Constraints on Hamiltonian lattice formulations of field theories in an expanding universe

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A consistent formulation of lattice field theories in Robertson-Walker spaces is not possible in comoving coordinates. For renormalizable lattice theories, physical results do not agree with results obtained using dimensional regularization. Coordinate systems where such a formulation may be possible are described. Problems with quantization with these "fixed-distance" coordinate systems are discussed.

INTRODUCTION

The study of phase transitions in the very early universe requires an understanding of the behavior of quantum field theories near their critical points. To properly treat this system, one should quantize these theories in a curved (Robertson-Walker) background characteristic of an expanding universe. Such a quantization has been studied extensively for free-field theories and much work has been done on perturbative calculations for interacting theories.[1] However, a study of the *critical* behavior of these theories requires nonperturbative methods. In flat spacetime, the most successful approach has been to consider a lattice version of the theory which then maps the problem into an analogue statistical-mechanics problem for which methods of solution are more common. There are two general approaches to the lattice method. The first, and by far the most successful, is the Euclidean Monte Carlo approach[2] which, unfortunately, is not easily generalized to Robertson-Walker spaces due to the explicit time dependence in these spaces. The second is the Hamiltonian approach[3] where only a spatial lattice is introduced and time remains continuous. This approach was, for example, the first to study the confining-deconfining transition in QCD.[4] Although it is not as powerful as the Euclidean Monte Carlo approach, its generalization to an expanding universe could prove useful in better understanding phase transitions in an expanding universe.

In formulating the lattice version of a field theory in a Robertson-Walker background, one has a wide range of choices including the choice of coordinates and a choice between many (nonequivalent) vacuum states[1] (or thermal states in case one is studying the theory at finite temperature[5]). In this paper, we show that the requirement of renormalizability, i.e., the existence of a continuum limit, severely restricts the possible lattice formulations when quantized in comoving coordinates. We begin by showing that scalar theories with quadratic divergences are nonrenormalizable when regulated on a lattice. The physical reasons for this will be discussed and we shall describe a fixed-distance coordinate system where this problem can be avoided. Lattice formulation in this coordinate system will have its own set of problems including the existence of a metric singularity and loss of translational invariance. We next discuss non-Abelian gauge theories which are conformally invariant. When regulated on a lattice in comoving coordinates, these theories behave very differently than if they were regulated via dimensional regularization. The physical model is thus regularization dependent. We shall conclude without an adequate formulation of the lattice problem but with several constraints on any future attempt at such a formulation.

SCALAR THEORIES

We start by considering a scalar field ϕ quantized in a curved background with an action

$$S=\int d^4x\,\sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)] \tag{1}$$

with

$$V(\phi)=\tfrac{1}{2}(m^2+\xi R)\phi^2+\frac{\lambda}{4}\phi^4\ ,$$

where R is the scalar curvature. We immediately specialize to spatially flat Robertson-Walker spaces and start by choosing comoving coordinates with conformal time η, so that the line element is given by[1]

$$ds^2=C(\eta)(d\eta^2-d\mathbf{r}^2)\ , \tag{2}$$

where $\sqrt{C(\eta)}$ is the scale factor of the expanding universe. In this coordinate system the action is given by

$$S=\int d\eta\, d^3r\, C(\eta)\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial\eta}\right]^2-\tfrac{1}{2}(\nabla\phi)^2-C(\eta)V(\phi)\right]\ . \tag{3}$$

It is useful to make a time-dependent canonical transformation

$$\chi=\sqrt{C}\phi\ . \tag{4}$$

Now

$$\frac{CR}{6}=\frac{\dot{D}}{2}+\frac{D^4}{4}\ , \tag{5}$$

where $D=\dot{C}/C$ and an overdot denotes $\partial/\partial\eta$. The action then becomes (apart from surface terms)

$$S=\int d\eta\, d^3r \left[\frac{\dot{\chi}^2}{2}-\frac{(\nabla\chi)^2}{2} -\frac{C}{2}[m^2+(\xi-\tfrac{1}{6})R]\chi^2-\frac{\lambda\chi^4}{4} \right] . \quad (6)$$

(In the conformal limit $m^2\to 0$, $\xi=\frac{1}{6}$, this action is identical in form to the flat-space-time action since it represents a conformally coupled theory in a conformally flat background.) The (time-dependent) Hamiltonian for this system is

$$H=\int d^3r \left[\frac{\pi^2}{2}+\frac{(\nabla\chi)^2}{2} +\frac{C}{2}[m^2+(\xi-\tfrac{1}{6})R]\chi^2+\frac{\lambda\chi^4}{4} \right] \quad (7)$$

with canonical commutation relations

$$[\chi(\mathbf{r}),\pi(\mathbf{r}')]=i\delta^3(\mathbf{r}-\mathbf{r}') . \quad (8)$$

Let us now introduce a spatial lattice with spacing a and lattice sites i. The natural choice for a lattice Hamiltonian H_L is

$$H_L=\frac{1}{a}\sum_{\mathrm{i}} \left[\tfrac{1}{2}P_{\mathrm{i}}{}^2+\tfrac{1}{2}(\Delta q_{\mathrm{i}})^2+\tfrac{1}{4}\lambda q_{\mathrm{i}}{}^4 +\frac{C(\eta)}{2}[m^2+(\xi-\tfrac{1}{6})R]q_{\mathrm{i}}{}^2 \right] ,$$
$$[q_{\mathrm{i}},P_{\mathrm{j}}]=i\delta_{\mathrm{ij}} . \quad (9)$$

(Note that $\int d^3r$ was replaced by $a^3\sum_{\mathrm{i}}$, π was replaced by P_{i}/a^2, and χ by q_{i}/a.) Here Δ represents the discretized (nearest-neighbor) Laplacian.

Before proceeding we note that there may be a temptation to allow the lattice spacing a to vary with time so that two adjacent lattice points have a fixed *proper* distance at any time. This would require $a\propto C(\eta)^{-1/2}$. Such an option is *not* possible. To see this, consider two "particles" of the system (9) separated by n lattice sites, where n is much greater than the correlation length (in lattice units) so that they interact only very weakly. Their distance z in comoving coordinates will vary in time as $a\eta$ so that $z\propto 1/\sqrt{C(\eta)}$. Thus their proper distance will be roughly constant. Such a choice completely defeats the purpose of working in a comoving frame in which free objects are receding from each other. We shall thus demand that a be constant in Eq. (9).

We now argue that the Hamiltonian (9) is not renormalizable, not even perturbatively. To see this let us specialize to the case $\xi=\frac{1}{6}$ for illustrative purposes and compute the renormalized mass m_R as a function of m, λ, and a in the one-loop approximation. For any suitable definition of m_R we certainly would expect m_R to be time dependent. Its divergent part, however, will have to be time independent if the theory is to be renormalizable. We shall see that this does not happen in theories with quadratic divergences.

The quantity m_R is evaluated via the self-energy tadpole graph. The leading one-loop divergence structure for the Hamiltonian (9) can be extracted by considering the continuum action (6) and evaluating the relevant Feynman graphs with a momentum cutoff $k_{\max}=\Lambda$. Λ will then correspond to $1/a$. Feynman rules are given in Ref. 5. The leading divergences are those of adiabatic order zero and are extracted by ignoring the time dependence of C. They are insensitive to the choice of vacuum. Thus, to extract the leading (quadratic) divergence, we can use the Feynman propagator $1/(k^2-Cm^2)$ so that

$$\begin{aligned} Cm_R{}^2&=Cm^2+\text{const}\times\lambda\int d^4k\frac{1}{k^2-Cm^2}\\ &=Cm^2+\text{const}\times\lambda\Lambda^2\\ &=Cm^2+\text{const}\times\lambda/a^2 . \end{aligned} \quad (10)$$

It follows that if $m_R{}^2$ is to be finite as $a\to 0$, the leading term in m^2 will have to be chosen as

$$m^2\sim m_R{}^2-\frac{\text{const}}{Ca^2} .$$

This contradicts the fact that the bare mass m^2 does not depend on time. [A time-dependent m^2 would violate the general covariance of the action (1).] Thus lattice (or momentum-cutoff) regularization does not work for massive scalar fields. (Choosing $a^2\propto 1/C$ is not viable, as discussed previously.)

The physical reason for this is as follows. The Hamiltonian (9) represents a set of coupled oscillators each in a local potential $(C/2)m^2q^2+(\lambda/4)q^4$. As C increases, both the q^4 term and the nearest-neighbor coupling term appear to become negligible. At large times we would conclude that Eq. (9) just describes a set of decoupled harmonic oscillators. We know, however, that such an interpretation is wrong in the continuum limit. The reason why the neighbor-oscillator coupling *seems* to be weakening is that the oscillators are moving physically farther apart, though their coordinate distance is unchanged. In the continuum limit there are many new lattice points in between the original neighbors with a much stronger interaction (since their lattice spacing is smaller). These interactions force correlations on the system which are not seen in the lattice version of Eq. (6). The mathematical manifestation of these ideas is the inability to renormalize this Hamiltonian, even perturbatively.

The previous discussion suggests that we might solve the problem by working in a fixed-distance coordinate system in which the spatial coordinates more closely represent the physical distance. If we start with Robertson-Walker coordinates

$$ds^2=dt^2-a^2(t)(dr^2+r^2d\Omega^2) , \quad (11)$$

and define a new coordinate

$$\rho=a(t)r \quad (12)$$

then

$$ds^2=(1-H^2\rho^2)dt^2+2H\rho\, d\rho\, dt-d\rho^2-\rho^2d\Omega^2$$

with (13)

$$H=\dot{a}/a\ .$$

This system has a coordinate singularity at $\rho=H^{-1}$. In this coordinate system the action (1) becomes

$$S=\int dt\,d\rho\,d\theta\,d\Phi\,\rho^2 \times\sin\theta\left[\frac{\dot{\phi}^2}{2}+H\rho\dot{\phi}\frac{\partial\phi}{\partial\rho}-\tfrac{1}{2}(1-H^2\rho^2)\left[\frac{\partial\phi}{\partial\rho}\right]^2 -\frac{1}{2\rho^2}\left[\frac{\partial\phi}{\partial\theta}\right]^2-\frac{1}{2\rho^2\sin^2\theta}\left[\frac{\partial\phi}{\partial\Phi}\right]^2 -V(\phi)\right],\qquad(14)$$

where the integral is only over the region $0\leq\rho<H^{-1}$. The Hamiltonian for this system (in "Cartesian" coordinates) is given by

$$H=\int d^3\rho\left[\frac{\pi^2}{2}+\frac{(\nabla\phi)^2}{2}+V(\phi)-H\pi\boldsymbol{\rho}\cdot\nabla\phi\right]$$

with (15)

$$[\phi(\rho),\pi(\rho')]=i\delta^3(\rho-\rho')\ .$$

(The operator ordering in the last term is irrelevant since the spatial integral of the commutation of π with $\boldsymbol{\rho}\cdot\nabla\phi$ is a c number.)

It is straightforward to check explicitly that the leading (quadratic) divergence, in this case, leads to a time-independent infinite renormalization, as required. [This can be seen intuitively by noting that the leading divergence comes from the lowest-order adiabatic term in the propagator which is obtained by ignoring the time dependence of $a(t)$. If we set $H=0$ in Eq. (9) we retrieve the flat-space-time Hamiltonian in which the renormalization works out properly.]

The lattice version of Eq. (15) seems like a promising possible lattice version for this theory (either in Cartesian or in spherical polar coordinates). There remain, however, several problems. The first (and least serious) is the lack of spatial translational invariance in Eq. (15). This may make the theory more difficult to analyze. A much more serious issue is the singularity in the metric leading to Eq. (15). Although this singularity is not evident in Eq. (15) it is easy to show that for $\rho>H^{-1}$ the Hamiltonian density is not positive semidefinite. (To see this, start by completing the square for π in spherical polar coordinates.) As a result, Eq. (19) only describes the dynamics of part of the quantum system. In fact we should wonder what boundary conditions to impose at $\rho=H^{-1}$. In the lattice version we must cope with the fact that this "boundary" typically grows with time. Furthermore it is not clear that the ground state of this Hamiltonian is a proper physical ground state for the system. In fact, if we were in de Sitter space ($\dot{H}=0$) then the de Sitter-invariant vacuum for this system would lead to a density matrix, rather than a pure state,[6] in these coordinates due to the lack (in principle) of information about the region $\rho>H^{-1}$. (In fact with a slight modification of the coordinates the de Sitter-invariant state would correspond to a thermal state at the Hawking temperature.) Thus the choice of initial state as well as the choice of the boundary and of the boundary conditions must be seriously dealt with before studying the lattice version of Eq. (15).

NON-ABELIAN GAUGE THEORIES

In this section we study the lattice versions of non-Abelian gauge theories in Robertson-Walker spaces. The conformal invariance of these theories will allow them to be renormalized in comoving coordinates but they will show some very unusual behavior which makes the lattice method of renormalization quite unacceptable.

Another goal of studying gauge theories in an expanding universe might be to start, at some initial time, in thermal equilibrium at a temperature $T\gg T_{\rm cr}$ where $T_{\rm cr}$ is the critical temperature at which the theory deconfines. The future development of the system will then be completely controlled by the equations of motion. It would be nice to follow the system through its critical point as the universe expands, simply by evolving the initial thermal state.

A pure gauge theory is conformally invariant. Thus in conformal coordinates (2) the action for such a theory is given by

$$S=\frac{1}{g^2}\int d\eta\,d^3r\,\tfrac{1}{2}(E^2-B^2)\ ,\qquad(16)$$

where

$$E_i=F_{0i}\ \text{ and }\ B_i=\tfrac{1}{2}\epsilon_{ijk}F_{jk}\ .$$

The Hamiltonian for this system, in the $A_0=0$ gauge, is given by

$$H=\int d^3r\left[\frac{g^2E^2}{2}+\frac{B^2}{2g^2}\right]\qquad(17)$$

with

$$[A_i^a(x),E_j^b(y)]=i\delta_{ij}\delta_{ab}\delta^3(x-y)\ .$$

The Hamiltonian lattice version H_L of H is given by[3]

$$H_L=\frac{1}{a}\left[\sum_{\substack{\text{sites}\\ \text{i}}}\frac{g^2E_{\rm i}^2}{2}+\frac{1}{2g^2}\sum_{\rm Pl}\text{Tr}\left[\prod_{\rm Pl}U+\text{H.c.}\right]\right],\qquad(18)$$

where a is the lattice spacing, $\sum_{\rm Pl}$ represents a sum over plaquettes, U is the link variable defined on a plaquette $[U\sim P\exp(i\int A\cdot dl)]$, and H.c. denotes Hermitian conjugation. As discussed in the previous section, we must choose a to be constant. Thus H_L is precisely the flat-space-time lattice-gauge-theory Hamiltonian.

The conformal invariance is, of course, anomalous. The introduction of a scale into this theory which is required when the theory is renormalized leads to a very surprising result. The continuum limit can be obtained by letting $a\rightarrow0$ and $g\rightarrow0$ in such a way that some physical quantity such as the mass gap, m, stays fixed (i.e., $m[g(a),a]$ is independent of a). If we do this at some time t_0, this mass gap (or the correlation length $l\propto m^{-1}$) will be independent of time, since H_L is time independent.

Thus the correlation length is constant in a comoving frame so that it grows with the scale of the universe. This is certainly contrary to expectations. Usually, in conformally noninvariant theories, the length scale stays fixed as the universe expands. The lattice formulation of the Hamiltonian has thus led us to an unusual result which we shall see, is *not* present in other regularization methods.

This unusual result does not hold in dimensional regularization. It is useful to review this argument at this point. In n space-time dimensions the action (16) is given by

$$S=\frac{1}{2g^2}\int d\eta\, d^3r(C\mu^2)^{(n-4)/2}(E^2-B^2)\ , \tag{19}$$

where the arbitrary mass μ is introduced to scale out the dimensions of g via $g\to(\mu^2)^{(4-n)/2}g$; and the factor of $C^{(4-n)/2}$ comes from the values of $\sqrt{-g}$ and $g^{\mu\nu}$ in n dimensions. Conformal invariance is now broken by the scale factor μ when $n\neq 4$. μ, however, always appears in the combination $C\mu^2$. Thus the mass gap m must be of the form

$$m^2\equiv C\mu^2F(g,n-4)\ , \tag{20}$$

where F is a dimensionless function of g and $n-4$. Thus m varies as $\sqrt{C}$ and the universe expands so that the correlation length $l\propto 1/m$ increases like $1/\sqrt{C}$ in comoving coordinates. Thus the physical length scale stays fixed as the universe expands. The naive lattice formulation (18) of the Hamiltonian (17) does *not* give the same answer.

We conclude that the physics of conformally invariant field theories in Robertson-Walker spaces is regularization dependent when quantized in comoving coordinates. In the dimensionally regulated theory the dynamical scale stays fixed as the universe expands, whereas in the lattice regulated theory it grows with the scale of the universe.

A similarly unusual situation arises if we add a term which explicitly breaks the conformal invariance. Consider, for example, a gauge theory coupled to a fermion of mass M. Unlike the scalar theory discussed in the previous section, this theory will be renormalizable when a lattice regulator is used, since only logarithmic (rather than quadratic) divergences are present. In dimensional regularization both the mass scale M and the dynamically generated scale m of the gauge theory stay fixed as the universe expands, whereas in the lattice theory, the scale M stays fixed whereas the length scale m^{-1} grows with the expanding universe. Interactions will, of course, mix these two scales leading to very different dynamics for the two regularization schemes.

One can show that the previously mentioned results apply in Robertson-Walker coordinates with proper time $[ds^2-a^2(t)d\mathbf{r}^2]$ as well; though the arguments are somewhat more complicated and we do not reproduce them here.

The next step is to attempt a lattice formulation in the coordinate system (13). The action (16) now becomes

$$S=\frac{1}{2g^2}\int dt\, d^3\rho[E^2-B^2+H\mathbf{E}\cdot(\boldsymbol{\rho}\times\mathbf{B})]\ , \tag{21}$$

where the dot and $\times$ denote *spatial* dot and cross products and $H=a^{-1}da/dt$. In the $A_0=0$ gauge the momentum π conjugate to A_i is

$$\pi=\frac{1}{g^2}(\mathbf{E}+\tfrac{1}{2}H\boldsymbol{\rho}\times\mathbf{B})\ . \tag{22}$$

Gauss's law becomes

$$D_i\pi_i=0\ , \tag{23}$$

where D_i is the group-covariant derivative.

The Hamiltonian is given by

$$H=\int d^3\rho\left[\frac{g^2}{2}\left[\pi-\frac{1}{2g^2}H\boldsymbol{\rho}\times\mathbf{B}\right]^2+\frac{B^2}{2g^2}\right]\ . \tag{24}$$

This Hamiltonian could be used as a starting point for formulating a lattice version of the theory although the problems previously mentioned regarding vacuum choice must still be dealt with.

SUMMARY

In this paper it is shown that formulation of lattice versions of field theories in Robertson-Walker spaces is not as straightforward as one might expect. Conformally invariant theories whose conformal invariance is broken by interactions have different physical behavior when regulated on a lattice as opposed to being regulated dimensionally. In the conventional (dimensional) regularization, the induced scale stays fixed as the universe expands whereas in the lattice approach, this scale grows with the scale of the universe. Massive scalar theories cannot be renormalized on a lattice in comoving coordinates if the lattice spacing is assumed time independent. It is shown that this must be the case since a variation of the lattice spacing in time will result in distant objects *not* receding and thus the universe *not* expanding. It is suggested that lattice versions should be formulated in a fixed-distance coordinate system where coordinate distances are more closely related to physical distances. These coordinate systems have singularities which make the choice of ground state and the boundary conditions difficult to specify. Furthermore, they only cover a time-dependent portion of the space ($\rho<H^{-1}$) near an observer. It seems an intractable problem to try to patch these (growing) "lattice formulated" spheres together to make up the whole space. We thus conclude with no adequate formulation of these theories. The lattice versions of the Hamiltonians (15) and (24) show the most promise.

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