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Regular reduction of relativistic theories of gravitation with a quadratic Lagrangian

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We consider those relativistic theories of gravitation which generalize Einstein's theory in the sense that their field equations derive from a scalar Lagrangian which, besides the matter term, contains a linear combination of the Ricci scalar, its square, and the square of the Ricci tensor. Using a generalization of a technique which has been used to deal with some dynamical systems, we regularly and covariantly reduce the corresponding fourth-order differential equations to second-order ones. We examine, in particular, at a low order of approximation, these reduced equations in cosmology, and for static and spherically symmetric interior solutions with constant density.

I. INTRODUCTION

Relativistic theories of gravitation which generalize Einstein's theory in the sense that their field equations derive from the Lagrangian

$$L=2\Lambda+R+(\beta_1/2)R^2+\beta_2 R_{ab}R^{ab}-8\pi GT\ , \qquad (1.1)$$

where R is the Ricci scalar, R_{ab} is the Ricci tensor, G is Newton's constant of gravitation, and T is the trace of the energy-momentum tensor, have been considered at different times of the history of these theories.[1] But no compelling physical argument was ever invoked to force such a generalization. More recently, however, the extra quadratic terms of (1.1) have been justified as counterterms in the process of a partial renormalization of quantized fields,[2] and have been used to discuss the possibility of constructing cosmological models without initial singularity and not having the horizon problem.[3]

The field equations which follow from (1.1) are

$$\begin{aligned} S_{mn}-\Lambda g_{mn}+\beta_1[S_{;m;n}-SS_{mn}-(\Box S-\tfrac{1}{4}S^2)g_{mn}] \\ +\beta_2\{\Box S_{mn}+S_{;m;n}-SS_{mn} \\ -[\Box S+\tfrac{1}{2}(S_{ab}S^{ab}-S^2)]g_{mn}+2R^{a}{}_{m}{}^{b}{}_{n}S_{ab}\} \\ =8\pi GT_{mn}\ , \end{aligned} \qquad (1.2)$$

where g_{mn} is the metric tensor, S_{mn} is Einstein's tensor, S its trace, and Λ the cosmological constant (the notation conventions are given in Appendix A), and from them one gets the conservation equations:

$$T^{mn}{}_{;n}=0\ . \qquad (1.2')$$

In the classical domain, besides the lack of physical justification, the main drawback of theories based on these differential equations is the fact that they are fourth-order ones, instead of second order like Poisson's or Einstein's equations, and this might be thought of as an unjustified qualitative departure of a framework which has been proved to be useful both in the nonrelativistic domain and in the relativistic one.

In this paper we point out first of all that there is at least one compelling physical reason why we should consider generalized fourth-order relativistic equations, and second that, independently of the justification which is invoked, equations such as (2.1) can always be considered as partial constraints of the theory to which it can be consistently added a principle of regularity of the solutions with respect to β_1 and β_2. As we shall see this amounts to considering as real field equations a system of second-order ones.

In 1967 one of us showed[4] that when one considers a medium composed of self-gravitating objects deformable by tidal effects and looks at it at a scale for which the medium appears as a continuum, Poisson's equation ceases to be a good approximation and should be replaced by a fourth-order equation (we give a few more details in Appendix C):

$$\Delta V+\beta\Delta\Delta V=4\pi G\rho\ , \qquad (1.3)$$

where β is a parameter depending on the deformability of the objects composing the medium which one considers. Such mediums can be thought of as quadrupolar gravitationally polarizable mediums.

We have no doubt that from a logical and completeness point of view it is necessary to derive the relativistic generalization of Eq. (1.3) to deal with polarizable mediums, and we pretend that while lacking a more detailed derivation we can consider Eqs. (1.2) as a natural generalization of Eq. (1.3), eventually with some relation holding between β_1 and β_2. The equations which were proposed in Ref. 4 correspond actually to $\beta_1=-\beta_2$. In any case with such an interpretation these coefficients would not be universal constants as was the case with the first justification coming from quantum gravity.

We believe also that it is not completely useless to consider theories for which we do not have at present a good justification but which are such that being, so to speak, close to the theory which we believe to be presently more correct would allow examination of the stability of qualitative or quantitative results following from the latter.

This could be useful in deciding the relevance of some theoretical results with the real world.

Anyway, we are going to consider Eqs. (1.2) with an open mind and suppose that they describe a polarizable medium or that they have a not yet thought of physical interpretation, or that they correspond to a new theory of gravitation. The main difference between these two points of view is that in the first case β_1 and β_2 are parameters and in the second case they are universal constants. And in both cases we are going to consider Eqs. (1.2) as partial constraints only and add to them the demand that only those solutions which are analytic functions of β_1 and β_2 in a neighborhood of 0 are relevant in the theory that we consider. Notice that this is more justified with the first point of view than with the second. But since, as we shall see, this amounts to reducing the equations to second order, this demand can be considered equivalently as a principle calling for second-order equations. After all this principle is used also when one wishes to derive Einstein's equations themselves. Notice also that this demand is stronger than just requiring that the solutions of the new theory tend to solutions of the old one when β_1 and β_2 both tend to zero.

The reason why requiring analyticity of the solutions with respect to β_1 and β_2 is a strong demand is because Eqs. (1.2) are singular equations with respect to them, in the sense that when they are both zero the equations change qualitatively, the order dropping from four to two. As a consequence of that, the generic solutions of these equation do not depend smoothly on β_1 and β_2 when either one or both tend to zero. There exist, however, a class of nongeneric solutions which do depend smoothly on these parameters and which is the general solution of a second-order system of differential equations. It is this latter system which we call the regular reduction of the initial fourth-order one. The method which we use to construct the regular reduction is a generalization of a method which has been introduced to deal with some dynamical systems.[5] Appendix B is a self-contained presentation of some previous work on this concept as it was used before and as we have generalized it to deal with systems of partial differential equations.

We consider a low-order approximation of the regular system thus defined in two cases: in cosmology and for static spherically symmetric interior solutions with constant density.

We plan to discuss later on the possibility of using the results of this introductory paper to ease the problem of the missing mass of the Universe and some lower-order structures such as, for instance, globular clusters. Recently a new theory of gravitation has been proposed in Ref. 6 to explain the missing mass in galaxies.

Appendixes B and C must be read before the main body of the paper.

II. COVARIANT REGULAR REDUCTION

As is obvious when $\Lambda=0$, but is also true for $\Lambda\neq 0$, if the energy-momentum tensor is zero then the solutions of vacuum Einstein's equations

$$S_{mn}=\Lambda g_{mn} \tag{2.1}$$

are also solutions of Eqs. (1.2), but of course the latter have other solutions, even in a vaccuum, than those of Eqs. (2.1). The point of view that we adopt in this paper amounts, in particular, to consider all these extra solutions as spurious.

A glance at Eqs. (1.2) and (B12), that we have considered in Appendix B, shows that some similarity exists between them. In this comparison S_{mn} would be the principal operator, the cofactors of β_1 and β_2 would be secondary operators and T_{mn} would be the source term. But there are some differences: First of all the unknowns of the problem, i.e., the gravitational potentials g_{mn}, appear both in the source term and in the secondary operators, contrary to what we have assumed in Appendix B. This is not a difficulty though, because these quantities have a zero covariant derivative and therefore they behave like constants in the process of regularly reducing the order of the equations. But, more importantly, the secondary operators contain derivatives of the gravitational potentials besides those which are part of the components of the Einstein tensor. These are the derivatives which are, so to speak, hidden in the Christoffel symbols involved in the convariant derivatives and in the Riemann tensor. Actually the origin of these derivatives is the same, because the latter terms come from applying the Ricci identity to terms containing second-covariant derivatives of the Einstein tensor.

Strictly speaking, therefore, we can apply the algorithm of Appendix B to Eqs. (1.2) only if we decide that in these equations the components of the connection involved in the covariant derivatives and the components of the Reimann tensor are known functions of the coordinates. Let us do that. We obtain then, neglecting terms nonlinear in β_1 and β_2,

$$S_{mn}-\Lambda g_{mn}+\beta_1 F_{mn}^{(1,0)}+\beta_2 F_{mn}^{(0,1)}=\chi T_{mn} , \tag{2.2}$$

where ($\chi=8\pi G$)

$$F_{mn}^{(1,0)}=\chi(T_{;m;n}-\Box T g_{mn})+\chi^2(\tfrac{1}{4}T^2 g_{mn}-TT_{mn})+\Lambda\chi(Tg_{mn}-4T_{mn}) , \tag{2.3}$$

$$F_{mn}^{(0,1)}=\chi(\Box T_{mn}+T_{;m;n}-\Box T g_{mn})+2\chi R^{a}{}_{m}{}^{b}{}_{n}T_{ab}+\chi^2(\tfrac{1}{2}T^2 g_{mn}-TT_{mn}-\tfrac{1}{2}T_{ab}T^{ab}g_{mn})+\Lambda\chi(Tg_{mn}-2T_{mn}) . \tag{2.4}$$

These equations, even when we drop the in-mind assumption that we made, are second-order equations and therefore, since they are nonsingular when β_1 and β_2 tend to zero in the sense that the order of the equations is the same as the order of the equations with $\beta_1=\beta_2=0$, we have covariantly reduced the order of the initial equations from four to two at a linear approximation with respect to the parameters β. These equations for $\beta_2=0$ have also been considered in Ref. 7.

At this first order of approximation the conservation equations are satisfied up to terms of order β^2 only:

$$T^{mn}{}_{;n}=0(\beta^2) . \tag{2.2'}$$

This process cannot be continued without modification because actually the equations that we would obtain would have a differential order which would increase with the order of the powers of β_1 and β_2 in the series expansions with respect to these parameters. In this paper we shall restrict ourselves to a partial study of the linearly reduced Eqs. (2.2), but should we desire to consider higher-order approximations with respect to these parameters we could do it using power expansions with respect to the gravitational constant G. In fact, if we assume that the solutions we are interested in can be developed in power expansions of the form

$$g_{mn}=g^{(0)}_{mn}+Gg^{(1)}_{mn}+\cdots \ , \tag{2.5}$$

where $g^{(0)}_{mn}$ is a solution of Eqs. (2.1) independent of G, then at each order of approximation s we would obtain equations of the form

$$S^{(s)}_{mn}-\Lambda g^{(s)}_{mn}+\beta_1 D^{(s)}_{1mn}(S^{(s)}_{mn})+\beta_2 D^{(s)}_{2mn}(S^{(s)}_{mn})=\chi T^{(s)}_{mn} \ , \tag{2.6}$$

where $S^{(s)}_{mn}$ and $T^{(s)}_{mn}$ are the sth-order Einstein and energy-momentum tensors and where the D's are two second-order operators which depend on potentials and derivatives of them of of orders lower than s. Therefore if one assumes that the potentials are known up to order $s-1$, then the equations (2.6) that one has to solve to obtain the sth-order potentials are exactly of the type of Eqs. (B12) and consequently they can be regularly reduced as indicated there. This proves that there exists at least a possible perturbative algorithm to covariantly and regularly reduce Eqs. (1.2). It follows in particular from this algorithm that, as we mentioned at the beginning of this paragraph, if the energy-momentum tensor is zero then the regular reduction of Eqs. (1.2) is Eqs. (2.1).

The following are some useful formulas which also make more precise the content of Eqs. (2.3) and (2.4). Assuming that the energy-momentum tensor is of the perfect-fluid type

$$T_{mn}=(\rho+p)u_m u_n - p g_{mn} \ , \tag{2.7}$$

where ρ is the density of the fluid, p the pressure, and u_m the unit tangent to the stream lines, we have

$$T_{mn}u^n=\rho u_m, \quad T=\rho-3p, \quad T_{ab}T^{ab}=\rho^2+3p^2 \ , \tag{2.8}$$

$$T_{mb}T_n{}^b=(\rho^2-p^2)u_m u_n+p^2 g_{mn} \ , \tag{2.9}$$

$$\Box T_{mn}=\Box(\rho+p)u_m u_n+2(u_m{}^{;a}u_n+u_m u_n{}^{;a})(\rho+p)_{,a} \\ +(\rho+p)(\Box u_m u_n+u_m\Box u_n+2u_{m;a}u_n{}^{;a}) \\ -\Box p g_{mn} \ . \tag{2.10}$$

With the usual definitions of the four-acceleration, the expansion, the rotation, and the shear rates

$$b_m=u^a u_{m;a}, \quad \theta=u_a{}^{;a} \ ,$$

$$\omega_{mn}=-\omega_{nm}=\tfrac{1}{2}(u_{n;m}-u_{m;n}+u_m b_n-u_n b_m) \\ (\omega_{mn}u^m=0) \ , \tag{2.11}$$

$$\sigma_{mn}=\sigma_{nm}=\tfrac{1}{2}(u_{n;m}+u_{m;n}+u_m b_n+u_n b_m) \quad (\sigma_{mn}u^m=0)$$

of the congruence associated with the vector field u_m, we have

$$u_{n;m}=\omega_{mn}+\sigma_{mn}+\tfrac{1}{3}\theta(g_{mn}-u_m u_n)+u_m b_n \tag{2.12}$$

and therefore

$$u_{m;a}u_n{}^{;a}=\omega_{am}\omega^a{}_n+\sigma_{am}\sigma^a{}_n+\omega_{am}\sigma^a{}_n+\sigma_{am}\omega^a{}_n \\ +\tfrac{2}{3}\theta\sigma_{mn}+\tfrac{1}{9}\theta^2(g_{mn}-u_m u_n)+b_m b_n \tag{2.13}$$

and

$$\Box u_n=\omega_{mn}{}^{;m}+\sigma_{mn}{}^{;m}+b_{n;m}u^m \\ +\tfrac{1}{3}\theta^{,m}(g_{mn}-u_m u_n)+\tfrac{1}{3}(4b_n-\theta u_n) \ . \tag{2.14}$$

Remark: Notice that with some additional assumptions Eqs. (1.2) are already second-order equations. This is the case if one assumes that $S=$constant and that either one or both of these conditions hold: the metric is conformally flat and/or $\beta_2=0$ (Macrae and Riegert, Ref. 3). The trace T is also constant and, in particular, if $T=0$ then Eqs. (1.2) reduce to

$$S_{mn}-\Lambda g_{mn}=[\chi/(1-4\beta\Lambda)]T_{mn} \ . \tag{2.15}$$

Reciprocally when T is constant and either one or both of the above conditions hold then from the regular reduction of the equation

$$S-4\Lambda-3\beta\Box S=\chi T \ , \tag{2.16}$$

which is the trace of Eqs. (1.2), it follows that the solutions will be such that $S=$constant.

III. UNIFORM COSMOLOGICAL MODELS

We consider here uniform cosmological models with a Robertson-Walker line element:

$$ds^2=dt^2-(1+kr^2/4)^{-2}S^2(t)[dr^2+r^2(d\theta^2+\sin\theta^2 d\phi^2)] \ . \tag{3.1}$$

We shall assume that units have been chosen such that the speed of light in a vacuum $c=1$. We shall assume also that the scale factor S (not to be confused in this paragraph with the trace of Einstein's tensor) is dimensionless, and that

$$S(t_0)=1 \ , \tag{3.2}$$

t_0 being the present time. As is well known this is not a restriction when the space curvature $k=0$. It is not a restriction either when $k\neq 0$ if one renounces to the normalization $|k|=1$.

It is known (Ref. 3) that for conformally flat line elements the scalar Lagrangian (1.1) is equivalent to

$$L=2\Lambda+R+\tfrac{1}{6}\beta R^2-\chi T \quad \text{with} \quad \beta=3\beta_1+2\beta_2 \ , \tag{3.3}$$

i.e., the corresponding densities differ by a divergence. Since Robertson-Walker line elements are conformally flat this remark substantially speeds up the calculation of the left-hand terms of Eqs. (1.2) because it makes unnecessary the calculation of the terms which contain β_2 as factor. The explicit form of Eqs. (1.2) reduces to the following pair of equations (Ref. 3):

$$S^{-2}(\dot{S}^2+k)-\Lambda/3-\beta S^{-4}(2S^2\dot{S}\dddot{S}+2S\dot{S}^2\ddot{S}-S^2\ddot{S}^2+k^2-2k\dot{S}^2-3\dot{S}^4)=\chi\rho/3 , \tag{3.4}$$

$$S^{-2}(2S\ddot{S}+\dot{S}^2+k)-\Lambda-\beta S^{-4}(2S^3\ddddot{S}+4S^2\dot{S}\dddot{S}+3S^2\ddot{S}^2-k^2+3\dot{S}^4+2k\dot{S}^2-12S\dot{S}^2\ddot{S}-4kS\ddot{S})=-\chi p , \tag{3.5}$$

where a dot means a derivative with respect to time.

Since from these equations it follows that

$$\dot{\rho}S+3(\rho+p)\dot{S}=0 , \tag{3.6}$$

which is the explicit form of the conservation equations (1.2′) for the particular metric and energy-momentum tensor that we are considering here; from now on we are going to consider, as basic equations of the models, Eqs. (3.4) and (3.6) complemented with an equation of compressibility defining p as function of ρ. We know of course that reciprocally Eq. (3.5) is a consequence of Eqs. (3.4) and (3.6).

We shall assume that the content of the Universe is a mixture of matter and radiation without interconversion. To be more precise, and just as an indication of what can be done, we shall assume, as has already been done by others (Ref. 8), that

$$\rho=\rho_m+\rho_r,\quad p=\rho_r/3 , \tag{3.7}$$

where ρ_m is the density of matter and ρ_r is the density of radiation. From these assumptions and from Eq. (3.6) we obtain

$$\chi\rho/3=S^{-4}(D_r+SD_m),\quad \chi p=S^{-4}D_r , \tag{3.8}$$

D_m and D_r being constants.

Using Eqs. (2.2) and (2.3) we can calculate the first-order regular reduction of Eqs. (1.2) for uniform cosmological models, i.e., the first-order regular reduction of Eqs. (3.4) and (3.5). The equation corresponding to Eq. (3.4) is

$$S^{-2}(\dot{S}^2+k)-\Lambda/3+\tfrac{1}{3}\beta\chi\{-\tfrac{1}{4}[\chi(\rho-3p)+4\Lambda])(\rho+p)-S^{-1}\dot{S}(\dot{\rho}-3\dot{p})\}=\chi\rho/3 \tag{3.9}$$

and using (3.8), or more precisely the approximation that follows from (2.2′), we can write this equation as

$$S^{-2}(\dot{S}^2+k)-\Lambda/3+\beta S^{-5}[-\tfrac{1}{4}(3S^{-3}D_m+4\Lambda)(4SD_r/3+S^2D_m)+3\dot{S}^2D_m]=S^{-4}(SD_m+D_r) \tag{3.10}$$

and using the zero-order expression of $\dot{S}^2$ inside the square brackets we obtain

$$S^{-2}(\dot{S}^2+k)-\Lambda/3+\beta S^{-5}[S^{-2}D_m(2D_r+9SD_m/4)-4SD_r\Lambda/3-3kD_m]=S^{-4}(SD_m+D_r) , \tag{3.11}$$

where we finally get the following first-order approximated evolution equation for the scale factor S:

$$\dot{S}=f^{(0)}+\beta f^{(1)} , \tag{3.12}$$

where

$$f^{(0)}=S^{-1}(SD_m+D_r+S^4\Lambda/3-S^2k)^{1/2} ,$$
$$f^{(1)}=-S^{-5}(D_mD_r+9SD_m{}^2/8-3S^2kD_m/2-2S^3\Lambda D_r/3)/f^{(0)} . \tag{3.13}$$

We shall define the relative inaccuracy of Eq. (3.12) for one of its solutions as

$$\eta=(3L/\chi-\rho)/\rho , \tag{3.14}$$

L being the left-hand side of Eq. (3.4) and ρ, given by Eq. (3.8), being the density corresponding to the solution of Eq. (3.12) which is being considered. As an indication of how this inaccuracy depends on β we consider the case where $k=\Lambda=D_r=0$ and we assume that $t=t_0$. We obtain then

$$\eta=x^2(22-97x+72x^2-54x^3) , \tag{3.15}$$

where $x=\beta D_m$ and where the coefficients are integer approximations.

It follows from the remark of Sec. II that in vacuum and with pure radiation ($p=\rho/3$) the solutions of the regularly reduced equations coincide with the constant scalar curvature solutions which are the solutions of Einstein's equations with a rescaled source [see Eqs. (2.15)]. Therefore, the initial singularity cannot be removed in the framework of these models with $D_m=0$.

IV. STATIC SPHERICALLY SYMMETRIC MODELS WITH CONSTANT DENSITY

We consider a static spherically symmetric line element

$$ds^2=e^{\nu(r)}dt^2-e^{\lambda(r)}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) . \tag{4.1}$$

The matter is at rest in the coordinate system

$$u^m=(e^{-\nu/2},0,0,0) . \tag{4.2}$$

We assume that ρ is a constant. From the energy-momentum conservation equation (1.2′) we obtain

$$u^m{}_{;m}=0 , \tag{4.3}$$

which means that the expansion vanishes and that

$$\rho+p=Ce^{-\nu/2} , \tag{4.4}$$

C being a constant.

Among the components of $u_{m;n}$ and $b_m u_n$ the nonzero one are

$$u_{1;0}=-\nu'/2e^{\nu/2},\quad b_1u_0=-\nu'/2e^{\nu/2} . \tag{4.5}$$

A prime means a derivative with respect to r. The shear and the rotation also vanish. $\Box T_{mn}$ becomes

$$\Box T_{mn} = \Box p(u_m u_n - g_{mn}) + 2p_{,a}(u_m{}^{;a}u_n + u_m u_n{}^{;a}) + (\rho+p)(\Box u_m u_n + u_m \Box u_n + 2u_{m;a}u_n{}^{;a}) , \quad (4.6)$$

where

$$\Box u_m = b_{m;a}u^a \quad (4.7)$$

and therefore,

$$\Box u_0 = \nu'^2/4e^{\nu/2-\lambda}, \quad \Box u_\mu = 0 , \quad (4.8)$$

and also

$$\Box p = (-g)^{-1/2}[(-g)^{1/2}p_{,m}g^{mn}]_{,n} = -e^{-(\nu+\lambda)/2}r^{-2}(e^{(\nu-\lambda)/2}r^2p')' . \quad (4.9)$$

The following components of the Riemann tensor are needed:

$$R^0{}_{101} = \lambda'\nu'/4 - \nu'^2/4 - \nu''/2, \quad R^0{}_{202} = r\nu'/2e^{-\lambda}, \quad R^0{}_{303} = -r\sin^2\theta\nu'/2e^{-\lambda} ,$$
$$R^1{}_{010} = -(\lambda'\nu'/4 - \nu'2/4 - \nu''/2)e^{\nu-\lambda}, \quad R^1{}_{212} = r\lambda'/2e^{-\lambda}, \quad R^1{}_{313} = r\sin^2\theta/2\lambda'e^{-\lambda} . \quad (4.10)$$

And finally the (0^0) and the (1^1) components of the field Eqs. (2.2) $(\Lambda=0)$ become

$$e^{-\lambda}(\lambda'/r - r^{-2}) + r^{-2} = \chi\rho + (\beta_1+\beta_2)[3\chi e^{-\lambda}(2p'/r - \lambda'p'/2 + p'') + 3\chi^2/4(\rho+p)(\rho-3p)] + \beta_2[-\chi\rho\nu'^2/2e^{-\lambda} + 2\chi p e^{-\lambda}(\nu'/r - \lambda'\nu'/4 + \nu''/2) + \chi^2/4(\rho^2 + 6\rho p - 3p^2)] , \quad (4.11)$$

$$-e^{-\lambda}(\nu'/r + r^{-2}) + r^{-2} = -\chi p + (\beta_1+\beta_2)[3\chi e^{-\lambda}(2p'/r + \nu'p'/2) - \chi^2/4(\rho+p)(\rho-3p)] + \beta_2\{-2\chi\rho e^{-\lambda}(-\lambda'\nu'/4 + \nu''/2) + \chi p e^{-\lambda}(-2\lambda'/r + \nu'^2/2) - \chi e^{-\lambda}[2p'/r + (\lambda'-\lambda')p'/2 + p''] + \chi^2(\rho^2/4 - 6\rho p/4 - 3p^2/4)\} . \quad (4.12)$$

They are of first order with respect to λ and second order with respect to ν.

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APPENDIX A: NOTATIONS AND CONVENTIONS

Latin indices range and sum over 0,1,2,3. Greek indices range and sum over 1,2,3. The signature of the metric is -2. Units are chosen such that $c=1$, $\chi = 8\pi G$. A comma corresponds to partial differentiation and covariant differentiation is denoted by a semicolon. The Riemann tensor and the Ricci tensor are taken, respectively, in the forms

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^m{}_{bd}\Gamma^a{}_{mc} - \Gamma^m{}_{bc}\Gamma^a{}_{md} ,$$

where the Γ's are the Christoffel symbols.

$$R_{bd} = R^a{}_{bad} .$$

The Einstein tensor is defined as

$$S_{mn} = R_{mn} - \tfrac{1}{2}Rg_{mn} .$$

The symbol □ is the d'Alembert operator; for any function F

$$\Box F = g^{ab}F_{;ab} .$$

Δ is the Laplacian in the three-dimensional flat space.

APPENDIX B: REGULAR REDUCTIONS OF DIFFERENTIAL EQUATIONS

Let us consider a secong-order differential equation of the following general form:

$$W(t,x,\dot{x},\ddot{x};\beta) = 0 , \quad (B1)$$

where x is the unknown function, a dot means a derivative with respect to the independent variable t, and β is a parameter. Equation (B1) is said to be singular for $\beta=0$ if for this value of β the function W does not depend on $\ddot{x}$; i.e., Eq. (B1) reduce to a first-order differential equation

$$W(t,x,\dot{x}) = 0 , \quad (B2)$$

which we call the zero-order reduction of Eq. (B1). For a general discussion of some of the problems related to singular equations of the type above or similar ones we refer the reader to Ref. 5.

We define the regular reduction of Eq. (B1) with respect to β as a first-order differential equation, parametrized by β, of the explicit from

$$\dot{x} = \xi(t,x,\beta) , \quad (B3)$$

having the following properties: (i) it is a reduction, i.e., all solutions of Eq. (B3) are solutions of Eq. (B1); (ii) it is analytic with respect to β and reduces to Eq. (B2) when

$\beta=0$, i.e., the function ξ can be developed in a neighborhood of this value as a power series

$$\xi(t,x;\beta)=\xi(t,x)^{(0)}+\beta\xi(t,x)^{(1)}+\cdots\;, \tag{B4}$$

and the zero-order term defines an equation equivalent to Eq. (B2).

Example 1: The equation

$$\dot{x}^3-\beta x^2\ddot{x}=0 \tag{B5}$$

has two regular reductions:

$$\dot{x}=0,\;\; \dot{x}=\beta x\;. \tag{B6}$$

Example 2: The inhomogeneous linear equation

$$\dot{x}+\beta\ddot{x}=t \tag{B7}$$

has one and only one regular reduction: namely,

$$\dot{x}=t-\beta\;. \tag{B8}$$

More generally, it can be proved that singular differential equations of the type

$$\dot{x}=L(t,x,\beta\ddot{x})\;, \tag{B9}$$

the function L being a smooth function of the argument $\beta\ddot{x}$, have at most one regular reduction. The corresponding function ξ can be obtained either by solving exactly, if this is possible, the partial differential equation

$$\xi=L[t,x,\beta(\partial\xi/\partial t+\xi\partial\xi/\partial x)] \tag{B10}$$

or obtaining from it, by a straightforward algorithm the successive terms of the expansion (B4). The two lowest-order terms are

$$\begin{aligned}&\xi^{(0)}=L(t,x,0)\;,\\&\xi^{(1)}=\frac{\partial L}{\partial(\beta\ddot{x})}(t,x,0)(\partial\xi^{(0)}/\partial t+\xi^{(0)}\partial\xi^{(0)}/\partial x)\;.\end{aligned} \tag{B11}$$

For further information concerning the application of the concept of regular reductions to some particular dynamical systems we refer the reader to Ref. 5 and references therein.

In Appendix C and in the main body of the paper we use a generalization of the concept of regular reduction in the domain of partial differential equations. The generalization that we introduce here is a fairly general one but not as general as it could be because we wanted to remain close to the applications that we had in mind.

Let us consider a system of partial differential equations of the type

$$P(v)+\sum_{i=1}^{p}\beta_i D_i[P(v)]=4\pi GT(y)\;, \tag{B12}$$

where y is a point of some manifold, v is a set of unknown function of y, P is a differential operator of order n, say, which we call the principal operator, where β_i are p parameters, G is a constant, and T are functions of y that we call collectively the source, and D_i are differential operators that we call the secondary operators. We shall assume that these secondary operators are at most the sum of a linear and a quadratic part. More precisely we shall assume that

$$\begin{aligned}&D_i(Z_1+Z_2)=D_i(Z_1)+D_i(Z_2)+G_i(Z_1,Z_2)\;,\\&D_i(kZ)=k(1-k)L(Z)+k^2D_i(Z)\;,\end{aligned} \tag{B13}$$

$G_i(Z_1,Z_2)$ being a set of symmetric bilinear operators, and L_i the linear parts of D_i.

Equations (B12) can then be written as

$$P(v)+\sum_{i=1}^{p}\beta_iD_i\left[4\pi GT-\sum_{i=1}^{p}\beta_jD_j[P(v)]\right]=4\pi GT \tag{B14}$$

and using Eqs. (B13)

$$P(v)+\sum_{i=1}^{p}\beta_iD_i(4\pi GT)+O(\beta^2)=4\pi GT\;, \tag{B15}$$

where $O(\beta^2)$ stands collectively for terms which have squares or products of two β's as factors. Iterating the process which led to the equation above it follows that Eq. (B12) can be written as

$$P(v)+\sum_{i_1+\cdots+i_p=1}^{S}\beta_1{}^{i}\cdots\beta_p{}^{i}D_{i_1\cdots i_p}(4\pi GT)=4\pi GT \tag{B16}$$

modulo terms which have monomials of products of β's of order $s+1$ as factors.

We call the system of equations (B16) which has for any s the order of the principal operator P, the sth-order approximated regular reduction of Eqs. (B12). And we call the limit, when s tends to infinity of this approximation, the regular reduction of Eqs. (B12). The existence of this limit, or the domain of validity of any order of approximation or of the exact reduction will depend of course on the system itself and the source term. Notice that in particular if the source is zero then the exact reduction of Eqs. (B12) is always

$$P(v)=0\;. \tag{B17}$$

Without pretending to any rigor we claim that by construction the regular reduction has as solutions all solutions of Eqs. (B12) which depend smoothly on β_i in a neighborhood of $\beta_i=0$, and only these. A rigorous proof of this statement is beyond our scope, but we have checked it in a few examples. Equation (B7) is one of them. In fact, this equation belongs to the general class of equations of type (B12). p is equal to 1; both the principal and the secondary operators are the derivative with respect to t, and the source term is t. In Appendix C we shall consider a second example.

APPENDIX C: QUADRUPOLAR GRAVITATIONAL POLARIZATION

Let us consider a medium composed of a large number of objects per unit volume in an appropriate scale. Let us assume that each of these objects is gravitationally characterized by a monopole and a quadrupole, and that therefore the medium can be considered as a superposition of a continuous distribution of monopoles with density ρ and a continuous distribution of quadrupoles with density D_{ij}. The Newtonian potential at any one point in the interior or exterior of the medium will then be given by the generalized Poisson integral

$$V(x^k) = -G\left[\int_\tau [R^{-1}\rho(\xi^i) + \tfrac{1}{2}R^{-5}D_{mn}(\xi^i)(x^m-\xi^m)(x^n-\xi^n)]d\tau\right], \tag{C1}$$

where τ is the volume of space occupied by the medium, G is Newton's constant, and

$$R = \left[\sum_{i=1}^{3}(x^i-\xi^i)\right]^{1/2}.$$

It follows from this formula after two integrations by parts that at any interior point the potential V will satisfy the equation

$$\Delta V - \tfrac{2}{3}\pi G D^{ij}{}_{,ij} = 4\pi G\rho . \tag{C2}$$

By quadrupolar gravitational polarization we mean, by analogy with the dielectric polarizability, the phenomenon by which the quadrupolar density is acquired by tidal effects due to the inhomogeneities of the global field and to the deformability of the elements of the medium, which would be otherwise just mass monopoles. It can be proved in this case, for small deformations, that the quadrupole density depends on the global field according to the relation

$$D_{ij} = -[3\beta/(4\pi G)](3V{,}ij - \delta ij\Delta V) , \tag{C3}$$

where $\beta=\epsilon\rho$ is the product of a parameter ϵ characterizing the quadrupolar deformability of the elements of the medium which is considered times the density ρ of elements in it. In particular, for incompressible objects with constant density $\epsilon=a^5/(2G)$, a being the radius of the object.

Then where β is constant Eq. (C2) becomes

$$\Delta V + \beta\Delta\Delta V = 4\pi G\rho . \tag{C4}$$

We refer the reader to Ref. 4 for some more details, connected in particular with the matching conditions which hold when there is a sharp border separating a medium from the vacuum.

Equation (C4) belongs to the class of equations which we have considered at the end of Appendix B. Here $p=1$ and both the principal operator and the secondary operator are the Laplacian. A straightforward application of Eq. (B16) leads then to the equation for the regular reduction of Eq. (C4),

$$\Delta V = 4\pi G\rho_{\text{eff}} , \tag{C5}$$

where the effective density is given by

$$\rho_{\text{eff}} = \sum_{n=0}^{\infty}\beta^n\rho_n \tag{C6}$$

with

$$\rho_0=\rho, \quad \rho_{n+1} = -\Delta\rho_n \tag{C7}$$

Notice that if ρ is zero then Eq. (C5) is the Laplace equation. This is what we had to have and not what it is obtained from Eq. (C4) without reducing its order.

Let us define σ by the equation

$$\Delta V = 4\pi G\sigma . \tag{C8}$$

According to Eq. (C4) σ must be a solution of the equation

$$\sigma + \beta\Delta\sigma = \rho . \tag{C9}$$

Therefore we can always solve Eq. (C4) in two steps. We can obtain first σ, the nonreduced effective density, solving Eq. (C9) and obtain then the potential V by solving Eq. (C8) which is the ordinary Poisson equation.

Let us assume that ρ is the following function of the radial polar coordinate:

$$\rho = \rho_c e^{-\alpha r}(1+\alpha r) , \tag{C10}$$

where ρ_c, the central density, and α are two positive parameters. This is a fairly general density to consider. For spherical symmetry Eq. (C9) reduces to

$$\sigma + \beta(\sigma'' + 2\sigma'/r) = \rho , \tag{C11}$$

where a prime means a derivative with respect to r. The general solution of this equation is

$$\sigma = \rho_{\text{eff}} + k e^{-\mu r}/r + l e^{\mu r}/r , \tag{C12}$$

where k and l are two constants of integration, $\mu=(-1/\beta)^{1/2}$, and where

$$\rho_{\text{eff}} = \rho_c\mu^2 e^{-\alpha r}[\alpha(\mu^2-\alpha^2)^2 r^2 + (\mu^2-\alpha^2)(\mu^2-5\alpha^2)r + 8\alpha^3]r^{-1}(\mu^2-\alpha^2)^{-3} . \tag{C13}$$

This expression is the analytic part of σ in a neighborhood of $\beta=0$. Therefore Eq. (C5) with ρ_{eff} defined by Eq. (C13) is the regular reduction of Eq. (C4) for the particular distribution of matter that we have considered. It could have been obtained more painfully by summing the series (C6).

Let us assume that we want to consider Eq. (C4) without *a priori* reducing its order. If β is negative, and therefore μ is real, we have to take $l=0$ to have solutions which make sense at infinity. k, on the other hand, could be fixed demanding that ρ be finite at the origin. In this case the effective density associated with the full equation and that corresponding to its regular reduction would differ substantially only close to the origin; how close depends of course on the value of μ. If β is positive, and therefore μ is pure imaginary, then to have a finite nonreduced effective mass we have to take both $k=l=0$. In other words, with this condition the solutions of the full equation and those of its regular reduction would be the same.

Notice that in this latter case the effective density behaves as $1/r$ near the origin. This is essentially true also in the first one, except that in this case the effective density eventually becomes finite at the origin.

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