

$$R_\phi-HR_t-D_rP+DP_r=0\ , \tag{10}$$

$$P_\phi-HP_t+D_rR=0\ , \tag{11}$$

where the subscripts denote partial derivatives.

Let us first consider the general solution to Eqs. (4)–(6). From (4) and (6) one obtains

$$T=z\theta(r,\phi)+\iota(r,\phi)\ , \tag{12}$$

$$Z=t\theta(r,\phi)+\kappa(r,\phi)\ , \tag{13}$$

where θ, ι, and κ are arbitrary functions of their arguments. Similarly, from (5) and (6) one finds

$$R=-z\epsilon(t,\phi)+\xi(t,\phi)\ , \tag{14}$$

$$Z=r\epsilon(t,\phi)+\pi(t,\phi)\ , \tag{15}$$

where we have set $m = +2\Omega$ for definiteness.

The corresponding Lie algebra has the following nonvanishing commutators:

$$[K_3,K_4] = -mK_5,\quad [K_4,K_5] = mK_3,\quad [K_5,K_3] = -mK_4\ ,$$
$$[K_1,K_6] = -mK_7,\quad [K_6,K_7] = mK_1,\quad [K_7,K_1] = -mK_6\ . \tag{49}$$

It seems worth emphasizing that the extension of the investigations of Raychaudhuri and Thakurta and Rebouças and Tiomno in this case has given rise to the present class of Gödel-type ST-homogeneous Riemannian manifolds with a maximal group of isometries containing seven parameters. Moreover, the present class corresponds to the unique causal Gödel-type solution found by Rebouças and Tiomno.[4] In other words, the breakdown of causality of Gödel-type was avoided through a more symmetric model.[5]

Second and third cases. II. $m^2 < 0$, $\Omega \neq 0$. *III.* $m = 0$, $\Omega \neq 0$. As far as these two cases are concerned, a similar but fairly lengthy analysis shows that in both cases there is no additional Killing vector field besides those discussed by Rebouças and Tiomno.

Fourth case. IV. $m \neq 0$, $\Omega = 0$. For completeness we shall examine the isometries of this degenerated Gödel-type manifold. By trivial coordinate transformation one can make $H = 0$. We start from the general expressions (16)–(18) and consider the Killing equations (7)–(11) sucessively, with $H = 0$.

Substituting (17) and (18) into (7) one finds that the Killing equation is satisfied only if $\lambda = 0$ and

$$\iota(r,\phi) = ra(\phi) + c(\phi)\ , \tag{50}$$

$$\xi(t,\phi) = ta(\phi) + e(\phi)\ , \tag{51}$$

with a, c, and e arbitrary functions. The components T,R,Z then become

$$T = z\epsilon(\phi) + ra(\phi) + c(\phi)\ , \tag{52}$$

$$R = -z\nu(\phi) + ta(\phi) + e(\phi)\ , \tag{53}$$

$$Z = t\epsilon(\phi) + r\nu(\phi) + \rho(\phi)\ , \tag{54}$$

while Eq. (8) gives immediately

$$P = -zD^{-1}(t\epsilon_\phi + r\nu_\phi + \rho_\phi) + n(t,r,\phi)\ , \tag{55}$$

with n an arbitrary function.

Now Eq. (9) implies

$$\epsilon = \text{const} \equiv \kappa_3\ , \tag{56}$$

$$n(t,r,\phi) = tD^{-1}(ra_\phi + c_\phi) + u(r,\phi)\ , \tag{57}$$

where u is an arbitrary function. The expressions (52), (54), and (55) are now easily simplified. If one introduces the simplified expressions into Eq. (10), one finds that this equation is only satisfied if

$$\nu = \text{const},\quad a = \text{const},\quad \rho = \text{const} \equiv \kappa_2\ , \tag{58}$$

$$c = \text{const} \equiv \kappa_1,\quad u(r,\phi) = D_r e_\phi + D\omega(\phi)\ , \tag{59}$$

where ω is an arbitrary function. Again expressions (58) and (59) can be used to simplify $K^{(A)} = (T,R,P,Z)$; in doing so and next using the remaining Killing equation one obtains

$$a = \nu = 0,\quad \omega = \text{const} \equiv \kappa_6\ , \tag{60}$$

$$e = \kappa_4 \cos\phi - \kappa_5 \sin\phi \quad (\kappa_4, \kappa_5 = \text{const})\ . \tag{61}$$

Therefore, the solution for $m \neq 0$ and $\Omega = 0$ turns out to be given by Eqs. (52)–(55) together with the conditions (56)–(61); in the coordinate basis, the six Killing vector fields can be written as

$$K_1 = \partial_t,\quad K_2 = \partial_z,\quad K_3 = z\partial_t + t\partial_z\ ,$$
$$K_4 = \cos\phi\,\partial_r - D_r D^{-1}\sin\phi\,\partial_\phi\ , \tag{62}$$
$$K_5 = -\sin\phi\,\partial_r - D_r D^{-1}\cos\phi\,\partial_\phi,\quad K_6 = \partial_\phi\ ,$$

with the nonvanishing commutators

$$[K_1,K_3] = K_2,\quad [K_2,K_3] = K_1\ ,$$
$$[K_4,K_5] = -m^2 K_6,\quad [K_5,K_6] = K_4,\quad [K_6,K_4] = K_5\ . \tag{63}$$

We notice that when $m = \Omega = 0$ the line element (1) is clearly Minkowskian. It admits the Poincaré group.

Finally, we should mention that the Killing equations, as well as all vector fields we obtained, were checked through the computer program KILLNF,[6] written in the symbolic manipulation language SHEEP.[7]

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[1] K. Gödel, Rev. Mod. Phys. **21**, 447 (1949).

[2] See, for example, Ref. 4, and references therein, quoted on Gödel-type solutions.

[3] A. K. Raychaudhuri and S. N. Guha Thakurta, Phys. Rev. D **22**, 802 (1980).

[4] M. J. Rebouças and J. Tiomno, Phys. Rev. D **28**, 1251 (1983).

[5] Professor J. Tiomno kindly informed us that he and Dr. I. D. Soares have recently been able to prove that a G_7 has to exist in the family of Gödel-type solutions; they are further investigating the subject, in a context completely different from ours.

[6] J. E. Åman, University of Stockholm Report, 1982 (unpublished).

[7] I. Frick, University of Stockholm, Institute of Physics, Report No. 77-15, 1979 (unpublished).

where ϵ, ξ, and π are arbitrary functions. Comparing (13) with (15) one has

$$Z = t[r\lambda(\phi)+\epsilon(\phi)]+r\nu(\phi)+\rho(\phi) \quad , \tag{16}$$

which, together with (13), enables us to rewrite (12) as

$$T = rz\lambda(\phi)+z\epsilon(\phi)+\iota(r,\phi) \quad . \tag{17}$$

Alternatively, (16) can also be used together with (15) to identify ϵ, and then we rewrite (14) as

$$R = -tz\lambda(\phi)-z\nu(\phi)+\xi(t,\phi) \quad . \tag{18}$$

Equations (16)–(18) do not depend on the metric functions $D(r)$ and $H(r)$, and are the general solution to Eqs. (4)–(6).

The function $P(t,r,\phi,z)$ is yet completely arbitrary. However, it was shown by Rebouças and Tiomno[4] that all ST-homogeneous Riemannian manifolds endowed with a Gödel-type metric (1) are given by

$$\text{(i)}\quad H=\frac{2\Omega}{m^2}[1-\cosh(mr)],\quad D=\frac{1}{m}\sinh(mr) \quad , \tag{19}$$

with $m^2=\text{const}>0$,

$$\text{(ii)}\quad H=\frac{2\Omega}{\mu^2}[\cos(\mu r)-1],\quad D=\frac{1}{\mu}\sin(\mu r) \quad , \tag{20}$$

whenever $m^2\equiv-\mu^2<0$,

$$\text{(iii)}\quad H=-\Omega r^2,\quad D=r,\quad \text{if } m=0 \quad , \tag{21}$$

where $\Omega=\text{const}$ in all cases. Therefore, for ST-homogeneous spacetimes we have $H_r=-2\Omega D$, and then (7) gives

$$2\Omega P=-\iota_r(r,\phi)+\xi_t(t,\phi)-2z\lambda(\phi) \quad . \tag{22}$$

Equations (16)–(18) and (22) are the general solution to the Killing equations (4)–(7) for all ST-homogeneous Gödel-type metrics. They depend on six arbitrary functions, which will be determined by the remaining Killing equations for each different class of Gödel-type ST-homogeneous Riemannian manifolds, as follows.

First case. I. $m^2>0$ *and* $\Omega\neq 0$. In this case the functions H and D are given by (19). Substituting (16) and (22) into (8), after some rearrangements, one learns that the Killing equation (8) is satisfied only if $\lambda=\epsilon=0$, $\nu=\text{const}$, and $\rho=\text{const}\equiv\kappa_2$. Equations (16)–(18) and (22) then simplify to

$$T=\iota(r,\phi),\quad R=-\nu z+\xi(t,\phi) \quad , \tag{23}$$

$$2\Omega P=-T_r(r,\phi)+\xi_t(t,\phi),\quad Z=\kappa_2+\nu r \quad . \tag{24}$$

Inserting (23) and (24) into (9) one finds that the last one is satisfied only if $\nu=0$ and

$$\xi(t,\phi)=f_\phi(\phi)+g(\phi)\sin(2\Omega t)+h(\phi)\cos(2\Omega t) \quad , \tag{25}$$

$$\iota(r,\phi)=T=2\Omega[Df(\phi)-v(r)] \quad , \tag{26}$$

where g, h, and v are arbitrary functions. The remaining components of $K^{(A)}$ can now be simplified to give

$$R=f_\phi(\phi)+g(\phi)\sin(2\Omega t)+h(\phi)\cos(2\Omega t) \quad , \tag{27}$$

$$P=v_r(r)-D_rf(\phi)+g(\phi)\cos(2\Omega t)-h(\phi)\sin(2\Omega t) \quad , \tag{28}$$

$$Z=\kappa_2 \quad . \tag{29}$$

Finally, we substitute (27) and (28) into (10); the resulting equation is satisfied only if

$$f_{\phi\phi}+f=-Dv_{rr}+D_rv_r=\text{const}\equiv q \quad , \tag{30}$$

and

$$g_\phi+(2\Omega/m)^2h=h_\phi-(2\Omega/m)^2g=[1-(2\Omega/m)^2]h=[1-(2\Omega/m)^2]g=0 \quad . \tag{31}$$

Integrating (30) we obtain

$$f(\phi)=\kappa_4\sin\phi+\kappa_5\cos\phi+q \quad (\kappa_4,\kappa_5=\text{const}) \quad , \tag{32}$$

$$v(r)=qD-\kappa_3D_r-\kappa_1/(2\Omega) \quad (\kappa_3,\kappa_1=\text{const}) \quad . \tag{33}$$

As far as (31) is concerned, we distinguish two different classes of solutions, namely,

$$\text{(Ia)}\quad g=h=0 \quad , \tag{34}$$

$$\text{(Ib)}\quad g=a\cos\phi+b\sin\phi,\quad h=a\sin\phi-b\cos\phi \quad ,$$

$$m^2=4\Omega^2,\quad \text{with } a,\ b=\text{const} \quad . \tag{35}$$

Class Ia. In this class Eqs. (26)–(29) together with (32)–(34) give

$$T=\kappa_1+2\Omega[\kappa_3D_r+(\kappa_4\sin\phi+\kappa_5\cos\phi)D] \quad , \tag{36}$$

$$R=\kappa_4\cos\phi-\kappa_5\sin\phi \quad , \tag{37}$$

$$P=-\kappa_3mD-(\kappa_4\sin\phi+\kappa_5\cos\phi)D_r \quad , \tag{38}$$

$$Z=\kappa_2 \quad . \tag{39}$$

The Killing equation (11) is identically satisfied. Thus the corresponding Killing vector fields can be written, in the coordinate basis defined by (1), as

$$K_1=\partial_t,\quad K_2=\partial_z,\quad K_3=(2\Omega/m)\partial_t-m\partial_\phi \quad , \tag{40}$$

$$K_4=-HD^{-1}\sin\phi\partial_t+\cos\phi\partial_r-D_rD^{-1}\sin\phi\partial_\phi \quad , \tag{41}$$

$$K_5=-HD^{-1}\cos\phi\partial_t-\sin\phi\partial_r-D_rD^{-1}\cos\phi\partial_\phi \quad . \tag{42}$$

The Lie algebra has the following nonvanishing commutators:

$$[K_3,K_4]=-mK_5,\quad [K_4,K_5]=mK_3,\quad [K_5,K_3]=-mK_4 \quad . \tag{43}$$

It should be noticed that the expressions for all Killing vector fields are time independent.

Class Ib. Making use of (26)–(29) together with (32), (33), and (35) we obtain the tetrad components $K^{(A)}$ of seven Killing vector fields, which in the coordinate basis can be written in the form

$$K_1=\partial_t,\quad K_2=\partial_z,\quad K_3=\partial_t-m\partial_\phi \; , \tag{44}$$

$$K_4=-HD^{-1}\sin\phi\partial_t+\cos\phi\partial_r-D_rD^{-1}\sin\phi\partial_\phi \; , \tag{45}$$

$$K_5=-HD^{-1}\cos\phi\partial_t-\sin\phi\partial_r-D_rD^{-1}\cos\phi\partial_\phi \; , \tag{46}$$

$$K_6=-HD^{-1}\cos(mt+\phi)\partial_t+\sin(mt+\phi)\partial_r+D^{-1}\cos(mt+\phi)\partial_\phi \; , \tag{47}$$

$$K_7=-HD^{-1}\sin(mt+\phi)\partial_t-\cos(mt+\phi)\partial_r+D^{-1}\sin(mt+\phi)\partial_\phi \; , \tag{48}$$