Finite-dimensional approximation of time evolution operators: A nonperturbative approach to quantum field theories

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We suggest a time-dependent, finite-dimensional approximation scheme for time evolution operators and the S matrix in relativistic quantum field theories. It can be solved by diagonalization of finite-dimensional Hermitian matrices. The method is presented for a ϕ^4-field theory. In the weak-coupling regime, where perturbation theory is applicable, our method tends to the results of perturbation theory in all orders, when the approximation parameters tend to infinity. Renormalization is achieved by introducing counterterms in the Lagrangian in the usual way. This method does not involve vertices and propagators and is a nonperturbative scheme. It is applicable to arbitrary coupling constants.

I. INTRODUCTION

In recent years several methods have been developed in order to overcome the limitations of standard perturbation theory for relativistic quantum field theories. One approach is based on finding the large-order behavior via functional integrals and construct the solution via Borel transformations.[1] Another approach is taken in lattice-field theory.[2] Discretization of coordinate space is suitable for the treatment of bound-state problems, where particles are essentially confined to a finite region in coordinate space.

Here we suggest a discretization in momentum space, which should be suitable for scattering problems, where the incoming and outgoing particles are essentially confined to a finite region in momentum space. On that basis a method has been proposed in Refs. 3–9 for the calculation of nonrelativistic few-body wave operators and the S matrix, in the following called the strong approximation of Moller (SAM) wave-operator approach. It is a time-dependent systematic approximation scheme based on approximated finite-dimensional Hermitian Hamilton matrices. Scattering states are calculated by diagonalizing these matrices. The scattering boundary conditions are included in the form of Moller wave operators; i.e., one approximates

$$\Omega^{(\pm)}=\operatorname*{s-lim}_{t\to\mp\infty}\exp(iHt)\exp(-iH^0t)\to$$
$$\Omega_n(\mp T)=\exp(\mp iH_nT)\exp(\pm iH_n^0T)\ , \tag{1}$$

where H and H^0 denote the full and the asymptotic Hamiltonian, respectively, and H_n, H_n^0 denote the corresponding finite-dimensional approximations. T is a real time parameter. It can be shown rigorously for quite general classes of potentials and different types of finite-dimensional approximations that $\Omega_n(\mp T)$ tends strongly to $\Omega^{(\pm)}$ and

$$S_n(+T,-T)=\Omega_n(+T)^\dagger\Omega_n(-T)$$

tends weakly to $S=\Omega^{(-)\dagger}\Omega^{(+)}$ for suitable n, $T\to\infty$.[3–5] It is possible to include long-range Coulomb forces by using suitably modified wave operators. The method has been tested in the two-body system for a variety of short-range potentials and the long-range Coulomb potential and was found to converge to the reference solution in all cases.[6,7] It has been applied to the $d+p\to p+p+n$ reaction close to the breakup threshold.[6,7] The following important features of the method should be noted.

(a) The calculation of $\Omega_n(\mp T)$ is straightforward by diagonalization of H_n, H_n^0.

(b) In contrast with time-independent formulations this method does not involve expansions in terms of vertices and propagators and it is free of singularities.

(c) Arbitrary coupling constants can be handled.

Based on these features we suggest here a generalization of the SAM approach to relativistic quantum field theories.

II. FORMALISM

We discuss the SAM approach for a neutral pseudoscalar-field theory with a ϕ^4 interaction in $3+1$ dimensions, given by the unrenormalized Lagrangian density (renormalization shall be discussed later)

$$\mathcal{L}^0=\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x)-\frac{m^2}{2}\phi^2(x)\ ,$$
$$\mathcal{L}^{\rm int}=-\frac{\lambda_4}{4!}\phi^4(x)\ , \tag{2}$$
$$\mathcal{L}=\mathcal{L}^0+\mathcal{L}^{\rm int}\ .$$

Let H^0, $H^{\rm int}$, H denote the corresponding Hamiltonians. Let ϕ^0 denote the free field. In the interaction representation the time evolution can be written

$$U(t,0)=\exp[iH^0\{\phi^0(\mathbf{x},0)\}t]\exp[-iH\{\phi(\mathbf{x},0)\}t] \tag{3}$$

and it is assumed that at $t=0$ the full and the free fields as well as their first-order time derivatives coincide, al-

lowing us to express H in terms of the free field operators[10]

$$H\{\phi(\mathbf{x},0)\}=H^0\{\phi^0(\mathbf{x},0)\}+H^{\rm int}\{\phi^0(\mathbf{x},0)\}\ . \tag{4}$$

The S matrix can be calculated from

$$\begin{aligned}U(t,t')&=U(t,0)U^\dagger(t',0)\\&=\exp(iH^0t)\exp[-iH(t-t')]\exp(-iH^0t')\end{aligned} \tag{5}$$

in the limit $t\to+\infty$, $t'\to-\infty$. In analogy to the nonrelativistic case, we suggest approximating those exponentials by finite-dimensional operators and calculating from those approximate S-matrix elements. We introduce four approximation parameters: (a) a cutoff Q in momentum space which defines a finite ball B_Q centered at the origin, (b) a partition of this ball in n cells, (c) a maximal number N of particles occurring in Fock space, and (d) a real, finite time T. The free field and functionals constructed from it such as the Hamiltonian, the time-evolution operator, and the S matrix are all substituted by a finite-dimensional approximation. We define creation and annihilation operators averaged over the small but finite cells of the ball B_Q,

$$\begin{aligned}A_i^\dagger&=\int d^3\mathbf{q}\,\chi_i(\mathbf{q})A^\dagger(\mathbf{q})\Big/\int d^3\mathbf{q}\,\chi_i(\mathbf{q})\ ,\\A_i&=\int d^3\mathbf{q}\,\chi_i(\mathbf{q})A(\mathbf{q})\Big/\int d^3\mathbf{q}\,\chi_i(\mathbf{q})\ ,\quad i=1,\ldots,n\ ,\end{aligned} \tag{6}$$

where $A^\dagger(\mathbf{q})$, $A(\mathbf{q})$ are the time-independent Fourier components of the decomposition of the free field $\phi^0(x)$ and $\chi_i(\mathbf{q})$ is the characteristic function of cell i (i.e., takes the value 1 inside the cell i and 0 outside). The "discretized" creation and annihilation operators $A_i^\dagger$, A_i obey the commutation rules

$$\begin{aligned}&[A_i,A_j]=[A_i^\dagger,A_j^\dagger]=0\ ,\\&[A_i,A_j^\dagger]=\delta_{i,j}/v_i\ ,\quad i,j=1,\ldots,n\ ,\end{aligned} \tag{7}$$

where v_i denotes the volume of cell i. We define the following subspaces of the Fock space, built from the generalized "particles" corresponding to the momentum-space cells. Let $\mathcal{F}_0$ be generated by the vacuum state $|0\rangle$, and let $\mathcal{F}_{k,n}$ be generated by $A_{i_1}^\dagger\cdots A_{i_k}^\dagger|0\rangle$, $i_1,\ldots,i_k\in1,\ldots,n$, which is a k-particle subspace. In order to obtain a finite-dimensional subspace, we restrict also the total particle number by N and define a subspace

$$\mathcal{F}_{\mathcal{N}}=\mathcal{F}_0\oplus\mathcal{F}_{1,n}\oplus\cdots\oplus\mathcal{F}_{N,n}\ , \tag{8}$$

where $\mathcal{N}=\mathcal{N}(N,n)$ is a multi-index. Because the Fock space is a Hilbert space, one can define $P_{k,n}$, $P_{\mathcal{N}}$ as orthogonal projectors on $\mathcal{F}_{k,n}$ and $\mathcal{F}_{\mathcal{N}}$, respectively. We define a finite-dimensional approximation of the full, the free, and the interaction Hamiltonian, respectively, via

$$\begin{aligned}&H\to H_{\mathcal{N}}=P_{\mathcal{N}}HP_{\mathcal{N}}\ ,\quad H^0\to H^0_{\mathcal{N}}=P_{\mathcal{N}}H^0P_{\mathcal{N}}\ ,\\&H^{\rm int}\to H^{\rm int}_{\mathcal{N}}=P_{\mathcal{N}}H^{\rm int}P_{\mathcal{N}}\ .\end{aligned} \tag{9}$$

Note that the finite-dimensional Hamiltonians are defined on the whole Fock space and that because of the Hermiticity of H, H^0, and $H^{\rm int}$, also $H_{\mathcal{N}}$, $H^0_{\mathcal{N}}$, and $H^{\rm int}_{\mathcal{N}}$ are Hermitian, which is convenient for numerical calculations. Now we define a finite-dimensional approximation of the time evolution operator,

$$\begin{aligned}&U(t,0)\to U_{\mathcal{N}}(t,0)=\exp(iH^0_{\mathcal{N}}t)\exp(-iH_{\mathcal{N}}t)\ ,\\&U(t,t')\to U_{\mathcal{N}}(t,t')=U_{\mathcal{N}}(t,0)U^\dagger_{\mathcal{N}}(t',0)\ ,\end{aligned} \tag{10}$$

and we suggest approximating $U(+\infty,-\infty)$, which enters in the S matrix, by

$$U(+\infty,-\infty)\to U_{\mathcal{N}}(+T,-T)\ , \tag{11}$$

where T is a real, finite-time parameter. The following remarks should be noted. (a) The operator $U_{\mathcal{N}}(T,-T)$ can be evaluated nonperturbatively by diagonalization of the finite-dimensional operators $H^0_{\mathcal{N}}$, $H_{\mathcal{N}}$,

$$\begin{aligned}&H^0_{\mathcal{N}}|\psi^0_\nu\rangle=E^0_\nu|\psi^0_\nu\rangle\ ,\\&H_{\mathcal{N}}|\psi_\nu\rangle=E_\nu|\psi_\nu\rangle\ ,\quad \nu=1,\ldots,r\ ,\\&U_{\mathcal{N}}(T,-T)=\sum_{\nu,\mu,\rho=1}^{r}|\psi^0_\nu\rangle\exp(iTE^0_\nu)\langle\psi^0_\nu|\psi_\mu\rangle\exp(-i2TE_\mu)\langle\psi_\mu|\psi^0_\rho\rangle\exp(iTE^0_\rho)\langle\psi^0_\rho|\ ,\end{aligned} \tag{11a}$$

where r is the rank of the matrices. Since by construction the finite-dimensional operators $H^0_{\mathcal{N}}$ and $H_{\mathcal{N}}$ are Hermitian, the approximate evolution operator $U_{\mathcal{N}}(T,-T)$ is unitary. The unitarity is preserved for any set of approximation parameters Q,n,N,T for any value of the coupling constant. (b) $U_{\mathcal{N}}(T,-T)$ is nonsingular for any set of finite approximation parameters. (c) As can be seen from Eq. (11a) the method relies on the calculation of exponentials. Contrary to perturbation theory we do not need the power-series expansion of the exponential in the coupling constant. This is due to the fact that our method is based on direct calculation of the exponential by diagonalizing the finite-dimensional matrices $H^0_{\mathcal{N}}$ and $H_{\mathcal{N}}$. Therefore the formalism is not restricted to a particular regime of the coupling constant. (d) The physical S matrix corresponds to the limit $Q,n,N,T\to\infty$. Some care is necessary for the order of the limits. It has been discussed in Ref. 7 for the nonrelativistic case, where the particle number parameter N is not involved. That experience tells us, e.g., that it is wrong to keep Q,n,N fixed but increase T, because $U_{\mathcal{N}}(T,-T)$ composed of exponentials of finite-dimensional matrices starts to oscillate as a function of T. The correct way is to choose T_1, then to increase Q,n,N until one obtains a stable limit $S(T_1)$. Then one picks

another $T_2 > T_1$, and increases Q,n,N until one obtains a limit $S(T_2)$ and so on. Finally, S is taken as the limit of $S(T_1)$, $S(T_2)$, ... Moreover, some correlation between Q and n is necessary. Increasing Q and thus the ball B_Q, the partition and the number of cells n have to be chosen such that the volume of the largest cells tends to 0.

III. EQUIVALENCE TO PERTURBATION THEORY

For small coupling constants standard perturbation theory gives the physical S-matrix elements. We claim for small coupling constants that the results of standard perturbation theory are obtained from $U_{\mathcal{N}}(+T,-T)$ if the approximation parameters tend to infinity. In particular $U_{\mathcal{N}}(+T,-T)$ can be expanded in orders of an expansion parameter in the same way as $U(+\infty,-\infty)$. Then $U_{\mathcal{N}}(+T,-T)$ tends to $U(+\infty,-\infty)$ in every order. We will demonstrate it explicitly up to second order and give an argument for higher orders. In the interaction representation one has

$$H^{\mathrm{int}}(t)=\exp(iH^0t)H^{\mathrm{int}}\,|_{t=0}\exp(-iH^0t)\,, \tag{12}$$

and the time evolution $U(t,0)$ obeys

$$i\frac{\partial}{\partial t}U(t,0)=H^{\mathrm{int}}(t)U(t,0)\,, \tag{13}$$

which yields the perturbation expansion

$$U(+\infty,-\infty)=1-i\int_{-\infty}^{\infty}dt\,H^{\mathrm{int}}(t)+\frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\mathcal{T}\{H^{\mathrm{int}}(t_1)H^{\mathrm{int}}(t_2)\}$$
$$+\frac{(-i)^3}{3!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\int_{-\infty}^{\infty}dt_3\mathcal{T}\{H^{\mathrm{int}}(t_1)H^{\mathrm{int}}(t_2)H^{\mathrm{int}}(t_3)\}+\cdots\,, \tag{14}$$

where $\mathcal{T}$ denotes time ordering. We define

$$H^{\mathrm{int}}_{\mathcal{N}}(T)=\exp(iH^0_{\mathcal{N}}T)H^{\mathrm{int}}_{\mathcal{N}}\Big|_{t=0}\exp(-iH^0_{\mathcal{N}}T) \tag{15}$$

and obtain from the finite-dimensional time evolution $U_{\mathcal{N}}(T,0)$

$$i\frac{\partial}{\partial T}U_{\mathcal{N}}(T,0)=H^{\mathrm{int}}_{\mathcal{N}}(T)U_{\mathcal{N}}(T,0)\,, \tag{16}$$

which yields the perturbation expansion

$$U_{\mathcal{N}}(+T,-T)=1-i\int_{-T}^{T}dt\,H^{\mathrm{int}}_{\mathcal{N}}(t)+\frac{(-i)^2}{2!}\int_{-T}^{T}dt_1\int_{-T}^{T}dt_2\mathcal{T}\{H^{\mathrm{int}}_{\mathcal{N}}(t_1)H^{\mathrm{int}}_{\mathcal{N}}(t_2)\}$$
$$+\frac{(-i)^3}{3!}\int_{-T}^{T}dt_1\int_{-T}^{T}dt_2\int_{-T}^{T}dt_3\mathcal{T}\{H^{\mathrm{int}}_{\mathcal{N}}(t_1)H^{\mathrm{int}}_{\mathcal{N}}(t_2)H^{\mathrm{int}}_{\mathcal{N}}(t_3)\}+\cdots\,. \tag{17}$$

Because H^0 is a diagonal Hamiltonian in the momentum representation and the characteristic functions χ_i are pairwise orthogonal, $H^0_{\mathcal{N}}$ is also diagonal. It takes the form

$$H^0_{\mathcal{N}}=\begin{cases}\sum_{i=1}^{n}v_i\omega_iA_i^{\dagger}A_i & \text{on } \mathcal{F}_{\mathcal{N}}\,,\\ 0 & \text{on } \mathcal{F}^{\perp}_{\mathcal{N}}\,,\end{cases} \tag{18}$$

where $\mathcal{F}^{\perp}_{\mathcal{N}}$ is the orthogonal complement of $\mathcal{F}_{\mathcal{N}}$ and

$$\omega_i=\int d\mathbf{k}\,\omega(\mathbf{k})\chi_i(\mathbf{k})\Big/\int d\mathbf{k}\,\chi_i(\mathbf{k})\,. \tag{19}$$

Hence,

$$\exp(iH^0_{\mathcal{N}}t)=\begin{cases}\exp\left[it\sum_{i=1}^{n}v_i\omega_iA_i^{\dagger}A_i\right] & \text{on } \mathcal{F}_{\mathcal{N}}\,,\\ 1 & \text{on } \mathcal{F}^{\perp}_{\mathcal{N}}\,,\end{cases} \tag{20}$$

We define

$$\delta_{i,\mathbf{k}}=[A_i,A^{\dagger}(\mathbf{k})]=[A(\mathbf{k}),A_i^{\dagger}]=\chi_i(\mathbf{k})/v_i\,,$$
$$\delta(i_1+i_2-i_3-i_4)=\int d\mathbf{k}_1\cdots d\mathbf{k}_4\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta_{i_1,\mathbf{k}_1}\delta_{i_2,\mathbf{k}_2}\delta_{i_3,\mathbf{k}_3}\delta_{i_4,\mathbf{k}_4}\,. \tag{21}$$

Let us consider for the ϕ^4 interaction the perturbation expansion of $U_{\mathcal{N}}(+T,-T)$ up to second order. In particular we consider only initial and final two-particle states. The zeroth order is 1 coinciding with the zeroth order of U. The first-order term can be written, for sufficiently small cells, i.e., sufficiently large n, as

$$\langle A_{i_1}{}^{\dagger}A_{i_2}{}^{\dagger} \mid U^{(1)}_{\mathcal{N}}(+T,-T) \mid A_{i_3}{}^{\dagger}A_{i_4}{}^{\dagger}\rangle = \alpha^{(1)} \frac{\delta(i_1+i_2-i_3-i_4)}{(\omega_{i_1}\omega_{i_2}\omega_{i_3}\omega_{i_4})^{1/2}} \int_{-T}^{T} dt \exp[i(\omega_{i_1}+\omega_{i_2}-\omega_{i_3}-\omega_{i_4})t] , \tag{22}$$

where $\alpha^{(1)}$ is an overall factor. Performing the limits $n\to\infty$, $Q\to\infty$, $T\to\infty$, one obtains

$$\alpha^{(1)} 2\pi \frac{\delta^4(p_1+p_2-p_3-p_4)}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}} = \langle \mathbf{p}_1,\mathbf{p}_2 \mid U^{(1)}(+\infty,-\infty) \mid \mathbf{p}_3,\mathbf{p}_4\rangle . \tag{23}$$

The corresponding graph is depicted in Fig. 1(a). Note that the order of the limits $n,Q,T\to\infty$ is important (see remark in Sec. II).

The second-order term reads

$$\begin{aligned}
&\langle A^{\dagger}_{i_1}A^{\dagger}_{i_2} \mid U^{(2)}_{\mathcal{N}}(+T,-T) \mid A^{\dagger}_{i_3}A^{\dagger}_{i_4}\rangle \\
&\quad = \alpha^{(2)} \Big[\int_{-T}^{T} dT_1 \int_{T_1}^{T} dT_2 \exp[i(\omega_{i_1}+\omega_{i_2})T_2 - i(\omega_{i_3}+\omega_{i_4})T_1] \langle A^{\dagger}_{i_1}A^{\dagger}_{i_2} \mid P_{\mathcal{N}}{:}\phi^4{:}P_{\mathcal{N}} e^{-iH^0_{\mathcal{N}}(T_2-T_1)} P_{\mathcal{N}}{:}\phi^4{:}P_{\mathcal{N}} \mid A^{\dagger}_{i_3}A^{\dagger}_{i_4}\rangle \\
&\qquad + \int_{-T}^{T} dT_1 \int_{-T}^{T_1} dT_2 \exp[i(\omega_{i_1}+\omega_{i_2})T_1 - i(\omega_{i_3}+\omega_{i_4})T_2] \\
&\qquad\qquad \times \langle A^{\dagger}_{i_1}A^{\dagger}_{i_2} \mid P_{\mathcal{N}}{:}\phi^4{:}P_{\mathcal{N}} e^{-iH^0_{\mathcal{N}}(T_1-T_2)} P_{\mathcal{N}}{:}\phi^4{:}P_{\mathcal{N}} \mid A^{\dagger}_{i_3}A^{\dagger}_{i_4}\rangle \Big] .
\end{aligned} \tag{24}$$

Because the external states are two-particle states, the $:\phi^4:$ interaction admits here as intermediary states only two-, four-, and six-particle states. The contribution with intermediary two-particle states reads for large n

$$C_{\text{discr}} = \alpha_c^{(2)} \frac{\delta(i_1+i_2-i_3-i_4)}{(\omega_{i_1}\omega_{i_2}\omega_{i_3}\omega_{i_4})^{1/2}} \sum_{k_1,k_2=1}^{n} v_{k_1}v_{k_2} \frac{\delta(i_1+i_2-k_1-k_2)}{\omega_{k_1}\omega_{k_2}} I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{k_1}+\omega_{k_2}) , \tag{25}$$

where

$$\begin{aligned}
I(\omega_a,\omega_b,\omega_c) &= \int_{-T}^{T} dT_1 \int_{T_1}^{T} dT_2 \exp[i\omega_a T_2 - i\omega_b T_1 - i\omega_c(T_2-T_1)] \\
&\quad + \int_{-T}^{T} dT_1 \int_{-T}^{T_1} dT_2 \exp[i\omega_a T_1 - i\omega_b T_2 - i\omega_c(T_1-T_2)] .
\end{aligned} \tag{26}$$

Performing the limit $n,Q,T\to\infty$, one obtains

$$C_{\text{cont}} = \alpha_c^{(2)} 2\pi \frac{\delta^4(p_1+p_2-p_3-p_4)}{\omega_1\omega_2\omega_3\omega_4} \int d\mathbf{k}_1 d\mathbf{k}_2 \frac{\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1-\mathbf{k}_2)}{\omega_{k_1}\omega_{k_2}(\omega_1+\omega_2-\omega_{k_1}-\omega_{k_2}+i0)} , \tag{27}$$

which gives a contribution to the graph depicted in Fig. 1(b). From intermediate four-particle states one obtains for large n

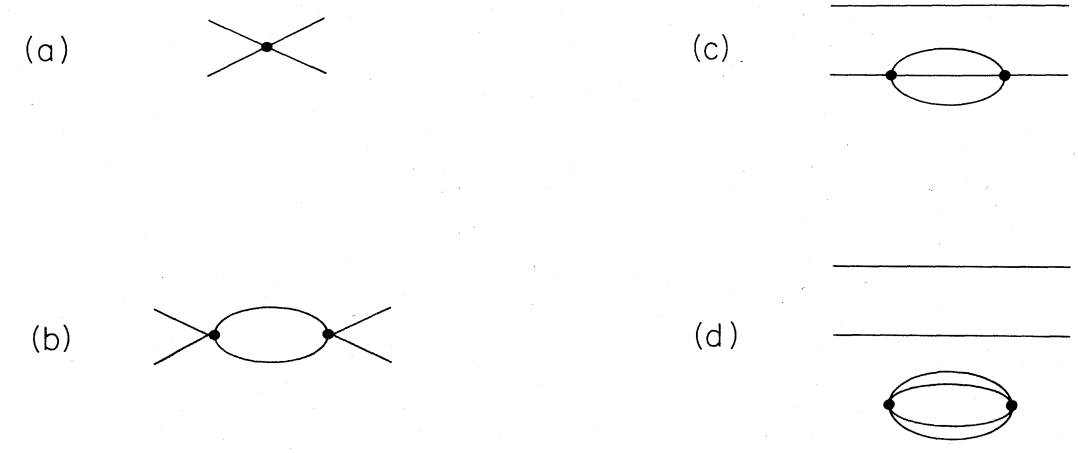

FIG. 1. ϕ^4-interaction graphs in (a) first order and (b)—(d) second order.

$$D^{i}_{\text{discr}}=\alpha^{(2)}_{Di}\sum_{\substack{i=1,2\\ j=3,4}}\frac{\delta_{i,j}\delta_{i',j'}}{v_i v_{i'}(\omega_i\omega_j)^{1/2}}\sum_{k_1,k_2,k_3=1}^{n} v_{k_1}\cdots v_{k_3}\frac{\delta(i-k_1-k_2-k_3)}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{j'}+\omega_{k_1}+\omega_{k_2}+\omega_{k_3}),$$

where $i'=1$ if $i=2$ and $i'=2$ if $i=1$, i.e., i' is the complement of i, and likewise for j':

$$D^{ii}_{\text{discr}}=\alpha^{(2)}_{Dii}\sum_{\substack{i=1,2\\ j=3,4}}\frac{\delta(i_1+i_2-i_3-i_4)}{(\omega_{i_1}\omega_{i_2}\omega_{i_3}\omega_{i_4})^{1/2}}\sum_{k_1,k_2=1}^{n} v_{k_1}v_{k_2}\frac{\delta(i-j'-k_1-k_2)}{\omega_{k_1}\omega_{k_2}}I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{i'}+\omega_{j'}+\omega_{k_1}+\omega_{k_2}). \tag{28}$$

Performing the limit, one obtains

$$D^{i}_{\text{cont}}=\alpha^{(2)}_{Di}2\pi\sum_{\substack{i=1,2\\ j=3,4}}\delta(\mathbf{p}_i-\mathbf{p}_j)\delta(\mathbf{p}_{i'}-\mathbf{p}_{j'})\frac{\delta(\omega_1+\omega_2-\omega_3-\omega_4)}{(\omega_i\omega_j)^{1/2}}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\frac{\delta(\mathbf{p}_j-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)}{\omega_{k_1}\omega_{k_2}\omega_{k_3}(\omega_j-\omega_{k_1}-\omega_{k_2}-\omega_{k_3}+i0)},$$

$$D^{ii}_{\text{cont}}=\alpha^{(2)}_{Dii}\sum_{\substack{i=1,2\\ j=3,4}}2\pi\frac{\delta^4(p_1+p_2-p_3-p_4)}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}\int d\mathbf{k}_1 d\mathbf{k}_2\frac{\delta(\mathbf{p}_i-\mathbf{p}_{j'}-\mathbf{k}_1-\mathbf{k}_2)}{\omega_{k_1}\omega_{k_2}(\omega_i-\omega_{j'}-\omega_{k_1}-\omega_{k_2}+i0)}. \tag{29}$$

D^{i}_{cont} gives a contribution to the graph depicted in Fig. 1(c), while D^{ii}_{cont} contributes to the graph of Fig. 1(b). From intermediate six-particle states one obtains for large n

$$E^{i}_{\text{discr}}=\alpha^{(2)}_{Ei}\sum_{\substack{i=1,2\\ j=3,4}}\frac{\delta_{i,j}\delta_{i',j'}}{v_i v_{i'}}\sum_{k_1,\ldots,k_4=1}^{n} v_{k_1}\cdots v_{k_4}\frac{\delta(k_1+\cdots+k_4)^2}{\omega_{k_1}\cdots\omega_{k_4}}I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{i_3}+\omega_{i_4}+\omega_{k_1}+\cdots+\omega_{k_4}),$$

$$E^{ii}_{\text{discr}}=\alpha^{(2)}_{Eii}\sum_{\substack{i=1,2\\ j=3,4}}\frac{\delta_{i,j}\delta_{i',j'}}{v_i v_{i'}}\frac{1}{(\omega_i\omega_j)^{1/2}}\sum_{k_1,k_2,k_3=1}^{n} v_{k_1}\cdots v_{k_3}\frac{\delta(i'+k_1+k_2+k_3)}{\omega_{k_1}\omega_{k_2}\omega_{k_3}}$$

$$\times I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{i_3}+\omega_{i_4}+\omega_{i'}+\omega_{k_1}+\omega_{k_2}+\omega_{k_3}), \tag{30}$$

$$E^{iii}_{\text{discr}}=\alpha^{(2)}_{Eiii}\frac{\delta(i_1+i_2-i_3-i_4)}{(\omega_{i_1}\cdots\omega_{i_4})^{1/2}}\sum_{k_1,k_2=1}^{n} v_{k_1}v_{k_2}\frac{\delta(i_1+i_2+k_1+k_2)}{\omega_{k_1}\omega_{k_2}}I(\omega_{i_1}+\omega_{i_2},\omega_{i_3}+\omega_{i_4},\omega_{i_1}+\cdots+\omega_{i_4}+\omega_{k_1}+\omega_{k_2}).$$

Performing the limit one obtains

$$E^{i}_{\text{cont}}=\alpha^{(2)}_{Ei}\sum_{\substack{i=1,2\\ j=3,4}}2\pi\delta(\mathbf{p}_1-\mathbf{p}_j)\delta(\mathbf{p}_{i'}-\mathbf{p}_{j'})\delta(\omega_1+\omega_2-\omega_3-\omega_4)\int d\mathbf{k}_1\cdots d\mathbf{k}_4\frac{\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)^2}{\omega_{k_1}\cdots\omega_{k_4}(-\omega_{k_1}-\cdots-\omega_{k_4}+i0)},$$

$$E^{ii}_{\text{cont}}=\alpha^{(2)}_{Eii}\sum_{\substack{i=1,2\\ j=3,4}}2\pi\delta(\mathbf{p}_i-\mathbf{p}_j)\delta(\mathbf{p}_{i'}-\mathbf{p}_{j'})\frac{\delta(\omega_1+\omega_2-\omega_3-\omega_4)}{(\omega_{i'}\omega_{j'})^{1/2}}\int d\mathbf{k}_1\cdots d\mathbf{k}_3\frac{\delta(\mathbf{p}_{i'}+\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)}{\omega_{k_1}\omega_{k_2}\omega_{k_3}(-\omega_{i'}-\omega_{k_1}-\omega_{k_2}-\omega_{k_3}+i0)}, \tag{31}$$

$$E^{iii}_{\text{cont}}=\alpha^{(2)}_{Eiii}2\pi\frac{\delta^4(p_1+p_2-p_3-p_4)}{(\omega_1\cdots\omega_4)^{1/2}}\int d\mathbf{k}_1 d\mathbf{k}_2\frac{\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1-\mathbf{k}_2)}{\omega_{k_1}\omega_{k_2}(-\omega_1-\omega_2-\omega_{k_1}-\omega_{k_2}+i0)}.$$

The term E^{i}_{cont} is represented by the graph of Fig. 1(d); E^{ii}_{cont} contributes to the graph of Fig. 1(c), and E^{iii}_{cont} contributes to the graph of Fig. 1(b). The graph of Fig. 1(b) is given by the sum $C_{\text{cont}}+D^{ii}_{\text{cont}}+E^{iii}_{\text{cont}}$, while the graph of Fig. 1(c) is given by $D^{i}_{\text{cont}}+E^{ii}_{\text{cont}}$.

Let us consider now the mth order. Comparison of Eq. (14) with Eq. (17) shows that each term in $U^{(m)}(+\infty,-\infty)$ has a one-to-one correspondence to a term in $U^{(m)}_{\mathcal{N}}(+T,-T)$. $U^{m}(+\infty,-\infty)$ is an m-fold product of factors of the form $:\phi^4:e^{iH^0 t}$, while $U^{(m)}_{\mathcal{N}}(+T,-T)$ is built analogously from $P_{\mathcal{N}}:\phi^4:P_{\mathcal{N}}e^{iH^0_{\mathcal{N}}t}$.

One can show that $e^{iH^0_{\mathcal{N}}t}$ tends strongly to $e^{iH^0 t}$ as $Q,n,N\to\infty$. In the discussion of first order we have shown that the matrix element of $P_{\mathcal{N}}:\phi^4:P_{\mathcal{N}}$ taken between two-particle states tends to that of $:\phi^4:$ [Eqs. (22) and (23)]. That is easily generalized to states of higher particle number. That suffices to establish that $U^{(m)}_{\mathcal{N}}(+T,-T)$ tends to $U^{(m)}(+\infty,-\infty)$ in every order m.

As we have shown $U_{\mathcal{N}}(+T,-T)$ is represented by the sum of all graphs including vacuum excitation graphs [as depicted in Fig. 1(d)] which do not contribute to the physical S matrix and therefore we eliminate them as usual by

dividing them out

$$\langle f \mid S_{\mathcal{N}}(+T,-T) \mid i \rangle = \frac{\langle f \mid U_{\mathcal{N}}(+T,-T) \mid i \rangle}{\langle 0 \mid U_{\mathcal{N}}(+T,-T) \mid 0 \rangle} . \quad (32)$$

This leads to the physical S matrix in the continuous limit.

As we have shown this method leads to standard perturbation theory in the continuous limit. Therefore there is a need for a proper renormalization procedure. The standard way of regularizing divergent integrals consists in introducing either a cutoff in momentum space or performing Pauli-Villars or dimensional regularization. Since our approximation scheme is already based on a cutoff parameter Q in momentum space it is natural to use a cutoff-dependent regularization scheme. Therefore the bare Lagrangian given by Eq. (2) has to be substituted by the renormalized Lagrangian

$$\mathcal{L}^{\text{ren}} = \mathcal{L}^{\text{unren}} + \frac{Z_1 - 1}{2}(\partial_\mu \phi)^2 - \frac{\delta m^2}{2}\phi^2 - (Z_2 - 1)\frac{\lambda_4}{4!}\phi^4 , \quad (33)$$

where Z_i and δm^2 are cutoff (Q) -dependent functions which will become infinite when the cutoff Q tends to infinity. When dealing with theories using a finite value of a momentum cutoff, a word of caution is in order. Calculating the quantum corrections to the Lagrangian of Eq. (33) either using a loop expansion[11] or another method[12] one encounters an infinite series of additional terms as ϕ^6, ϕ^8, etc., with coefficients which are cutoff dependent. These terms are present as long as the cutoff is kept finite; however, in the physical limit, when the cutoff tends to infinity, these contributions vanish and all infinities are absorbed as usual in Z_i and δm^2. Since this method requires a discretization of fields, we will absorb the counterterms into renormalized masses and coupling constants in order to continue to work with bare fields.

IV. NUMERICAL EXAMPLE

Finally let us present the results of a numerical calculation. We have considered the ϕ^4 model in $1+1$ dimensions in the weak-coupling regime with the parameters $(\hbar = c = 1)m = 140$ MeV, $g = \lambda_4/4! = 1$ MeV^2; i.e., $g \ll m^2$ means a small coupling constant. We neglect renormalization. We have calculated $\langle \psi_{fi} \mid U_{\mathcal{N}}(T,-T) \mid \psi_{in} \rangle$ and compared it with $\langle \psi_{fi} \mid S \mid \psi_{in} \rangle$ obtained from standard perturbation theory up to 1 order. For ψ_{fi}, ψ_{in} we have taken two-particle Fock states given by

$$\begin{aligned} \mid \psi_{fi} \rangle &= \int dq\, dq' \Omega_1(q)\Omega_2(q') A^\dagger(q) A^\dagger(q') \mid 0 \rangle , \\ \mid \psi_{in} \rangle &= \int dq\, dq' \Omega_3(q)\Omega_4(q') A^\dagger(q) A^\dagger(q') \mid 0 \rangle , \quad (34) \\ \Omega_i(q) &= \kappa_i \left[1 - \cos\left[2\pi \frac{q_i^{\text{up}} - q}{q_i^{\text{up}} - q_i^{\text{low}}} \right] \right] \theta(q_i^{\text{up}} - q)\theta(q - q_i^{\text{low}}) , \end{aligned}$$

where the functions Ω are bell-shaped wave packets, vanishing identically for $q < q^{\text{low}}$, $q > q^{\text{up}}$. We have chosen $q_1^{\text{low}} = 100$ MeV, $q_1^{\text{up}} = 120$ MeV, $q_2^{\text{low}} = 140$ MeV, $q_2^{\text{up}} = 160$ MeV, $q_3^{\text{low}} = 110$ MeV, $q_3^{\text{up}} = 130$ MeV, $q_4^{\text{low}} = 135$ MeV, $q_4^{\text{up}} = 150$ MeV. The κ_i were chosen such to normalize ψ_{fi}, ψ_{in} to unity. We have set the approximation parameters $Q = 160$ MeV, which is insensitive in the weak-coupling regime, we put $n = 2$ and varied $\Delta q = Q/n$ and T. The results are shown in Table I. We find convergence towards the reference value.

V. DISCUSSION AND CONCLUSION

We have proposed a method for the calculation of S-matrix elements. We pointed out in Sec. II that the formalism is equally applicable to both the weak- and the strong-coupling regimes. For the weak-coupling regime we have shown the equivalence of the method in the continuous limit to standard perturbation theory. A numerical test calculation is presented for the weak-coupling regime.

TABLE I. The approximate matrix element $\langle \Omega_1, \Omega_2 \mid S \mid \Omega_3, \Omega_4 \rangle$ dependent on the parameters time T and the mesh Δq of momentum-space cells ($\hbar = c = 1$). Reference $4.747 \times 10^{-2} - 9.1698 \times 10^{-5} i$.

T (MeV^{-1}) \ Δq (MeV)	1.0	0.8	0.6	0.4
4	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-2.7923 \times 10^{-4} i$	$-2.7964 \times 10^{-4} i$	$-2.7898 \times 10^{-4} i$	$-2.782 \times 10^{-4} i$
8	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-8.0581 \times 10^{-5} i$	$-8.1123 \times 10^{-5} i$	$-8.0633 \times 10^{-5} i$	$-8.0365 \times 10^{-5} i$
12	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-9.0301 \times 10^{-5} i$	$-9.0918 \times 10^{-5} i$	$-9.0353 \times 10^{-5} i$	$-9.0026 \times 10^{-5} i$
20	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-9.1706 \times 10^{-5} i$	$-9.2360 \times 10^{-5} i$	$-9.1750 \times 10^{-5} i$	$-9.1438 \times 10^{-5} i$
30	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-9.1698 \times 10^{-5} i$	$-9.2352 \times 10^{-5} i$	$-9.1743 \times 10^{-5} i$	$-9.1483 \times 10^{-5} i$
40	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-9.1706 \times 10^{-5} i$	$-9.2367 \times 10^{-5} i$	$-9.1743 \times 10^{-5} i$	$-9.1528 \times 10^{-5} i$
50	4.727×10^{-2}	4.759×10^{-2}	4.731×10^{-2}	4.727×10^{-2}
	$-9.2516 \times 10^{-5} i$	$-9.2367 \times 10^{-5} i$	$-9.1750 \times 10^{-5} i$	$-9.1669 \times 10^{-5} i$

As mentioned in the Introduction the method has been tested successfully for nonrelativistic few-nucleon scattering in the strong nuclear potential. On the basis of this calculation and the fact that the method does not involve an expansion in the coupling constant it should be applicable to the relativistic strong-coupling regime, too. From the point of view of a numerical calculation the strong-coupling regime requires the diagonalization of large but finite matrices. Investigations are under way.

Finally we would like to mention that Lorentz invariance and four-momentum conservation are violated in the discrete approximation but are recovered in the continuous limit. For simplicity we have chosen in this paper to demonstrate the method for a scalar field, but we do not anticipate any problem in extending the method to include other fields.

In particular case of a fermionic field the effective dimension of the Fock space in the discrete approximation is reduced due to the Fermi statistics, which may facilitate practical calculations.

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