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Is there a solution to the Rarita-Schwinger wave equation in the presence of an external electromagnetic field?

Teymour Darkhosh
Department of Physics, Bryn Mawr College, Bryn Mawr, Pennsylvania 19010
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The method of characteristics is applied in order to obtain the retarded Green's function for the Rarita-Schwinger (RS) wave equation coupled minimally to an external electromagnetic field. The retarded Green's function is Taylor expanded in terms of singularities around the characteristic surfaces and the coefficients of singular terms are calculated. In general, when the above method is applied to a hyperbolic partial differential equation, the equations satisfied by the coefficients are singular but compatible in the sense that iteration is possible. In this paper, we shall show that in the case of the RS equation in the presence of an external electromagnetic field, this compatibility does not exist implying that the existence of the solution is doubtful.

I. INTRODUCTION

The Rarita-Schwinger (RS) wave equation, a vector-spinor which describes massive spin-$\frac{3}{2}$ particles when coupled minimally to an external electromagnetic potential, exhibits certain ill effects. It was shown by Johnson and Sudarshan that the field anticommutator is not positive definite in all Lorentz frames.[1] This implies that the anticommutator is frame dependent and that the Lorentz invariance of the theory is violated. Later this problem was related by Velo and Zwanziger to the noncausal behavior of the RS wave equation in the presence of an external electromagnetic field.[2]

Wave equations are in general hyperbolic partial differential equations with well-defined characteristic surfaces whose slopes determine the speeds of the propagation of the solutions. In the case of the RS equation, there are two distinct surfaces, one of which falls outside the light cone for a range of the external field. This implies that the solutions propagate faster than light and that the Lorentz invariance of the wave equation is questionable. The Poincaré invariance of the quantized RS field coupled to a classical electromagnetic potential was investigated by Mainland and Sudarshan.[3] They found that the generators of the Poincaré group satisfy the appropriate commutation relations with the field components. Therefore, when the equation is examined as a hyperbolic partial differential equation, one is faced with the problem of noncausal propagation, but when the methods of quantum field theory are applied, there seem to be no ill effects. Therefore, it is fair to say that the difficulty lies in the structure of the classical wave equation with constraints. The constraint equations, in the free-field case, are imposed to eliminate all the unwanted components of the field. However, when the interaction is present, the constraint equations do not seem to eliminate the unwanted components completely. In the RS equation, the difficulty arises with the spin-$\frac{1}{2}$ component of the field.[4]

In this paper, we shall investigate the existence of the solution to the RS equation when an external electromagnetic field is present by calculating the fundamental solutions, the retarded Green's functions.

We shall begin by discussing the general method of characteristics to calculate the retarded Green's functions for hyperbolic partial differential equations where we obtain a set of singular ordinary differential equations along the bicharacteristic lines. To ensure a solution, these equations have to be compatible. In the case of the RS wave equation, we will show that this compatibility is lacking. Therefore, one is forced to conclude that the solution does not exist.

We shall use the notation of Bjorken and Drell,[5] and the γ matrices have the usual representation. We shall take γ^5 to be $i\gamma^0\gamma^1\gamma^2\gamma^3$ so that $(\gamma^5)^2=1$.

II. CONSTRUCTION OF THE FUNDAMENTAL SOLUTIONS

Given a hyperbolic partial differential equation

$$L\psi=0\ , \tag{2.1}$$

the retarded Green's function satisfies

$$LG^R(x)=\delta^4(x) \tag{2.2}$$

with the boundary condition

$$G^R(x)=0 \quad \text{for } x^0<0\ . \tag{2.3}$$

L is a well-defined hyperbolic partial differential operator with definite characteristic surfaces, $u_i(x)=0$. In general, ψ has many components, L is a matrix, and (2.1) is a system of partial differential equations. Here by x we mean space and time.

According to the theory of the hyperbolic partial differential equations,[6] we can assume that the solution to (2.2) has the following form:

$$G^R(x)=\sum_{n=N}^{0}\delta^n(u)\epsilon^n(x)+R(x)\ , \tag{2.4}$$

where $u=0$ is the characteristic surface and $\delta^n(u)$ is the

nth derivative of the Dirac δ function. If there is more than one characteristic surface, then (2.4) has to be summed over the number of the surfaces. The remainder term R can be Taylor expanded around the characteristic surfaces, i.e.,

$$R(x)=\Theta(u)\sum_{n=0}^{\infty}\frac{u^n}{n!}g^n(x)\ , \tag{2.5}$$

where $\Theta(u)$ is the step function.

By substituting (2.4) into (2.2) and separating the coefficients of different singularities, we obtain the following set of equations:

$$\begin{aligned}
&A\epsilon^N=0\ ,\\
&A\epsilon^{N-1}+O\epsilon^N=0\ ,\\
&A\epsilon^{N-2}+O\epsilon^{N-1}+P\epsilon^N=0\ ,\\
&A\epsilon^{N-3}+O\epsilon^{N-2}+P\epsilon^{N-1}+Q\epsilon^N=0\ ,\\
&\ldots,\\
&Ag^0+O\epsilon^0+P\epsilon^1+Q\epsilon^2+\cdots=0\ .
\end{aligned} \tag{2.6}$$

A is a matrix, O is a first-order, P is a second-order, and Q is a third-order differential operator.

The matrix A has to be a singular matrix for ϵ^N to exist, i.e.,

$$\det|A|=0 \tag{2.7}$$

which determines the characteristic surfaces.

For first-order or second-order partial differential equations, A solely depends on the highest-order terms. Such is not the case for higher-order (order >2) partial differential equations. Since in field equations one deals with either first- or second-order partial differential equations, one can simply insert n^μ for ∂^μ in the highest-order terms to obtain the characteristic matrix. The vector $n^\mu=\partial^\mu u$ is perpendicular to the characteristic surface $u=0$.

Therefore, the singular matrix A guarantees the solution, ϵ^N. However, to be able to obtain the other terms of the expansion, one must solve equations of the form

$$A\epsilon=F\ . \tag{2.8}$$

Unless F has a certain form, this equation has no solution. If F can be written as

$$F=Af \tag{2.9}$$

then (2.8) becomes

$$A(\epsilon-f)=0 \tag{2.10}$$

and ϵ can be found. This is what is meant by the compatibility condition. Field equations for spin-0, spin-$\frac{1}{2}$, and spin-1 particles all satisfy this condition.[7,8] However, as we shall show in the next section, the Rarita-Schwinger wave equation in the presence of an external electromagnetic potential lacks this internal consistency.

III. THE RARITA-SCHWINGER WAVE IN AN EXTERNAL ELECTROMAGNETIC POTENTIAL

Consider the Lagrangian density

$$\mathscr{L}=\bar{\psi}_\alpha[g^{\alpha\beta}(\not\pi-m)-(\gamma^\alpha\pi^\beta+\pi^\alpha\gamma^\beta)+\gamma^\alpha(\not\pi-m)\gamma^\beta]\psi_\beta\ , \tag{3.1}$$

where ψ_α is the Rarita-Schwinger vector-spinor field. We have suppressed the spinor indices, i.e., ψ has 16 components. π_μ is defined as $i\partial_\mu+eA_\mu$ where A_μ is the external vector potential.

The equation satisfied by the retarded Green's function is

$$[g^{\alpha\beta}(\not\pi-m)-(\gamma^\alpha\pi^\beta+\pi^\alpha\gamma^\beta)+\gamma^\alpha(\not\pi-m)\gamma^\beta]\times G^R_{\beta\lambda}(x-y)=g^\alpha{}_\lambda\delta^4(x-y)$$

and

$$G^R=0\ \text{ for } (x^0-y^0)<0\ . \tag{3.2}$$

The differential operator in (3.2) is not a well-defined hyperbolic operator since the determinant of the characteristic matrix is identically equal to zero, i.e.,

$$\det|g^{\alpha\beta}n\cdot\gamma-(\gamma^\alpha n^\beta+n^\alpha\gamma^\beta)+\gamma^\alpha n\cdot\gamma\gamma^\beta|=0\ . \tag{3.3}$$

This seems to be a general phenomenon for wave equations with constraints, i.e., for higher spin, $s\geq1$, wave equations. To get around this difficulty, we contract (3.2) by γ_α, and then by π_α, and then substitute the results back into (3.2). The final result is

$$\left[g^{\alpha\beta}(\not\pi-m)+\frac{2e}{3m^2}\left[\pi^\alpha+\frac{m}{2}\gamma^\alpha\right]\Sigma^\beta\right]G^R_{\beta\lambda}(x-y)=\left[g^\alpha{}_\lambda-\frac{2}{3m^2}\left[\pi^\alpha\pi_\lambda+\frac{m}{2}(\gamma^\alpha\pi_\lambda-\pi^\alpha\gamma_\lambda)+\frac{m^2}{2}\gamma^\alpha\gamma_\lambda\right]\right]\delta^4(x-y)\ , \tag{3.4}$$

where

$$\Sigma^\beta=i\gamma_\alpha F^{\alpha\beta}-\tfrac{1}{2}\sigma_{\alpha\lambda}F^{\alpha\lambda}\gamma^\beta\ .$$

$\sigma_{\alpha\beta}$ is defined as $(i/2)[\gamma_\alpha,\gamma_\beta]$ and $F^{\alpha\beta}$ is the external electromagnetic field.

Now, instead of (3.4), we consider the related equation

$$\left[g^{\alpha\beta}(\not\pi-m)+\frac{2e}{3m^2}\left[\pi^\alpha+\frac{m}{2}\gamma^\alpha\right]\Sigma^\beta\right]D^R_{\beta\lambda}(x-y)=g^\alpha{}_\lambda\delta^4(x-y)\ . \tag{3.5}$$

The solution to Eq. (3.5), D, is related to the solution of

(3.4) by the operator on the right-hand side of Eq. (3.4) (Ref. 7) and the existence of G depends on the existence of D. The partial differential equation (3.5) is hyperbolic with a well-defined characteristic matrix and is equivalent to Rarita-Schwinger equation (3.2).

The function D in general has the following form:

$$D^R(x)=\sum_{i=1}^{m}\left[\sum_{n=N}^{0}\delta^n(u_i)\epsilon_i^n(x)+\Theta(u_i)\sum_{n=0}^{\infty}\frac{u_i^n}{n!}g^n(x)\right], \tag{3.6}$$

where $u_i=0$ are the characteristic surfaces where the singularities occur.

If we substitute (3.6) into (3.5) and separate the coefficients of the singularities, we obtain

$$\begin{aligned}&A\epsilon^N=0\,,\\&A\epsilon^{N-1}+O\epsilon^N=0\,,\\&A\epsilon^{N-2}+O\epsilon^{N-1}=0\,,\\&\ldots,\\&Ag^0+O\epsilon^0=0\,,\\&Ag^1+Og^0=0\,,\\&\ldots,\end{aligned} \tag{3.7}$$

where A and O are 16×16 matrices and are given by

$$\begin{aligned}A^{\alpha\beta}&=n\cdot\gamma g^{\alpha\beta}+\frac{2e}{3m^2}n^\alpha\Sigma^\beta\,,\\O^{\alpha\beta}&=g^{\alpha\beta}(\not\pi-m)+\frac{2e}{3m^2}\left[\pi^\alpha+\frac{m}{2}\gamma^2\right]\Sigma^\beta\,.\end{aligned} \tag{3.8}$$

For simplicity, we take the external field to be a uniform magnetic field in the z direction. Then we have

$$\Sigma^0=-B_z\gamma^5\gamma^3,\ \ \Sigma^1=\Sigma^2=0,\ \ \Sigma^3=-B_z\gamma^5\gamma^0\,. \tag{3.9}$$

Since A is singular, it satisfies

$$\det|A|=(n^2)^6\left[n^2+\left[\frac{2e}{3m^2}\right]^2(n\cdot\Sigma)^2\right]^2=0\,, \tag{3.10}$$

where $n^2=n\cdot n$.

There are two distinct characteristic surfaces: $(n^2)^6=0$ which is a sixfold degenerate and

$$\left[n^2+\left[\frac{2e}{3m^2}\right]^2(n\cdot\Sigma)^2\right]^2=0$$

which is a twofold degenerate.

The first surface is just the light cone along which there are six linearly independent solutions given by

$$r^1=\begin{bmatrix}0\\n\cdot\gamma\hat e_0\\0\\0\end{bmatrix},\ r^2=\begin{bmatrix}0\\0\\n\cdot\gamma\hat e_0\\0\end{bmatrix},\ r^3=\begin{bmatrix}[n\cdot\gamma(2n^0+n^0H-n_3H\gamma^5)+2n^0H(n_0-n_3\gamma^5)]\hat e_2\\ [n\cdot\gamma+(n_0-n_3\gamma^5)]2n_1H\hat e_2\\ [n\cdot\gamma+(n_0-n_3\gamma^5)]2n_2H\hat e_2\\ [n\cdot\gamma(2n_3+n_3H-n_0H\gamma^5)+2n_3H(n_0-n_3\gamma^5)]\hat e_2\end{bmatrix},$$
$$r^4=\begin{bmatrix}0\\n\cdot\gamma\hat e_1\\0\\0\end{bmatrix},\ r^5=\begin{bmatrix}0\\0\\n\cdot\gamma\hat e_1\\0\end{bmatrix},\ r^6=\begin{bmatrix}[n\cdot\gamma(2n_0-n_0H-n_3H\gamma^5)-2n^0H(n_0+n_3\gamma^5)]\hat e_3\\ -[n\cdot\gamma+(n_0+n_3\gamma^5)]2n_1H\hat e_3\\ -[n\cdot\gamma+(n_0+n_3\gamma^5)]2n_2H\hat e_3\\ [n\cdot\gamma(2n_3-n_3H-n_0H\gamma^5)-2n_3H(n_0+n_3\gamma^5)]\hat e_3\end{bmatrix}, \tag{3.11}$$

where H is defined as $(2e/3m^2)B_z$ and $\hat e_0$, $\hat e_1$, $\hat e_2$, and $\hat e_3$ are Dirac spinors given by

$$\hat e_0=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ \hat e_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad \hat e_2=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ \hat e_3=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{3.12}$$

The vectors r^i are the right-hand solutions of the matrix A. There are six linearly independent left-hand solutions:

$$\begin{aligned}l^1&=\left[n\cdot\gamma\hat e_0^T,\frac{n_0}{n_1}n\cdot\gamma\hat e_0^T,0,0\right],\\l^2&=\left[n\cdot\gamma\hat e_0^T,0,\frac{n_0}{n_2}n\cdot\gamma\hat e_0^T,0\right],\\l^3&=\left[n\cdot\gamma\hat e_0^T,0,0,\frac{n_0}{n_3}n\cdot\gamma\hat e_0^T\right],\\l^4&=\left[n\cdot\gamma\hat e_1^T,\frac{n_0}{n_1}n\cdot\gamma\hat e_1^T,0,0\right],\end{aligned} \tag{3.13}$$

$$l^5 = \left[n\cdot\gamma\hat{e}\,_1^T, 0, \frac{n_0}{n_2} n\cdot\gamma\hat{e}\,_1^T, 0 \right] ,$$

$$l^6 = \left[n\cdot\gamma\hat{e}\,_1^T, 0, 0, \frac{n_0}{n_3} n\cdot\gamma\hat{e}\,_1^T \right] .$$

$\hat{e}^T$ is the transpose of the $\hat{e}$ vector.

The solution ϵ^N has the following form:

$$\epsilon^N_{\alpha\beta} = \sum_{i=1}^{6} r^i_\alpha \sigma^{iN}_\beta . \tag{3.14}$$

σ^{iN}_β are unknown functions to be determined. Contracting the second equation in (3.7) by l^i yields

$$lO\epsilon^N = 0 \tag{3.15}$$

which determines σ^N within a constant. When the right-hand and the left-hand vectors, (3.11) and (3.12), are used to compute (3.15), the final expression is

$$\sum_{j=1}^{6} M^{ij}[2n\cdot\pi + (\partial\cdot n)]\sigma^{jN} = 0 , \tag{3.16}$$

where M^{ij} is a 6×6 nonsingular matrix. Hence the solution to (3.16) is given by

$$\sigma^{jN} = \frac{c^{jN}}{r} \exp\left[\frac{i}{2} \int_0^r A\cdot n\, dr' \right] , \tag{3.17}$$

where r is measured from the origin to a point on the characteristic surface, $u = t - r = 0$, and c^{jN} is a constant of integration determined from the initial condition, the coefficient of the δ function on the right-hand side of equation (3.5). In the case of a uniform magnetic field, we can choose a gauge where $A\cdot n$ is equal to zero.

To determine ϵ^{N-1}, we consider

$$A\epsilon^{N-1} + O\epsilon^N = 0 . \tag{3.18}$$

This is a singular equation and there is a solution if $O\epsilon^N$ has the following form:

$$O\epsilon^N = A\chi^N . \tag{3.19}$$

If such is the case, then we can write

$$A(\epsilon^{N-1} - \chi^N) = 0 \tag{3.20}$$

and the solution is

$$\epsilon^{N-1} = \chi^N + \sum_{i=1}^{6} r^i \sigma^{iN-1} . \tag{3.21}$$

For the solutions that propagate along the light cone, i.e., causal solutions, Eqs. (3.7) are compatible, and (3.19) is satisfied. However, we shall show below that for the solutions propagating along

$$\left[n^2 + \left[\frac{2e}{3m^2} \right]^2 (n\cdot\Sigma)^2 \right]^2 = 0 \tag{3.22}$$

Eqs. (3.7) are not compatible and ϵ^{N-1} does not exist.

The second surface which satisfies

$$[n^2 - H^2(n_0{}^2 - n_3{}^2)]^2 = 0$$

is given by $u' = Vt - r'$, where V is

$$V = \frac{1}{(1-H^2)^{1/2}} \tag{3.23}$$

and r' is defined as $(x^2 + y^2 + V^2 z^2)^{1/2}$.

When H satisfies

$$H^2 < 1 \tag{3.24}$$

then V is greater than one and the solutions propagate noncausally in the x-y plane.

The right-hand and the left-hand solutions to the characteristic matrix are

$$r'^1 = \begin{bmatrix} (n\cdot\gamma - n^0 H + n_3 H\gamma^5)\hat{e}_0 \\ \frac{n_1}{n_0}(n\cdot\gamma - n^0 H + n_3 H\gamma^5)\hat{e}_0 \\ \frac{n_2}{n_0}(n\cdot\gamma - n^0 H + n_3 H\gamma^5)\hat{e}_0 \\ \frac{n_3}{n_0}(n\cdot\gamma - n^0 H + n_3 H\gamma^5)\hat{e}_0 \end{bmatrix}, \quad r'^2 = \begin{bmatrix} (n\cdot\gamma + n^0 H + n_3 H\gamma^5)\hat{e}_1 \\ \frac{n_1}{n_0}(n\cdot\gamma + n^0 H + n_3 H\gamma^5)\hat{e}_1 \\ \frac{n_2}{n_0}(n\cdot\gamma + n^0 H + n_3 H\gamma^5)\hat{e}_1 \\ \frac{n_3}{n_0}(n\cdot\gamma + n^0 H + n_3 H\gamma^5)\hat{e}_1 \end{bmatrix},$$

and (3.25)

$$l'^1 = \left[[n\cdot\gamma - n_0 H - (2-H) n_3 \gamma^5] \frac{n\cdot\gamma}{n\cdot n} \hat{e}\,_0^T, O, O, [n\cdot\gamma\gamma^5 - n_3 H - (2-H) n_0 \gamma^5] \frac{n\cdot\gamma}{n\cdot n} \hat{e}\,_0^T \right] ,$$

$$l'^2 = \left[[n\cdot\gamma + n_0 H + (2+H) n_3 \gamma^5] \frac{n\cdot\gamma}{n\cdot n} \hat{e}\,_1^T, O, O, [\gamma^5 n\cdot\gamma + n_3 H + (2+H) n_0 \gamma^5] \frac{n\cdot\gamma}{n\cdot n} \hat{e}\,_1^T \right] .$$

Following the same procedure as before, we calculate $lO\epsilon^N$:

$$2\begin{bmatrix} -\left[\dfrac{1-H}{H}\right]\left[\widetilde{n\cdot\pi}+\dfrac{i}{r'}+\dfrac{m}{n_0}H(n_0{}^2-n_3{}^2)\right] & \dfrac{m}{n_0}n_3n_- \\ -\dfrac{m}{n_0}n_3n_+ & \left[\dfrac{1+H}{h}\right]\left[\widetilde{n\cdot\pi}+\dfrac{i}{r'}-\dfrac{m}{n_0}H(n_0{}^2-n_3{}^2)\right]\end{bmatrix}\begin{bmatrix}\sigma^1\\ \sigma^2\end{bmatrix}=0\,, \tag{3.26}$$

where $\widetilde{n\cdot\pi}$ is defined as $(1-H^2)n^0\pi^0-n_1\pi_1-n_2\pi_2-(1-H^2)n_3\pi_3$ and $n_\pm$ as $n_1\pm in_2$.

The solutions to (3.26) are

$$\sigma_\lambda^1=\frac{1}{r'}(A_\lambda e^{im|n|r'}+B_\lambda e^{-im|n|r'}),\quad \sigma_\lambda^2=\frac{1}{r'}\left[\frac{(|n|-n_0H)}{n_3(1+H)}\frac{n_+}{|n|}A_\lambda e^{im|n|r'}+\frac{(|n|+n_0H)}{n_3(1+H)}\frac{n_+}{|n|}B_\lambda e^{-im|n|r'}\right], \tag{3.27}$$

where $|n|=(n\cdot n)^{1/2}$ and A_λ and B_λ are constants independent of r'. Now if Eqs. (3.7) are compatible, one can always write

$$O^{\alpha\beta}\epsilon_{\beta\lambda}^N=A^{\alpha\beta}\chi_{\beta\lambda}^N\,. \tag{3.19'}$$

For (3.19′) to satisfy, $\chi_{0\lambda}^N$ for any λ has to satisfy the following equation:

$$(n_3H+n^0H\gamma^0\gamma^3-\gamma^0\gamma^5n\cdot\gamma)\{\chi_{0\lambda}^N+[(\not\pi+m)+H(\pi_0-\pi_3\gamma^5)](\hat e_0\sigma_\lambda^1+\hat e_1\sigma_\lambda^2)\}$$
$$=\gamma^5\left[\left[2\widetilde{n\cdot\pi}+\frac{2i}{r'}\right](\hat e_0\sigma_\lambda^1+\hat e_1\sigma_\lambda^2)+2mH(n_0+n_3\gamma^5)(\hat e_0\sigma_\lambda^1-\hat e_1\sigma_\lambda^2)\right]. \tag{3.28}$$

The matrix $(n_3H+n^0H\gamma^0\gamma^3-\gamma^0\gamma^5n\cdot\gamma)$, is a singular matrix along $u'=0$. Therefore, to be able to determine $\chi_{0\lambda}^N$, it is necessary for the right-hand side of Eq. (3.28) to be equal to zero. This condition is not consistent with the solutions (3.27) and the differential equations (3.26). We therefore conclude that χ^N and consequently ϵ^{N-1} do not exist and Eqs. (3.7) are not compatible. Note that when $H=0$, i.e., when the interaction is turned off, the solutions, σ^1 and σ^2, are no longer coupled and the right-hand side of (3.28) is zero.

Therefore, it seems that the Rarita-Schwinger wave equation is highly sensitive to the introduction of the external field and in fact ceases to have a solution.

[1]K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) **13**, 126 (1961).

[2]G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).

[3]G. B. Mainland and E. C. G. Sudarshan, Phys. Rev. D **10**, 3343 (1974).

[4]A. Aurilia, M. Kobayashi, and Y. Takahashi, Phys. Rev. D **22**, 1368 (1980).

[5]J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[6]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley, New York, 1962), Vol. 2, p. 618.

[7]T. Darkhosh, J. Math. Phys. **20**, 1466 (1979).

[8]D. Zwanziger, in *Invariant Wave Equations,* proceedings of the "Ettore Majorana" International School of Mathematical Physics, Erice, Italy, 1977 (Lecture Notes in Physics, Vol. 73), edited by G. Velo and A. S. Wightman (Springer, New York, 1978), p. 160.