Meson electric form factor on the lattice

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Theoretical aspects of calculating the electric form factor of lattice hadrons are presented. We use the staggered formulation of lattice fermions and deal specifically with SU(2) color; however, the techniques described are easily adaptable to other situations.

I. INTRODUCTION

Recently there has been considerable interest in Monte Carlo investigations of hadron size and internal structure.[1-4] Electromagnetic properties provide clean and experimentally accessible information for this purpose.[1,5] We have shown, in Ref. 1, that a lattice computation of the electric form factor and the rms charge radius of hadronic states is feasible. Results were presented for the electric form factor of the pseudo-Goldstone meson within the staggered formulation[6-8] of lattice fermions. It was found that the quarks in a lattice meson are indeed localized in a compact object significantly smaller than the lattice volume. In this paper we present a detailed derivation of the formulas for the three-point function and the electric form factor used in Ref. 1. We consider only the pseudo-Goldstone meson state and use SU(2) color. There are no difficulties, in principle, to extending these considerations to SU(3) color and to other lattice hadron states.

An advantage of the staggered fermion scheme is that a remnant global chiral symmetry is preserved.[7,8] The pseudo-Goldstone boson associated with the spontaneous breakdown of the global axial symmetry (in the massless limit) can be interpreted as the generic pion. The construction of correlation functions for hadronic states with good quantum numbers in this formalism is not straightforward. It is usually done by using *nonlocal* flavored Dirac quark fields made up of staggered fermion fields on hypercubes in the lattice.[8-10] For some states, in particular, for the pseudo-Goldstone (pion) state, it is known that the mass can be determined from a two-point function of operators which are *local* combinations of staggered fermion fields. This is advantageous for numerical calculations. The main result of this paper is that pionic matrix elements of the electromagnetic current for low-momentum states can also be calculated using operators which are local in the staggered fermion fields.

II. DERIVATION

In this study, we will want to model a non-flavor-singlet meson with a net nondynamic electric charge. Specifically, we will be considering the pseudo-Goldstone meson (the generic pion) associated with the remnant continuous axial global symmetry present in the staggered scheme of lattice fermions.[8] The usual flavor structure associated with the staggered fermions is not useful for constructing charged states since an "electric charge" defined within these flavors is not conserved. There does exist a conserved vector current within this formulation; it has the effect of assigning identical charges to all four staggered fermion flavors. We therefore introduce two sets of the four flavors, labeled by u and d, with charges q^u and q^d, $q^u-q^d=1$. The staggered fermion action with gauge fields and a two-component χ field is

$$S_F(U)=\frac{1}{2}\sum_{x,\mu,f=u,d}\alpha_\mu(x)\bar{\chi}^f(x)\times[U_\mu(x)\chi^f(x+a_\mu)-U_\mu^\dagger(x-a_\mu)\chi^f(x-a_\mu)]+(ma)\sum_{x,f}\bar{\chi}^f(x)\chi^f(x)\,, \qquad (1)$$

where

$$\alpha_\mu(x)=\alpha_\mu(x\pm a_\mu)=\begin{cases}1, & \mu=1\\ (-1)^{x_1}, & \mu=2\\ (-1)^{x_1+x_2}, & \mu=3\\ (-1)^{x_1+x_2+x_3}, & \mu=4\,.\end{cases}$$

The particle content of the theory is usually given in terms of flavored Dirac quark fields (q fields). These are related to the χ fields by a linear transformation:

$$q_A^{\alpha a}(z)=\frac{1}{2^{3/2}}\sum_\eta \Gamma_\eta^{\alpha a}[U_\eta(z)]^{AA'}\chi_{A'}(2z+\eta)\,, \qquad (2)$$

$$\bar{q}_A^{\alpha a}(z)=\frac{1}{2^{3/2}}\sum_{\eta'}\bar{\chi}_B(2z+\eta')[U_{\eta'}^\dagger(z)]^{BA}\Gamma_{\eta'}^{*\alpha a}\,, \qquad (3)$$

where ($\gamma_\mu^\dagger=\gamma_\mu,\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$)

$$\Gamma_\eta=\gamma_1^{\eta_1}\gamma_2^{\eta_2}\gamma_3^{\eta_3}\gamma_4^{\eta_4}\,, \qquad (4)$$

and

$$U_\eta(z)=[U_1(2z)]^{\eta_1}\cdots[U_4(2z+\eta_1+\eta_2+\eta_3)]^{\eta_4}\,. \qquad (5)$$

η_μ is a lattice vector such that $\eta_\mu=0$ or 1 for each μ independently. Grouping the χ variables into hypercubes as in (2) and (3) puts the action (1) in a form that allows one to identify interpolating fields with given quantum numbers (up to parity mixing).

A. The two-point function

A π^+ interpolating field[11] is given by

$$\phi(z)=\bar{q}^{\,d}_A(z)(\gamma_5\otimes\gamma_5^*)q^u_A(z)$$
$$=\tfrac{1}{2}\sum_\eta(-1)^\eta\bar{\chi}^{\,d}_A(2z+\eta)\chi^u_A(2z+\eta)\ , \qquad (6)$$

where $(-1)^\eta\equiv(-1)^{\sum_\mu\eta_\mu}$ and $z=(\mathbf{z},t_z)$. Consider this pion's zero-momentum lattice propagator:

$$M_{\pi^+}(t_z)=\sum_{\mathbf{z}}\langle 0\,|\,T(\phi(z)\phi^\dagger(0))\,|\,0\rangle\ . \qquad (7)$$

Treating the $\bar{q},q$ as field-theoretic variables, we have

$$(\bar{q}^{\,d}_B(\gamma_5\otimes\gamma_5^*)q^u_B)^\dagger=-\bar{q}^{\,u}_B(\gamma_5\otimes\gamma_5^*)q^d_B\ . \qquad (8)$$

We now use the fundamental identity for zero-temperature field theory[12]

$$\langle 0\,|\,T(\psi_\alpha(-it_A)\bar{\psi}_B(-it_B)\cdots)\,|\,0\rangle$$
$$=Z^{-1}\int dU\,d\bar{\xi}\,d\xi\,e^{-S_g-S_F}(\xi_\alpha(t_A)\bar{\xi}_B(t_B)\cdots)\ , \qquad (9)$$

where $\bar{\psi}=\psi^\dagger\gamma_4$, Z is the normalization integral and

$$\{\psi_\alpha,\psi^\dagger_\beta\}=\delta_{\alpha\beta},\quad \{\psi_\alpha,\psi_\beta\}=0\ . \qquad (10)$$

In contrast, the independent Grassman integration variables $\bar{\xi}$ and ξ anticommute:

$$\{\bar{\xi}_\alpha,\xi_\beta\}=\{\xi_\alpha,\xi_\beta\}=\{\bar{\xi}_\alpha,\bar{\xi}_\beta\}=0\ . \qquad (11)$$

The latin indices are generic including spacetime, Dirac, flavor and color. Since the transformation from q to χ is linear, (9) leads to

$$M_{\pi^+}(t_z)=-\tfrac{1}{4}Z^{-1}\sum_{\mathbf{z},\eta,\eta'}(-1)^{\eta+\eta'}\times\langle\bar{\chi}^{\,d}_A(2z+\eta)\chi^u_A(2z+\eta)\times\bar{\chi}^{\,u}_B(\eta')\chi^d_B(\eta')\rangle\ , \qquad (12)$$

where

$$\langle\cdots\rangle\equiv\int dU\,d\bar{\chi}\,d\chi\,e^{-S_g-S_F}(\cdots)\ .$$

The $\bar{\chi},\chi$ fields are now being treated as integration variables.

Let us now shift the position on all the starting points of the propagator (12) to the origin. This shift entails the assumption that one may do an even-odd lattice-point redefinition on unit hypercubes. This is allowed in the quenched lattice vacuum with periodic or antiperiodic boundary conditions on the quarks. However, coupling the time boundaries together leads to difficulties when the charge operator is introduced since the amount of charge that flows in the forward time direction now depends on the value of t_z. Thus, we will prefer to adopt nonperiodic boundary conditions in the time directions for our numerical simulations. For the following derivations we will assume, for simplicity, a lattice of infinite time extent.

Performing the shift to the origin on starting positions as well as the η_4,η_4' sum, (12) becomes

$$M_{\pi^+}(t_z)=-2Z^{-1}\sum_{\mathbf{x}}(-1)^{\mathbf{x}}\langle[2\bar{\chi}^{\,d}_A(\mathbf{x},2t_z)\chi^u_A(\mathbf{x},2t_z)-\bar{\chi}^{\,d}_A(\mathbf{x},2t_z+1)\chi^u_A(\mathbf{x},2t_z+1)$$
$$-\bar{\chi}^{\,d}_A(\mathbf{x},2t_z-1)\chi^u_A(\mathbf{x},2t_z-1)]\bar{\chi}^{\,u}_B(0)\chi^d_B(0)\rangle\ , \qquad (13)$$

with $(-1)^{\mathbf{x}}\equiv(-1)^{\sum_i x_i}$, $\mathbf{x}$ denoting spatial positions in the original lattice.

We write the action (1) as

$$S_F(U)=\sum_{\alpha,\beta}\bar{\chi}_\alpha M_{\alpha\beta}\chi_\beta\ , \qquad (14)$$

where the fermion matrix $M_{\alpha\beta}$ is proportional to the unit matrix in u,d flavor space. Explicitly,

$$M_{xA;yB}=(ma)\delta_{xy}\delta_{AB}+\tfrac{1}{2}\sum_{\hat{\mu}}\alpha_\mu(x)\{[U_\mu(x)]^{AB}\delta_{x,y-a_\mu}-[U^\dagger_\mu(x-a_\mu)]^{AB}\delta_{x,y+a_\mu}\}\ , \qquad (15)$$

in the space-time (x,y) and color (A,B) indices. From (15) one may show that

$$M^*_{xA;yB}=(-1)^{x+y}M_{yB;xA}\ . \qquad (16)$$

This component statement may be written as a matrix relation and inverted to yield

$$[M^{-1}{}_{xA;yB}]^*=(-1)^{x+y}M^{-1}{}_{yB;xA}\ . \qquad (17)$$

We also have that[13]

$$\sigma_2 M\sigma_2=M^* \qquad (18)$$

where σ_2 acts in color space.

Using standard fermion integration formulas, setting $\det M=$constant, replacing the gauge field integral by a Monte Carlo average over configurations, and using (17) above now gives

$$M_{\pi^+}(t_z)=\frac{2}{N_c}\sum_{\mathbf{x},c}\{2\,\mathrm{tr}M^{-1}(\mathbf{x},2t_z;0)[M^{-1}(\mathbf{x},2t_z;0)]^\dagger+\mathrm{tr}M^{-1}(\mathbf{x},2t_z+1;0)[M^{-1}(\mathbf{x},2t_z+1;0)]^\dagger$$
$$+\mathrm{tr}M^{-1}(\mathbf{x},2t_z-1;0)[M^{-1}(\mathbf{x},2t_z-1;0)]^\dagger\}\ , \tag{19}$$

where the trace and Hermitian-conjugation symbol refer to color space.

We see from (19) that the pion propagator can be expressed in terms of the quantity

$$G(0;t)\equiv\frac{1}{N_c}\sum_{\mathbf{x},c}\mathrm{tr}M^{-1}(\mathbf{x},t;0)[M^{-1}(\mathbf{x},t;0)]^\dagger \tag{20}$$

which is just the correlation function for the local interpolating field $(-1)^{\mathbf{x}+t}\overline{\chi}^d(\mathbf{x},t)\chi^u(\mathbf{x},t)$. It can be shown for this correlation function that[10,14]

$$G(0;t)\underset{t\gg1}{\sim}Z(0)e^{-m_\pi t}+(-1)^tZ'(0)e^{-m_e t}\ . \tag{21}$$

The mass m_e denotes the exotic state 0^{+-}. Since evidence for such a state is absent both in the physical world as well as in our Monte Carlo simulations, we will drop the second term in (21). Then by inserting a complete set of states in (7), and by using (21) in (19), we establish that

$$Z(0)=\frac{N_s\,|\langle 0\,|\,\phi(0)\,|\,\pi^+(0)\rangle\,|^2}{2(1+e^{-m_\pi a})(1+e^{m_\pi a})}\ , \tag{22}$$

where N_s is the number of elementary spatial cubes on the doubled lattice and the ket $|\pi(0)\rangle$ denotes a zero-momentum single-pion state.

We also need the two-point function for states with nonzero momentum[15] $\mathbf{p}$

$$M^{\mathbf{p}}_{\pi^+}(t_z)=\sum_{\mathbf{z}}\langle 0\,|\,T[e^{-i\mathbf{p}\cdot(2\mathbf{z})}\phi(z)\phi^\dagger(0)]\,|\,0\rangle\ . \tag{23}$$

However, upon shifting of positions to the origin this propagator becomes awkward to work with. It is easier to deal with an expression which assigns a different phase factor to each point in the original lattice. Therefore in (23) we make the replacement for the fields $\phi(z),\phi(0)$

$$e^{-i\mathbf{p}\cdot(2\mathbf{z})}\phi(z)$$
$$\rightarrow\tfrac{1}{2}\sum_{\eta}(-1)^{\eta}e^{-i\mathbf{p}\cdot(2\mathbf{z}+\boldsymbol{\eta})}\overline{\chi}^d_A(2z+\eta)\chi^u_A(2z+\eta)\ . \tag{24}$$

In the continuum limit for $|\mathbf{p}|\ll\pi/2a$ the two descriptions should coincide since the spatial cubes over which the different phase factors are distributed are small compared to all significant length scales. In practice, this limits our Monte Carlo form factor results to low momentum values,[1] which is sufficient to extract the charge radius. In the future it will be important to learn how to develop methods in the Kogut-Susskind formalism to enable higher momenta measurements.

Following steps similar to the above with the definition

$$G(\mathbf{p};t)\equiv\frac{1}{N_c}\sum_{\mathbf{x},c}e^{-i\mathbf{p}\cdot\mathbf{x}}\mathrm{tr}M^{-1}(\mathbf{x},t;0)[M^{-1}(\mathbf{x},t;0)]^\dagger\ , \tag{25}$$

we expect to have

$$G(\mathbf{p};t)\underset{t\gg1}{\sim}Z(p)e^{-E_p t}\ , \tag{26}$$

assuming the continuum dispersion relation $E_p=(m_\pi{}^2+\mathbf{p}^2)^{1/2}$. This gives

$$Z(p)=\frac{N_s\,|\langle 0\,|\,\phi(0)\,|\,\pi^+(\mathbf{p})\rangle\,|^2}{2(1+e^{-E_p a})(1+e^{E_p a})}\ . \tag{27}$$

B. The three-point function

In this section we discuss matrix elements of the conserved vector (electromagnetic) current. The current operator[8] can be derived using the fact that the action (1) is invariant under the global transformation

$$\chi(x)\rightarrow e^{i\omega}\chi(x)\ , \tag{28a}$$

$$\overline{\chi}(x)\rightarrow\overline{\chi}(x)e^{-i\omega} \tag{28b}$$

for each flavor f. The equivalent transformation for the q fields is

$$q(z)\rightarrow e^{i\Omega(1\otimes1)}q(z)\ , \tag{29a}$$

$$\overline{q}(z)\rightarrow\overline{q}(z)e^{-i\Omega(1\otimes1)}\ . \tag{29b}$$

When the transformation (29) is made local on the doubled lattice by assigning distinct phase factors, $\Omega(z)$, to each hypercube, we may use

$$\hat{j}_\mu(z)\equiv-\frac{\delta S_F(u)}{i\delta[\Delta_\mu\Omega(z)]}\ ,$$

where $\Delta_\mu\Omega(z)=\Omega(z+a_\mu)-\Omega(z)$. We give the explicit expression for the current operator only for the time component (the charge density):

$$\hat{j}_4(z)=-\sum_{\boldsymbol{\eta}}\tfrac{1}{2}\alpha_4(\boldsymbol{\eta})[\overline{\chi}(2\mathbf{z}+\boldsymbol{\eta},2t_z+1)U_4(2\mathbf{z}+\boldsymbol{\eta},2t_z+1)\chi(2\mathbf{z}+\boldsymbol{\eta},2t_z+2)$$
$$+\overline{\chi}(2\mathbf{z}+\boldsymbol{\eta},2t_z+2)U_4^\dagger(2\mathbf{z}+\boldsymbol{\eta},2t_z+1)\chi(2\mathbf{z}+\boldsymbol{\eta},2t_z+1)]\ . \tag{30}$$

Notice the nonlocality of this charge density as well as the fact that it is positioned *between* hypercubes in time.

As was mentioned previously in connection with the nonzero-momentum two-point function, it is easier to deal with expressions that assign phase factors to points in the original lattice. We therefore use (24) and make the replacement

$$\sum_{\mathbf{z}_1} e^{i\mathbf{q}\cdot(2\mathbf{z}_1)}\hat{j}_4(z_1) \to \sum_{\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}} j_4(\mathbf{x},2t_{z_1}+1) , \tag{31}$$

$$j_4(\mathbf{x},t) \equiv -\tfrac{1}{2}\alpha_4(\mathbf{x})[\bar{\chi}(\mathbf{x},t)U_4(\mathbf{x},t)\chi(\mathbf{x},t+1)+\bar{\chi}(\mathbf{x},t+1)U_4^\dagger(\mathbf{x},t)\chi(\mathbf{x},t)] . \tag{32}$$

Such replacements should not affect the low-momentum physics.

The three-point function from which we will extract the electric form factor is

$$\hat{A}(\mathbf{p},\mathbf{q},t_{z_2},t_{z_1}) = \left\langle 0 \left| T\left[\sum_{\mathbf{z}_2} e^{-i\mathbf{p}\cdot(2\mathbf{z}_2)}\phi(z_2)\sum_{\mathbf{z}_1,f} e^{i\mathbf{q}\cdot(2\mathbf{z}_1)} q_f \hat{j}_4^f(z_1)\phi^\dagger(0)\right]\right| 0\right\rangle . \tag{33}$$

Using (31) and (24) we get

$$\hat{A}(\mathbf{p},\mathbf{q},t_{z_1},t_{z_2}) = -\tfrac{1}{4}Z^{-1}\sum_{\substack{\mathbf{z}_2,\mathbf{x}_1,\\ \eta,\eta'}} e^{-i\mathbf{p}\cdot(2\mathbf{z}_2+\eta-\eta')+i\mathbf{q}\cdot\mathbf{x}_1}(-1)^{\eta+\eta'}\langle \bar{\chi}_A^d(2z_2+\eta)\chi_A^u(2z_2+\eta)\rho(\mathbf{x}_1,2t_{z_1}+1)\bar{\chi}_B^u(\eta')\chi_B^d(\eta')\rangle , \tag{34}$$

with $\rho(\mathbf{x},t)=\sum_f q_f j_4^f(\mathbf{x},t)$. The usual steps of shifting starting positions and summing on η_4,η_4' are now performed. One can now imagine doing the $\boldsymbol{\eta},\boldsymbol{\eta}'$ sums with the constraint that $(\boldsymbol{\eta}-\boldsymbol{\eta}')$ is a constant vector, yielding

$$\begin{aligned}\hat{A}(\mathbf{p},\mathbf{q};t_{z_1},t_{z_2}) = -2Z^{-1}\sum_{\mathbf{x}_2,\mathbf{x}_1} e^{-i\mathbf{p}\cdot\mathbf{x}_2+i\mathbf{q}\cdot\mathbf{x}_1}(-1)^{\mathbf{x}_2}\langle\{&\bar{\chi}_A^d(\mathbf{x}_2,2t_{z_2})\chi_A^u(\mathbf{x}_2,2t_{z_2})[\rho(\mathbf{x}_1,2t_{z_1}+1)+\rho(\mathbf{x}_1,2t_{z_1})]\\ &-\bar{\chi}_A^d(\mathbf{x}_2,2t_{z_2}+1)\chi_A^u(\mathbf{x}_2,2t_{z_2}+1)\rho(\mathbf{x}_1,2t_{z_1}+1)\\ &-\bar{\chi}_A^d(\mathbf{x}_2,2t_{z_2}-1)\chi_A^u(\mathbf{x}_2,2t_{z_2}-1)\rho(\mathbf{x}_1,2t_{z_1})\}\bar{\chi}_B^u(0)\chi_B^d(0)\rangle .\end{aligned} \tag{35}$$

Consider the generic form of a typical term in Eq. (35). We define

$$A(\mathbf{p},\mathbf{q};t_2,t_1) \equiv -Z^{-1}(-1)^{t_2}\sum_{\mathbf{x}_2} e^{-i\mathbf{p}\cdot\mathbf{x}_2}(-1)^{\mathbf{x}_2}\left\langle \bar{\chi}_A^d(\mathbf{x}_2,t_2)\chi_A^u(\mathbf{x}_2,t_2)\sum_{\mathbf{x}_1} e^{i\mathbf{q}\cdot\mathbf{x}_1}\rho(\mathbf{x}_1,t_1)\bar{\chi}_B^u(0)\chi_B^d(0)\right\rangle . \tag{36}$$

Notice that this is just the matrix element of the charge-density operator with local interpolating fields. Expressing $\hat{A}$, Eq. (35), in terms of A we get

$$\begin{aligned}\hat{A}(\mathbf{p},\mathbf{q};t_{z_2},t_{z_1}) = 2[&A(\mathbf{p},\mathbf{q};2t_{z_2},2t_{z_1}+1)\\ &+A(\mathbf{p},\mathbf{q};2t_{z_2},2t_{z_1})\\ &+A(\mathbf{p},\mathbf{q};2t_{z_2}+1,2t_{z_1}+1)\\ &+A(\mathbf{p},\mathbf{q};2t_{z_2}-1,2t_{z_1})] .\end{aligned} \tag{37}$$

Let us insert lattice completeness

$$\sum_{\mathbf{p},n} |n\mathbf{p}\rangle\langle n\mathbf{p}| = 1 \tag{38}$$

two places in expression (33). It is easy to show then that for t_{z_1}, $(t_{z_2}-t_{z_1}) \gg 1$

$$\begin{aligned}\hat{A}(\mathbf{p},\mathbf{q};t_{z_2},t_{z_1}) \to (N_s)^2 &\langle 0|\phi(0)|\pi^+(\mathbf{p})\rangle\langle\pi^+(\mathbf{p})|\phi^\dagger(0)|0\rangle\\ &\times\langle\pi^+(\mathbf{p})|\rho(0)|\pi^+(\mathbf{p}')\rangle\\ &\times e^{-E_p(2t_{z_2}-2t_{z_1})}e^{-E_{p'}(2t_{z_1})} ,\end{aligned} \tag{39}$$

with $\mathbf{p}'=\mathbf{p}-\mathbf{q}$. Assuming the amplitude (33) is real on the lattice (true to the extent that the summand is an even function of $\mathbf{z}_1$ and $\mathbf{z}_2$) we have from (27) that

$$\begin{aligned}N_s\langle 0|\phi(0)|\pi^+(\mathbf{p})\rangle\langle\pi^+(\mathbf{p}')|\phi^\dagger(0)|0\rangle\\ =2\{Z(p)Z(p')(1+e^{-E_p a})\\ \times(1+e^{E_p a})(1+e^{-E_{p'}a})(1+e^{E_{p'}a})\}^{1/2} .\end{aligned} \tag{40}$$

In order to extract the form factor from (39), it is necessary to relate the lattice-charge-density matrix element to the continuum expression. We write continuum completeness as

$$\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|n\mathbf{p})(n\mathbf{p}| = 1 . \tag{41}$$

The states $|n\mathbf{p})$ produce Lorentz-covariant matrix elements. The continuum limits

$$\frac{1}{N_s(2a)^3}\sum_{\mathbf{p}} \to \int \frac{d^3p}{(2\pi)^3} , \tag{42a}$$

$$(2a)^3\sum_{\mathbf{z}} \to \int d^3z \tag{42b}$$

show that the lattice states and spin-$\frac{1}{2}$ fields used here have the following correspondences:

$$|n\mathbf{p}\rangle \to [N_s(2a)^3 2E_p]^{-1/2}|n\mathbf{p}) , \tag{43a}$$

$$q(z) \to (2a)^{3/2}\psi(z) , \tag{43b}$$

where $\psi(z)$ is a continuum spin-$\frac{1}{2}$ field. Therefore,

$$\langle \pi^+(\mathbf{p}) | \rho(0) | \pi^+(\mathbf{p}') \rangle \rightarrow \frac{1}{N_s(2E_p 2E_{p'})^{1/2}} (\pi^+(\mathbf{p}) | \rho^c(0) | \pi^+(\mathbf{p}')) , \quad (44)$$

where $\rho^c(0)$ is the continuum charge-density operator. The electric form factor[16] is defined as

$$(\pi^+(\mathbf{p}) | \rho^c(0) | \pi^+(\mathbf{p}')) = (E_p + E_{p'}) F_+(q) . \quad (45)$$

Since we now know the relation of $\hat{A}(\mathbf{p},\mathbf{q};t_{z_2},t_{z_1})$ to $F_+(q)$, we may solve for the local quantity $A(\mathbf{p},\mathbf{q};t_2,t_1)$ from (37) when $t_1, (t_2-t_1) \gg 1$

$$A(\mathbf{p},\mathbf{q};t_2,t_1) \rightarrow \left[Z(p)Z(p') \frac{(1+e^{E_p a})(1+e^{-E_{p'} a})}{(1+e^{-E_p a})(1+e^{E_{p'} a})} \right]^{1/2} \times e^{-E_p(t_2-t_1)} e^{-E_{p'} t_1} \frac{(E_p+E_{p'})}{2(E_p E_{p'})^{1/2}} F_+(q) . \quad (46)$$

This is the result we are looking for. It shows that the pion-electric form factor can be calculated from a three-point function involving local interpolating fields. When $\mathbf{q}=0$ the operator in the three-point function A, Eq. (36), reduces to the total charge. Charge conservation yields a sum rule (for $t_2 > t_1$)

$$A(\mathbf{p},0;t_2,t_1) = G(\mathbf{p};t_2) , \quad (47)$$

a relation noted before in Ref. 3. If periodic or antiperiodic boundary conditions were used in time Eq. (47) would not have such a simple form.

To extract the form factor from Monte Carlo data it is convenient to form the combination

$$\left[\frac{A(0,\mathbf{q};t_2,t_1) A(\mathbf{q},\mathbf{q};t_2,t_1)}{G(0;t_2) G(\mathbf{q};t_2)} \right]^{1/2} \rightarrow \frac{(E_q + m_\pi)}{2(E_q m_\pi)^{1/2}} F_+(q) . \quad (48)$$

Notice that the $Z(p), Z(p')$ factors and the time dependence have all dropped out leaving only the matrix element of interest. Since it is the factors $Z(p)$ and $Z(p')$ which contain contributions from states other than the vacuum when we do our calculations with nonperiodic time-boundary conditions on a finite lattice,[17] we expect the ratio to be free of time-boundary effects. Finally, we note that it is important in calculating the statistical error in $F_+(q)$ from Monte Carlo data to include the covariances between the various factors in (48).

We conclude with a discussion of how the three-point function can be calculated as the derivative of a two-point function, a technique which has been used in other applications.[18,19] Define a new action

$$S_F(U,\alpha_t^{u,d},\mathbf{q}) = S_F(U) - \sum_{\mathbf{x},f} \alpha_t^f e^{i\mathbf{q}\cdot\mathbf{x}} j_4^f(\mathbf{x},t) . \quad (49)$$

This gives a new fermion matrix

$$M(\alpha_t,\mathbf{q})_{xA;yB} = M_{xA;yB} + \tfrac{1}{2}\alpha_t e^{i\mathbf{q}\cdot\mathbf{x}} \alpha_4(\mathbf{x}) \times \delta_{\mathbf{x},\mathbf{y}} \{ \delta_{t_x,t} \delta_{t_y,t+1} [U_4(x)]^{AB} + \delta_{t_x,t+1} \delta_{t_y,t} [U_4^\dagger(x-a_4)]^{AB} \} , \quad (50)$$

for *each* flavor, f. This new matrix has the properties

$$M^*(\alpha_t,\mathbf{q})_{xA;yB} = (-1)^{x+y} M(\alpha_t,-\mathbf{q})_{yB;xA} , \quad (51)$$

and

$$\sigma_2 M(\alpha_t,\mathbf{q}) \sigma_2 = M^*(-\alpha_t,-\mathbf{q}) \quad (52)$$

for σ_2 in color space. One can show as in (17) that (51) gives

$$[M^{-1}(\alpha_t,\mathbf{q})_{xA;yB}]^* = (-1)^{x+y} M^{-1}(\alpha_t,-\mathbf{q})_{yB;xA} . \quad (53)$$

In addition, (52) implies

$$\sigma_2 \frac{\partial}{\partial \alpha_t} M^{-1}(\alpha_t,\mathbf{q}) \sigma_2 \Big|_{\alpha_t=0} = - \frac{\partial}{\partial \alpha_t} [M^{-1}(\alpha_t,-\mathbf{q})]^* \Big|_{\alpha_t=0} . \quad (54)$$

We now go back to the three-point function Eq. (36). This can be written

$$A = -Z^{-1}(-1)^{t_2} \sum_f q_f \frac{\partial}{\partial \alpha_{t_1}^f} \sum_{\mathbf{x}_2} e^{-i\mathbf{p}\cdot\mathbf{x}_2} (-1)^{x_2} \langle\langle \bar{\chi}_A^d(\mathbf{x}_2,t_2) \chi_A^u(\mathbf{x}_2,t_2) \bar{\chi}_B^u(0) \chi_B^d(0) \rangle\rangle , \quad (55)$$

where it is understood the right-hand side is evaluated at $\alpha_{t_1}=0$ after differentiation. The notation here is

$$\langle\langle \cdots \rangle\rangle \equiv \int dU\, d\bar{\chi}\, d\chi\, e^{-S_g - S_F(U,\alpha,\mathbf{q})} (\cdots) . \quad (56)$$

We can now do the fermion integrations in (55):

$$A = Z^{-1}(-1)^{t_2} \sum_f q_f \frac{\partial}{\partial \alpha_{t_1}^f} \int dU\, e^{-S_g} \det[M(\alpha_{t_1}^u,\mathbf{q})] \times \det[M(\alpha_{t_1}^d,\mathbf{q})] \sum_{\mathbf{x}_2} e^{-i\mathbf{p}\cdot\mathbf{x}_2} (-1)^{x_2} \mathrm{tr} M^{-1}(\alpha_{t_1}^u,\mathbf{q} | \mathbf{x}_2,t_2;0) M^{-1}(\alpha_{t_1}^d,\mathbf{q} | 0;\mathbf{x}_2,t_2) . \quad (57)$$

Notice the factors of $\det[M(\alpha^{u,d},\mathbf{q})]$ above. The appearance of these determinantal factors, which would be implemented in the Monte Carlo part of the simulation, makes physical sense since the method we are describing characterizes the effect of the charge operator as a source. Therefore to be consistent, one must expect to include the effect of the source on the vacuum. Schematically, such effects come from current self-contractions. Since one can show that

$$\langle j_4(\mathbf{x},t)\rangle = 0$$

for SU(2), configuration by configuration,[20] we assume it is safe to neglect such effects.

After dropping the determinantal factors and the distinction between u and d matrices, we find upon using (53) and (54), the Monte Carlo statement

$$A(\mathbf{p},\mathbf{q};t_2,t_1) = \frac{1}{N_c}(q_u - q_d)\sum_{\mathbf{x}_2,c} e^{-i\mathbf{p}\cdot\mathbf{x}_2}\,\mathrm{tr}[M^{-1}(\mathbf{x}_2,t_2;0)]^\dagger \times \frac{\partial}{\partial\alpha_{t_1}} M^{-1}(\alpha_{t_1},\mathbf{q}\,|\,\mathbf{x}_2,t_2;0)\,. \quad (58)$$

Notice the physically necessary factor of $(q_u - q_d) = 1$ which has emerged.

III. SUMMARY

We have outlined here the theoretical ingredients necessary for an extraction of the charged pseudo-Goldstone meson form factor on the lattice. We have shown that the necessary amplitudes involve only local combinations of staggered fermion fields and are thus quite simple to calculate. In addition, we have discussed the application of the source method in our context and pointed out the usefulness of the formula (48) in correcting for time-boundary effects. Of course in these numerical investigations, one must make sure that continuum physics is being correctly represented and that the results are independent of boundary conditions and other systematic effects. The corresponding measurements using the Wilson formulation of lattice fermions will be simpler theoretically, but of course more computationally intensive.

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[1]W. Wilcox and R. M. Woloshyn, Phys. Rev. Lett. **54**, 2653 (1985).

[2]B. Velikson and D. Weingarten, Nucl. Phys. **B249**, 433 (1985).

[3]K. Barad, M. Ogilvie, and C. Rebbi, Phys. Lett. **143B**, 222 (1984).

[4]O. Martin, K. Moriarty, and S. Samuel, Columbia University report (unpublished).

[5]R. M. Woloshyn and W. Wilcox, TRIUMF Report No. TRI-PP-84-91 (unpublished).

[6]J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975); L. Susskind, *ibid.* **16**, 3031 (1977).

[7]N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981).

[8]H. Kluberg-Stern, A. Morel, O. Napoly, and B. Petersson, Nucl. Phys. **B220**, 447 (1983).

[9]J. Kogut, M. Stone, H. W. Wyld, S. H. Shenker, J. Shigemitsu, and D. K. Sinclair, Nucl. Phys. **B225**, 326 (1983).

[10]A. Billoire, R. Lacaze, E. Marinari, and A. Morel, Nucl. Phys. **B251**, 581 (1985).

[11]Because of parity mixing within the staggered formulation, another pion interpolating field is $\bar{q}(z)(\gamma_4\otimes\gamma_4^*)q(z)$, assuming the numerical absence of a signal for the 0^{+-} state. The results (46) and (48) do not depend on which interpolating field is used.

[12]D. Soper, Phys. Rev. D **18**, 4590 (1978).

[13]Such a relation does not exist for SU(3). This will be one of the main differences in generalizing the present SU(2) color derivation.

[14]M. F. L. Golterman, University of Amsterdam Report No. ITFA-85-05 (unpublished).

[15]The description of states in terms of q fields (defined on hypercubes) allows quark momenta with the values [labeling positions on the original lattice by $x_\mu = (-L_\mu+1,\ldots,L_\mu)$]

$$p_\mu = \frac{\pi n}{L_\mu a},\quad n = -\frac{L_\mu}{2}+1,\ldots,0,\ldots,L_\mu/2$$

for the L_μ = even case.

[16]See, for example, C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 160.

[17]C. Bernard, T. Draper, K. Olynyk, and M. Rushton, Nucl. Phys. **B220**, 508 (1983).

[18]S. Gottlieb, P. B. Mackenzie, H. B. Thacker, and D. Weingarten, Phys. Lett. **134B**, 346 (1984).

[19]C. Bernard, in *Gauge Theory on a Lattice: 1984,* edited by C. Zachos, W. Celmaster, E. Kovacs, and D. Sivers (National Technical Information Service, Springfield, Virginia, 1984).

[20]This will not be true for SU(3) color. Then use of this source technique along with the quenched approximation requires some extra precautions.