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Aspects of quasi-Riemannian Kaluza-Klein theory

K. S. Viswanathan and Brenden Wong
Simon Fraser University, Department of Physics, Burnaby, British Columbia, Canada V5A 1S6
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We consider the applications of quasi-Riemannian geometry in Kaluza-Klein theories. We find that such theories cannot be implemented for all choices of the tangent group G_T and internal space G/H for reasons of gauge invariance. Coupling of fermions to gravity poses further problems in these theories.

I. INTRODUCTION

The standard Kaluza-Klein approach to unification[1] is beset by several difficulties. Amongst them are the lack of massless chiral fermions in the four-dimensional effective theory, the incompatibility of the ground-state ansatz $M_4\times G/H$ with the d-dimensional gravitational field equations and the largeness of the cosmological constant. A possible resolution of these problems may be provided by the departure from Riemannian geometry. For example, parallelizing torsion on the internal manifold can give rise to massless fermion modes and a vanishing cosmological term.[2] A particular class of non-Riemannian theories is that where the tangent group is a subgroup of the standard d-dimensional Lorentz group $SO(1,d-1)$. These are the quasi-Riemannian theories first suggested by Weinberg.[3,4] Such theories can, in general, give rise to chiral fermions in four dimensions.[3] Compactifying solutions may also be possible.[5] Hence it appears interesting to strive for "realistic" application of these theories. In this paper, we comment on the incorporation of quasi-Riemannian geometry in Kaluza-Klein theory. We construct a d-dimensional gravitational action suitable for Kaluza-Klein-type compactification (that is, one which gives a gauge-invariant Yang-Mills plus gravitational action in the four-dimensional effective theory with the gauge symmetry corresponding to the isometries of the internal manifold). We also consider the dimensional reduction of such theories with and without fermions. This article is a generalization and extension of the results contained in an earlier publication.[6]

II. GRAVITATIONAL LAGRANGIAN

In a d-dimensional quasi-Riemannian spacetime, the tangent group G_T is taken to be of the form $SO(1,k-1)\times G_T'$ where $G_T'\subset SO(d-k)$ and $d>k\geq 4$. This form is necessary in order to recover the usual four-dimensional Lorentz invariance. In the following, we consider specifically the case $G_T'=SO(d-k)$. This choice appears to be sufficient in illustrating the general features. It also is the most economical choice since it gives rise to the fewest number of possible terms in the prospective Lagrangian. Under a local $SO(1,k-1)\times SO(d-k)$ tangent-space rotation, the vielbein $e_M{}^A(z)$ transforms as the representation $(k,1)+(1,d-k)$. Hence the tangent-space indices split naturally into the sets $1,\dots,4$; $5,\dots,k$; $k+1,\dots,d$. We denote tangent-space indices by early letters of the alphabet as follows: $a,b,\dots=1,\dots,4$; $a',b',\dots=5,\dots,k$; $\alpha,\beta,\dots=k+1,\dots,d$; $\tilde a,\tilde b=1,\dots,k$; $\tilde\alpha,\tilde\beta,\dots=5,\dots,d$; $A,B,\dots=1,\dots,d$. Similarly, we denote world indices using middle letters of the alphabet.

In constructing a Lagrangian which is invariant under general coordinate and local G_T transformations, we have available to work with the vielbein $e_M{}^A(z)$ and the spin-connection components $B_M{}^{AB}(z)$. Under a local G_T transformation, these quantities transform as a vector and a connection, respectively. The spin connection B_M takes values in the Lie algebra of G_T. Hence, for example, in the standard Riemannian geometry where $G_T=SO(1,d-1)$, $B_M=B_M{}^{AB}\Sigma_{AB}$ [where Σ_{AB} are the $SO(1,d-1)$ generators in the fundamental representation] has $\frac{1}{2}d(d-1)$ independent components. In quasi-Riemannian geometry, B_M has fewer than $\frac{1}{2}d(d-1)$ independent components. In particular, for $G_T=SO(1,k-1)\times SO(d-k)$ which is embedded in $SO(1,d-1)$ in a trivial way the independent components are $B_M{}^{\tilde a\tilde b}$, $B_M{}^{\alpha\beta}$ or $B_M=B_M{}^{\tilde a\tilde b}\Sigma_{\tilde a\tilde b}+B_M{}^{\alpha\beta}\Sigma_{\alpha\beta}$. It has been demonstrated[4,5] however, that torsion must be nonvanishing for a non-block-diagonal vielbein (as required in the Kaluza-Klein ansatz) to be admissible. Hence torsion (a tensor under G_T) is an additional geometrical quantity which we must incorporate. Torsion can perhaps be introduced as an ansatz although we will take a route where it is determined dynamically. Whichever the case, it appears that a minimal theory will necessarily contain a tensor field $b_M{}^{\tilde a\beta}(z)$ in addition to the usual vielbein and spin-connection fields $e_M{}^A(z)$, $B_M{}^{\tilde a\tilde b}(z)$, $B_M{}^{\alpha\beta}$ (Refs. 6 and 7). In some sense, $b_M{}^{\tilde a\beta}$ has the role of replacing the "missing" components of the spin connection. Therefore it is reasonable to regard b_M and B_M on equal footing so we include "kinetic" terms for both. Ultimately they are determined in terms of $e_M{}^A$. Hence a general gravitational Lagrangian invariant under general coordinate and local $SO(1,k-1)\times SO(d-k)$ transformations (with at most two derivatives of the vielbein) is given by

$$\mathscr{L}=(c_0R_{\tilde{a}\tilde{b}}{}^{\tilde{a}\tilde{b}}+c_0'R_{\alpha\beta}{}^{\alpha\beta}+\lambda r_{\tilde{a}\beta}{}^{\tilde{a}\beta}+c_1b_{[\tilde{a}\tilde{b}]\gamma}b^{[\tilde{a}\tilde{b}]\gamma}+c_1'b_{[\alpha\beta]\tilde{c}}b^{[\alpha\beta]\tilde{c}}+c_2b_{\{\tilde{a}\tilde{b}\}\gamma}b^{\{\tilde{a}\tilde{b}\}\gamma}+c_2'b_{\{\alpha\beta\}\tilde{c}}b^{\{\alpha\beta\}\tilde{c}}$$
$$+c_3b_{\tilde{a}}{}^{\tilde{a}}{}_{\gamma}b_{\tilde{b}}{}^{\tilde{b}\gamma}+c_3'b_{\alpha}{}^{\alpha}{}_{\tilde{c}}b_{\beta}{}^{\beta\tilde{c}}+c_4)(\det e)\ , \quad (1)$$

where

$$R_{\tilde{a}\tilde{b}}{}^{\tilde{c}\tilde{d}}=E_{\tilde{a}}{}^{M}E_{\tilde{b}}{}^{N}(\partial_MB_N{}^{\tilde{c}\tilde{d}}-\partial_NB_M{}^{\tilde{c}\tilde{d}}-B_M{}^{\tilde{c}}{}_{\tilde{e}}B_N{}^{\tilde{e}\tilde{d}}+B_N{}^{\tilde{c}}{}_{\tilde{e}}B_M{}^{\tilde{e}\tilde{d}})\ ,$$
$$R_{\alpha\beta}{}^{\gamma\delta}=E_{\alpha}{}^{M}E_{\beta}{}^{N}(\partial_MB_N{}^{\gamma\delta}-\partial_NB_M{}^{\gamma\delta}-B_M{}^{\gamma}{}_{\epsilon}B_N{}^{\epsilon\delta}+B_N{}^{\gamma}{}_{\epsilon}B_M{}^{\epsilon\delta})\ ,$$
$$r_{\tilde{a}\beta}{}^{\tilde{c}\delta}=E_{\tilde{a}}{}^{M}E_{\beta}{}^{N}(\partial_Mb_n{}^{\tilde{c}\delta}-\partial_Nb_M{}^{\tilde{c}\delta}-B_M{}^{\tilde{c}}{}_{\tilde{e}}b_N{}^{\tilde{e}\delta}+B_N{}^{\tilde{c}}{}_{\tilde{e}}b_M{}^{\tilde{e}\delta}-B_M{}^{\delta}{}_{\epsilon}b_N{}^{\tilde{c}\epsilon}+B_N{}^{\delta}{}_{\epsilon}b_M{}^{\tilde{c}\epsilon})\ ,$$
$$b_{[\tilde{a}\tilde{b}]\gamma}=\tfrac{1}{2}(b_{\tilde{a}\tilde{b}\gamma}-b_{\tilde{b}\tilde{a}\gamma})\ ,\quad b_{[\alpha\beta]\tilde{c}}=\tfrac{1}{2}(b_{\alpha\beta\tilde{c}}-b_{\beta\alpha\tilde{c}})\ ,$$
$$b_{\{\tilde{a}\tilde{b}\}\gamma}=\tfrac{1}{2}(b_{\tilde{a}\tilde{b}\gamma}+b_{\tilde{b}\tilde{a}\gamma})-\frac{1}{k}\eta_{\tilde{a}\tilde{b}}b_{\tilde{d}}{}^{\tilde{d}\gamma}\ ,\quad b_{\{\alpha\beta\}\tilde{c}}=\tfrac{1}{2}(b_{\alpha\beta\tilde{c}}+b_{\beta\alpha\tilde{c}})-\frac{1}{k'}\eta_{\alpha\beta}b_{\delta}{}^{\delta\tilde{c}}\ ,$$

and $k'=d-k$ with $c_0,c_0',\ldots,c_4$ numerical constants, one of which may be disposed of by an overall rescaling. The G_T connections $B_M{}^{\tilde{a}\tilde{b}}$,$B_M{}^{\alpha\beta}$ and the tensor field $b_M{}^{\tilde{a}\beta}$ are determined by the respective variations of (1). The results are

$$B_{\tilde{a}\tilde{b}\tilde{c}}=-\tfrac{1}{2}(\Omega_{\tilde{a}\tilde{b}\tilde{c}}+\Omega_{\tilde{c}\tilde{b}\tilde{a}}+\Omega_{\tilde{c}\tilde{a}\tilde{b}})+d\,(\eta_{\tilde{a}\tilde{c}}\Omega_{\tilde{b}\delta}{}^{\delta}-\eta_{\tilde{a}\tilde{b}}\Omega_{\tilde{c}\delta}{}^{\delta})\ ,$$
$$B_{\alpha\beta\gamma}=-\tfrac{1}{2}(\Omega_{\alpha\beta\gamma}+\Omega_{\gamma\beta\alpha}+\Omega_{\gamma\alpha\beta})+d'(\eta_{\alpha\gamma}\Omega_{\beta\tilde{d}}{}^{\tilde{d}}-\eta_{\alpha\beta}\Omega_{\gamma\tilde{d}}{}^{\tilde{d}})\ ,$$
$$B_{\alpha\tilde{b}\tilde{c}}=-\tfrac{1}{2}(\Omega_{\alpha\tilde{b}\tilde{c}}+\Omega_{\tilde{c}\alpha\tilde{b}}+\Omega_{\tilde{c}\tilde{b}\alpha})+e\Omega_{\tilde{c}\tilde{b}\alpha}\ ,\quad B_{\tilde{a}\beta\gamma}=-\tfrac{1}{2}(\Omega_{\tilde{a}\beta\gamma}+\Omega_{\gamma\tilde{a}\beta}+\Omega_{\gamma\beta\tilde{a}})+e'\Omega_{\gamma\beta\tilde{a}}\ , \quad (2)$$
$$b_{\tilde{a}\tilde{b}\gamma}=-\frac{c_0}{\lambda}(\Omega_{\tilde{a}\tilde{b}\gamma}+\Omega_{\gamma\tilde{a}\tilde{b}}+\Omega_{\gamma\tilde{b}\tilde{a}})+f\,(\Omega_{\gamma\tilde{a}\tilde{b}}+\Omega_{\gamma\tilde{b}\tilde{a}})+g\eta_{\tilde{a}\tilde{b}}\Omega_{\gamma\tilde{d}}{}^{\tilde{d}}\ ,$$
$$b_{\alpha\beta\tilde{c}}=-\frac{c_{0'}}{\lambda}(\Omega_{\alpha\beta\tilde{c}}+\Omega_{\tilde{c}\alpha\beta}+\Omega_{\tilde{c}\beta\alpha})+f'(\Omega_{\tilde{c}\alpha\beta}+\Omega_{\tilde{c}\beta\alpha})+g'\eta_{\alpha\beta}\Omega_{\tilde{c}\delta}{}^{\delta}\ ,$$

where $\Omega_{ABC}=E_A{}^ME_B{}^N(\partial_Me_{NC}-\partial_Ne_{MC})$,

$$d=\frac{(1-k')\lambda^2+4c_0'c_3'k'}{k'[(k-1)\lambda^2-4c_0c_3'(k-2)]}\ ,\quad d'=\frac{(1-k)\lambda^2+4c_0c_3k}{k[(k'-1)\lambda^2-4c_0'c_3(k'-2)]}\ ,$$
$$e=\frac{1}{2}-\frac{2c_0c_1}{\lambda^2}\ ,\quad e'=\frac{1}{2}-\frac{2c_0'c_1'}{\lambda^2}\ ,\quad f=\frac{\lambda}{4c_2}+\frac{c_0}{\lambda}\ ,\quad f'=\frac{\lambda}{4c_2'}+\frac{c_0'}{\lambda}\ ,$$
$$g=-\frac{1}{k\lambda}\left[\frac{\lambda^2}{2c_2}+2c_0+2c_0'(k'-2)d'\right]\ ,\quad g'=-\frac{1}{k'\lambda}\left[\frac{\lambda^2}{2c_2'}+2c_0'+2c_0(k-2)d\right]\ .$$

It can be seen in (2) that the G_T connections are, in general, non-Riemannian. The gravitational field equations are obtained by varying $\mathscr{L}$ with respect to the vielbein and we get

$$\eta_c{}^D\mathscr{L}+\eta_{\tilde{b}}{}^D(2c_0R_{c\tilde{a}}{}^{\tilde{a}\tilde{b}}+\lambda r_{c\alpha}{}^{\alpha\tilde{b}})+\eta_{\beta}{}^D(2c_0'R_{c\alpha}{}^{\alpha\beta}+\lambda r_{c\tilde{a}}{}^{\tilde{a}\beta})=0\ . \quad (3)$$

It has been shown in a somewhat different context that quasi-Riemannian theories may admit compactifying solutions.[7] In the present case, (3) may be simplified to read

$$c_0\left[R_{\tilde{a}\tilde{c}}{}^{\tilde{b}\tilde{c}}-\frac{1}{k}\eta_{\tilde{a}}{}^{\tilde{b}}R_{\tilde{c}\tilde{d}}{}^{\tilde{c}\tilde{d}}\right]=-\frac{\lambda}{2}\left[r_{\tilde{a}\gamma}{}^{\tilde{b}\gamma}-\frac{1}{k}\eta_{\tilde{a}}{}^{\tilde{b}}r_{\tilde{c}\delta}{}^{\tilde{c}\delta}\right]\ ,$$
$$c_0'\left[R_{\alpha\gamma}{}^{\beta\gamma}-\frac{1}{k'}\eta_{\alpha}{}^{\beta}R_{\gamma\delta}{}^{\gamma\delta}\right]=-\frac{\lambda}{2}\left[r_{\alpha\tilde{c}}{}^{\beta\tilde{c}}-\frac{1}{k'}\eta_{\alpha}{}^{\beta}r_{\gamma\tilde{d}}{}^{\gamma\tilde{d}}\right]\ ,$$
$$c_0R_{\alpha\tilde{c}}{}^{\tilde{b}\tilde{c}}=-\frac{\lambda}{2}r_{\alpha\gamma}{}^{\tilde{b}\gamma}\ ,\quad c_0'R_{\tilde{a}\gamma}{}^{\beta\gamma}=-\frac{\lambda}{2}r_{\tilde{a}\tilde{c}}{}^{\beta\tilde{c}}\ , \quad (4)$$

so that it seems possible that the $r_{AB}{}^{\tilde{c}\delta}$ provide source terms for compactification. We are more interested, however, in applying the Kaluza-Klein ansatz to (1).

III. KALUZA-KLEIN ANSATZ AND DIMENSIONAL REDUCTION

As in standard Kaluza-Klein theory, we take the vielbein to be of the form

$$e_M{}^A(z)=\begin{bmatrix} e_m{}^a(x) & -A_m{}^{\hat{\beta}}(x)D_{\hat{\beta}}{}^{\tilde{\alpha}}(L_y) \\ 0 & e_{\tilde{\mu}}{}^{\tilde{\alpha}}(y)\end{bmatrix}\ , \quad (5)$$

where L_y is an element in the coset space G/H, $D_{\hat{\alpha}}{}^{\hat{\beta}}$ are matrices in the adjoint representation of $G(\hat{\alpha},\hat{\beta},\ldots$

$=5,\ldots,d,\ldots,d+\dim H$ label all generators of G) and $A_m{}^{\hat\alpha}(x)$ are the four-dimensional gauge potentials. The left action of G on G/H is defined by $gL_y=L_{y'}h$ ($g\in G, h\in H$). The transformation laws of the various components of $e_M{}^A$ under a left translation are as follows:[1]

$$\begin{aligned}
e'^{a}_{m}(x') &= e_m{}^a(x)\,,\\
e'^{\alpha}_{m}(x',y') &= \frac{\partial y^{\tilde\mu}}{\partial x'^{m}} e_{\tilde\mu}{}^{\beta}(y) D_{\beta}{}^{\alpha}(h^{-1})\,,\\
e'^{\alpha}_{\tilde\mu}(y') &= \frac{\partial y^{\tilde\nu}}{\partial y'^{\tilde\mu}} e_{\tilde\nu}{}^{\beta}(y) D_{\beta}{}^{\alpha}(h^{-1})\,.
\end{aligned} \tag{6}$$

In particular, substituting the ansatz (5) into the second equation of (6) gives the four-dimensional Yang-Mills gauge transformation law

$$A'_m(x') = gA_m(x)g^{-1} - g\partial_m g^{-1}\,, \tag{7}$$

where $A_m=A_m{}^{\hat\alpha}Q_{\hat\alpha}$ with the $Q_{\hat\alpha}$ generators of G. We follow standard Kaluza-Klein theory and demand that the d-dimensional Lagrangian be invariant under left translation. This will be true for any Lagrangian [such as (1)] which is invariant under the spacetime symmetries provided that $D_{\bar\alpha}{}^{\beta}(h^{-1})$ may be considered as a tangent-space rotation in the internal space. That is, we must have $H\subset SO(k-4)\times G'_T$. For our case $G'_T=SO(d-k)$, the $D_{\bar\alpha}{}^{\beta}(h^{-1})$ is an orthogonal block-diagonal matrix (with $k-4\times k-4$ and $d-k\times d-k$ blocks) provided that $D_{a'}{}^{\beta}(h^{-1})=0$ or in infinitesimal form

$$c_{a'\beta\bar\gamma}=0\,, \tag{8}$$

where $\bar\alpha,\bar\beta,\ldots=d+1,\ldots,d+\dim H$ run over the subgroup indices. Alternatively, the embedding of H generators in $SO(k-4)\times SO(d-k)$ [in analogy to the usual $SO(d-4)$ in Riemannian theories] via

$$Q_{\bar\alpha}=\tfrac{1}{2}c_{\bar\alpha b'c'}\Sigma^{b'c'}+\tfrac{1}{2}c_{\bar\alpha\beta\gamma}\Sigma^{\beta\gamma} \tag{9}$$

succeeds only if $c_{a'\beta\bar\gamma}=0$. Otherwise $Q_{\bar\alpha}$ will not verify the Lie algebra of H because the $\Sigma^{a'\beta}$ are missing. The implication of (8) is that in quasi-Riemannian Kaluza-Klein theory, we are not free to choose any G/H as internal space since, in general, (8) is not satisfied. Evidently, the generators of G/H also have to be partitioned into a' and α categories appropriately, for (8) may be satisfied for some choices of partitioning but not for others. For internal spaces of low dimensionality, the possibilities may be readily categorized. However, for higher numbers of dimensions, the available choices grow rapidly. As an example, consider $G/H=SO(3)\times SO(2)/SO(2)$ where the SO(2) generator in the denominator is taken to be a linear combination of the SO(2) and diagonal SO(3) generators in the numerator. The tangent group in this seven-dimensional spacetime is taken to be SO(1,5). (This is the particular example given by Weinberg and shown to give rise to chiral fermions in four dimensions.) We associate $a'=5,6$ with τ_1,τ_2 of SO(3) and $\beta=7$, $\bar\gamma$ with the two orthogonal combinations of the remaining (diagonal) generators so that (8) is clearly satisfied.

Now to see the four-dimensional content of (1), we substitute (5) into (2) and then into (1). The first step gives

$$\begin{aligned}
B_{abc} &= -\tfrac{1}{2}(\Omega_{abc}+\Omega_{cab}+\Omega_{cba})\,,\\
B_{a'b'c'} &= \tfrac{1}{2}c_{a'b'c'}+\pi_{a'}{}^{\bar\delta}c_{\bar\delta b'c'}\,,\\
B_{ab'c'} &= -A_a{}^{\hat\delta}(D_{\hat\delta}{}^{\bar\alpha}-\pi_{\hat\epsilon}{}^{\bar\alpha}D_{\hat\delta}{}^{\hat\epsilon})c_{\bar\alpha b'c'}\,,\\
B_{a'bc} &= \tfrac{1}{2}F_{cb}{}^{\hat\delta}D_{\hat\delta a'}\,,\\
B_{abc'} &= \tfrac{1}{2}F_{ab}{}^{\hat\delta}D_{\hat\delta c'}\,,\\
B_{a'b'c} &= 0\,,\\
B_{\alpha\beta\gamma} &= \tfrac{1}{2}c_{\alpha\beta\gamma}+\pi_{\alpha}{}^{\bar\delta}c_{\bar\delta\beta\gamma}\,,\\
B_{a\beta\gamma} &= -A_a{}^{\hat\delta}(D_{\hat\delta}{}^{\bar\alpha}-\pi_{\hat\epsilon}{}^{\bar\alpha}D_{\hat\delta}{}^{\hat\epsilon})c_{\bar\alpha\beta\gamma}\,,\\
B_{a'\beta\gamma} &= c_{a'\beta\gamma}+\pi_{a'}{}^{\bar\delta}c_{\bar\delta\beta\gamma}-(e'-\tfrac{1}{2})c_{\gamma\beta a'}\,,\\
B_{\alpha bc} &= -(e-\tfrac{1}{2})F_{cb}{}^{\hat\delta}D_{\hat\delta\alpha}\,,\\
B_{\alpha b'c'} &= c_{\alpha b'c'}+\pi_{\alpha}{}^{\bar\delta}c_{\bar\delta b'c'}-(e-\tfrac{1}{2})c_{c'b'\alpha}\,,\\
B_{\alpha bc'} &= 0\,,\\
b_{ab\gamma} &= \frac{c_0}{\lambda}F_{ab}{}^{\hat\delta}D_{\hat\delta\gamma}\,,\\
b_{a'b'\gamma} &= \frac{c_0}{\lambda}c_{a'b'\gamma}\,,\\
b_{a'b\gamma} &= 0\,,\\
b_{ab'\gamma} &= 0\,,\\
b_{\alpha\beta c} &= 0\,,\\
b_{\alpha\beta c'} &= \frac{c'_0}{\lambda}c_{\alpha\beta c'}\,,
\end{aligned} \tag{10}$$

where $\pi_{\bar\alpha}{}^{\beta}(y)=E_{\bar\alpha}{}^{\tilde\mu}(y)e_{\tilde\mu}{}^{\beta}(y)$ (with $L_y dL_y{}^{-1}=dy^{\tilde\mu}e_{\tilde\mu}{}^{\hat\alpha}Q_{\hat\alpha}$) and

$$F_{ab}{}^{\hat\gamma}(x)=E_a{}^m(x)E_b{}^n(x)[\partial_m A_n{}^{\hat\gamma}(x)-\partial_n A_m{}^{\hat\gamma}(x)-c^{\hat\gamma}{}_{\hat\delta\hat\epsilon}A_m{}^{\hat\delta}(x)A_n{}^{\hat\epsilon}(x)]\,.$$

Use of (8) has been made in simplifying the expressions in (10). Then the Lagrangian (1) reduces to

$$\mathscr{L}=(\det e)\left[c_0R_4(x)+\tfrac{1}{4}c_0F_{ab}{}^{\hat\alpha}(x)F^{ab\hat\beta}(x)D_{\hat\alpha c'}(y)D_{\hat\beta}{}^{c'}(y)+\frac{c_0{}^2c_1}{\lambda^2}F_{ab}{}^{\hat\alpha}(x)F^{ab\hat\beta}(x)D_{\hat\alpha\gamma}(y)D_{\hat\beta}{}^{\gamma}(y)\right.$$
$$\left.+\,y\text{-dependent curvature and torsion terms}\right], \tag{11}$$

where

$$R_4(x)=E_a{}^m E_b{}^n(\partial_m B_n{}^{ab}-\partial_n B_m{}^{ab}-B_m{}^a{}_c B_n{}^{cb}+B_n{}^a{}_c B_m{}^{cb})$$

is none other than the four-dimensional Einstein-Hilbert Lagrangian since $B_m{}^{ab}$ are all Riemannian valued. It is comforting to see that the four-dimensional effective theory is torsion-free. For a general G_T, this will not be the case. However, one may then perhaps impose some condition on the arbitrary coefficients in the Lagrangian in order that the four-dimensional torsion vanishes. The second and third terms in (11) require comment. In standard Kaluza-Klein theory, left translation invariance of the Lagrangian leads to four-dimensional Yang-Mills gauge invariance since dimensional reduction gives the term involving the gauge potentials as proportional to $F_{ab}{}^{\hat{\alpha}}(x)F^{ab\hat{\beta}}(x)D_{\hat{\alpha}\tilde{\gamma}}(y)D_{\hat{\beta}}{}^{\tilde{\gamma}}(y)$ and the integral over y of $D_{\hat{\alpha}\tilde{\gamma}}(y)D_{\hat{\beta}}{}^{\tilde{\gamma}}(y)$ is a constant times $\delta_{\hat{\alpha}\hat{\beta}}$. The integrals of $D_{\hat{\alpha}c'}(y)D_{\hat{\beta}}{}^{c'}(y)$ and $D_{\hat{\alpha}\gamma}(y)D_{\hat{\beta}}{}^{\gamma}(y)$ are not separately covariant G tensors, however. Hence, to get gauge invariance we must demand

$$\frac{1}{4}-\frac{c_0c_1}{\lambda^2}=\tfrac{1}{2}e=0\ . \tag{12}$$

In that case, we have for the effective four-dimensional Lagrangian

$$\mathscr{L}_4=[\det e_m{}^a(x)]\times\left[c_0R_4(x)+\tfrac{1}{4}c_0F_{ab}{}^{\hat{\alpha}}(x)F^{ab}{}_{\hat{\alpha}}(x)\left[\frac{d-4}{\dim G}\right]+\text{cosmological term}\right]. \tag{13}$$

IV. SPINOR FIELDS

We now turn to the inclusion of fermions. Here because $G_T=\mathrm{SO}(1,k-1)\times\mathrm{SO}(d-k)$, a d-dimensional field is characterized by representations of $\mathrm{SO}(1,k-1)$ and $\mathrm{SO}(d-k)$, that is, as a tensor or spinor under each piece of G_T. The spin-statistics connection of these general fields should be elucidated. However, from the standpoint of obtaining four-dimensional spinors in the harmonic expansion it is clear that we need to start with a field $\Psi(z)$ which transforms as a spinor under $\mathrm{SO}(1,k-1)$. The simplest choice would seem to have $\Psi(z)$ transforming as a scalar under $\mathrm{SO}(d-k)$. The obvious expression for an invariant Lagrangian involving $\Psi(z)$ is then

$$\mathscr{L}_\Psi=\frac{i}{2}(\det e)\overline{\Psi}\Gamma^{\tilde{a}}E_{\tilde{a}}{}^M\nabla_M\Psi+\text{H.c.}\ , \tag{14}$$

where $\nabla_M=\partial_M+\frac{1}{2}B_{M\tilde{a}\tilde{b}}\Sigma^{\tilde{a}\tilde{b}}$, $\Gamma^{\tilde{a}}$ are $2^{[k/2]}$-dimensional Dirac matrices and $\Sigma^{\tilde{a}\tilde{b}}=-\frac{1}{4}[\Gamma^{\tilde{a}},\Gamma^{\tilde{b}}]$. To see the four-dimensional content of (14), we make a harmonic expansion for $\Psi(z)$. Now for a general field transforming as $\phi_i(z)\to D_{ij}(h)\phi_j(z)$, the appropriate expansion is

$$\Psi_i(z)=\sum_{n,\zeta}\left[\frac{d_n}{d_D}\right]^{1/2}D^n_{i\zeta,p}(L_y{}^{-1})\psi^n_{p\zeta}(x)\ , \tag{15}$$

where the sum is over all irreducible representations of G which contain D on restriction to H; ζ is a supplementary label in the case where D occurs more than once in D^n. Generically, we will simply write $\phi(z)=D(L_y{}^{-1})\phi(x)$ as a typical term in the harmonic expansion. Then making use of (5), (10), and the identify

$$\partial_{\tilde{\mu}}D(L_y{}^{-1})=e_{\tilde{\mu}}{}^{\hat{\alpha}}D(Q_{\hat{\alpha}})D(L_y{}^{-1}) \tag{16}$$

we get for (14)

$$\mathscr{L}_\psi=\frac{i}{2}\overline{\psi}(x)D(L_y)\{\Gamma^a[E_a{}^m\partial_m+\tfrac{1}{2}B_{abc}\Sigma^{bc}-A_a{}^{\hat{\beta}}D_{\hat{\beta}}{}^{\hat{\gamma}}D(Q_{\hat{\gamma}})+\tfrac{1}{4}F_{ab}{}^{\hat{\delta}}D_{\hat{\delta}c'}(L_y)\Sigma^{bc'}]$$
$$+\Gamma^{a'}[\tfrac{1}{4}c_{a'b'c'}\Sigma^{b'c'}-D(Q_{a'})]\}D(L_y{}^{-1})\psi(x)+\text{H.c.} \tag{17}$$

This result is similar to the usual one in Riemannian theory [where $\Psi(z)$ is an $\mathrm{SO}(d)$ spinor] except for the important difference that the summations in the last three terms in (17) do not run over the entire internal space. Upon integrating over y, these terms are just the Pauli moment and mass terms in four dimensions. The incomplete summations imply a dependence on the basis of the G/H generators. This is basically a reflection of condition (8) which enforces a separation into the a',α indices of the internal space. A similar situation occurred in the pure gravitational case in (11). The condition (8) can be satisfied more than one way in general (or perhaps not at all for some choices of $G/H,G_T$). It appears physically reasonable (and also economical) that the four-dimensional theory should not depend on the details of the split into a',α indices even though for consistency (8) must be possible. Hence we require that summations in the four-dimensional theory be over $\alpha=(a',\alpha)$ indices. For this reason, we reject the d-dimensional field transforming as a spinor under $\mathrm{SO}(1,k-1)$ and scalar under $\mathrm{SO}(d-k)$ and the Lagrangian (14) as a possibility in obtaining four-dimensional spinor theories. Clearly, the missing terms in (17) are due at least in part to the fact that there are no $\Sigma^{\alpha\beta}$ matrices in (14). The above comments are also true for all d-dimensional fields transforming as a spinor under $\mathrm{SO}(1,k-1)$ and as a tensor under $SO(d-k)$. We are thus prompted to try the d-dimensional field transforming as a spinor under both $SO(1,k-1)$ and $SO(d-k)$ since the Lagrangian will then contain all the Dirac matrices. Indeed, the minimal expression is

$$\mathscr{L}_\Psi = \frac{i}{2}(\det e)\bar{\Psi}\Gamma^A E_A{}^M \nabla_M \Psi + \text{H.c.}\,, \tag{18}$$

where $\nabla_M = \partial_M + \frac{1}{2}B_{M\tilde{a}\tilde{b}}\Sigma^{\tilde{a}\tilde{b}} + \frac{1}{2}B_{M\alpha\beta}\Sigma^{\alpha\beta}$. The Γ^A are $2^{[k/2]}\times 2^{[d-k/2]}$-dimensional matrices given by $\Gamma^{\tilde{a}} = \gamma^{\tilde{a}}\times 1$, $\Gamma^\alpha = 1\times\gamma^\alpha$ where $\Gamma^{\tilde{a}},\Gamma^\alpha$ are, respectively, the k- and $(d-k)$-dimensional Dirac matrices. Substituting the harmonic expansion for Ψ in (18) then gives for a typical term

$$\begin{aligned}\mathscr{L}'_\psi = \frac{i}{2}\bar{\psi}(x)D(L_y)\Big[&\Gamma^a[E_a{}^m\partial_m + \tfrac{1}{2}B_{abc}\Sigma^{bc} - A_a{}^{\hat{\beta}}D_{\hat{\beta}}{}^{\hat{\gamma}}D(Q_{\hat{\gamma}})] + \tfrac{1}{4}F_{ab}{}^{\hat{\delta}}D_{\hat{\delta}c'}\Gamma^{c'}\Sigma^{ab}\\ &+\left[\frac{e}{2}-\frac{1}{4}\right]F_{ab}{}^{\hat{\delta}}D_{\hat{\delta}\gamma}\Gamma^\gamma\Sigma^{ab} - \Gamma^{\tilde{\alpha}}D(Q_{\tilde{\alpha}}) + \tfrac{1}{4}c_{\alpha\beta\gamma}\Gamma^\alpha\Sigma^{\beta\gamma} + \tfrac{1}{4}c_{a'b'c'}\Gamma^{a'}\Sigma^{b'c'}\\ &+\left[\frac{e'}{2}+\frac{1}{4}\right]c_{a'\beta\gamma}\Gamma^{a'}\Sigma^{\beta\gamma} + \left[\frac{e}{2}+\frac{1}{4}\right]c_{\alpha b'c'}\Gamma^\alpha\Sigma^{b'c'}\Big]D(L_y{}^{-1})\psi(x) + \text{H.c.}\end{aligned} \tag{19}$$

Unfortunately, there are still incomplete summations in (19) so (18) cannot be a satisfactory Lagrangian. Since we still have the fields $b_M{}^{\tilde{a}\beta}$ at our disposal, we exhaust the possibilities and couple the $b_M{}^{\tilde{a}\beta}$ to Ψ through the Γ^A matrices. The Lagrangian below [an extended version of (18)],

$$\mathscr{L}''_\Psi = \frac{i}{2}\bar{\Psi}\Gamma^A E_A{}^M\nabla_m\Psi + \frac{i}{2}\left[\frac{\lambda}{2}\right]\bar{\Psi}\left[\frac{t}{c_{0'}}b_{\beta\gamma\tilde{a}}\Gamma^a\Sigma^{\beta\gamma} + \frac{t'}{c_0}b_{\tilde{b}\tilde{c}\alpha}\Gamma^\alpha\Sigma^{\tilde{b}\tilde{c}}\right]\Psi + \text{H.c.} \tag{20}$$

reduces to terms like

$$\begin{aligned}\mathscr{L}''_\psi = \frac{i}{2}\bar{\psi}(x)D(L_y)\Big[&\Gamma^a[E_a{}^m\partial_m + \tfrac{1}{2}B_{abc}\Sigma^{bc} - A_a{}^{\hat{\beta}}D_{\hat{\beta}}{}^{\hat{\gamma}}D(Q_{\hat{\gamma}}) + \tfrac{1}{4}F_{ab}{}^{\hat{\delta}}D_{\hat{\delta}\tilde{\gamma}}\Sigma^{b\tilde{\gamma}}] - \Gamma^{\tilde{\alpha}}D(Q_{\tilde{\alpha}}) + \tfrac{1}{8}c_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}\Gamma^{\tilde{\alpha}}\Gamma^{\tilde{\beta}}\Gamma^{\tilde{\gamma}}\\ &+\left[\frac{e}{2}+\frac{t'-1}{2}\right]F_{ab}{}^{\tilde{\delta}}D_{\hat{\delta}\gamma}\Gamma^\gamma\Sigma^{ab} + \left[\frac{e}{2}+\frac{t'}{2}\right]c_{\alpha b'c'}\Gamma^\alpha\Sigma^{b'c'}\\ &+\left[\frac{e'}{2}+\frac{t}{2}\right]c_{a'\beta\gamma}\Gamma^{a'}\Sigma^{\beta\gamma}\Big]D(L_y{}^{-1})\psi(x) + \text{H.c.}\,,\end{aligned} \tag{21}$$

where t,t' is an arbitrary constant. The last three terms prevent a complete summation over internal-space indices. Recall that in the last section we had obtained $e=0$. Even if we now set $e'=t=0$, there would still be a term $c_{\alpha b'c'}\Gamma^\alpha\Sigma^{b'c'}$ left over. Hence, it seems that in order to get unambiguous results (for fermion masses, for instance), we must consider, in general, direct-product groups $G_1\times G_2$ with a trivial embedding of the a',α indices into the two separate factors.

We should remark that the above results were arrived at using the expressions for B_M,b_M in (10) which were obtained from a variational principle. With the inclusion of fermions, perhaps the effect in terms of bilinears must be considered and somehow incorporated in (10). Of course, one could also set B_M,b_M by hand as in[7] $B_M^{\text{Riem}} = B_M + b_M$ although we have checked that this choice leads to similar conclusions as the above.

V. CONCLUSIONS

We have found that in formulating a quasi-Riemannian version of Kaluza-Klein theory, not all choices of coset spaces as internal manifolds are admissible. Left-translation invariance is compatible with the spacetime symmetries only if $c_{a'\beta\bar{\gamma}}=0$. This condition is nontrivial unless one wishes to consider internal spaces of the form $G_1\times G_2/H$. The general d-dimensional Lagrangian has 10 undetermined constants [for $G_T = SO(1,k-1)\times SO(d-k)$]. The requirement of gauge invariance gives one condition on these constants as left-translation invariance does not lead directly to four-dimensional gauge invariance (as in the usual Riemannian theory). In this context, the tensor field $b_M{}^{\tilde{a}\beta}$ which was initially required (or introduced) to generate torsion is found to be necessary. When spinor fields are coupled to gravity we encounter further restrictive features. In the Kaluza-Klein ansatz, four-dimensional gauge invariance is in general not compatible with a general gauge group G unless it is of the form $G_1\otimes G_2$ where $SO(k-4)$ acts on G_1/H_1 and $SO(d-k)$ acts on G_2/H_2. Note, however, that the above difficulties do not occur for the case $k=4$. It is not clear if the above results would represent a serious blow to a successful application of quasi-Riemannian theories in the context of a supergravity theory based on graded Lie algebras.[4]

Note added. The general structure of $G_T = \mathrm{SO}(1,k-1) \times \mathrm{SO}(d-k)$ theories and the problem of gauge invariance upon compactification have also been considered in Refs. 8 and 9, respectively. We thank S. Weinberg for pointing this out to us.

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