On parametrizing the N-generation quark mixing matrix

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The invariant-phase approach previously introduced for describing the Kobayashi-Maskawa mixing matrix is generalized to N generations. The work is simplified by using the fact that the invariant phase is a 2-cocycle. Finally, we give the connection with Greenberg's recent approach.

The Kobayashi-Maskawa (KM) matrix,[1] which describes the charged-current weak interactions in the standard model of elementary particles, has a "geometrical" aspect in that it determines the nature of observed CP violations. Hence its parametrization is interesting both from theoretical and practical points of view. Recently, the concept of an "invariant CP phase" for this matrix was introduced[2] and applied in detail to the usual three-generation case. Here we would like to show how to count and specify the independent invariant phases for the N-generation case. This work is simplified by noting that the invariant phase is a mathematical object called a 2-cocycle. The invariant phase intrinsically spans three generations, which makes it natural that three generations are needed for CP violation.

The KM matrix U may be "gauge transformed,"

$$U \rightarrow PUQ \ , \tag{1}$$

where P and Q are diagonal matrices of phases, without changing any physical predictions. Hence the N^2 parameters of the unitary U can be reduced by $(2N-1)$ (an overall phase for P and for Q is redundant) to yield $(N-1)^2$ independent parameters. We wish to find objects which conveniently parametrize U and which are invariant under rephasings. We observe that the trivial physical situation with no intergenerational mixing and no CP violation not only corresponds to $U=1$ but also to $U=\text{diag}\ (\exp(i\tau_1),\exp(i\tau_2),\ \ldots\ )$. It is convenient to consider $U=1$ to be the representative of this class and hence to find rephasings which leave $U=1$ invariant. This amounts to restricting one's attention to the transformations

$$U \rightarrow PUP^{\dagger},\ \ P=\text{diag}(e^{i\alpha_1},e^{i\alpha_2},\ \ldots\ ) \ . \tag{2}$$

We shall introduce invariant CP phases which remain unchanged under the transformation (2) and then later show that they are, in fact, invariant under the full rephasing transformation (1).

The actual parametrization of U is, of course, nonunique. Writing U as a product of "complex" 2×2 rotations connecting each pair of generations guarantees unitarity with a simple finite analytical form, but is not required for our considerations. The generic complex rotation between the first and second generations,

$$\omega_{12}(\theta_{12},\phi_{12})=\begin{bmatrix} \cos\theta_{12} & e^{i\phi_{12}}\sin\theta_{12} & 0 & \cdots \\ -e^{-i\phi_{12}}\sin\theta_{12} & \cos\theta_{12} & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{3}$$

is described by a mixing angle θ_{12} and a CP phase ϕ_{12}. The KM matrix is then written as

$$U=\prod_{i<j}\omega_{ij} \ , \tag{4}$$

which also satisfies the restriction $\det U=1$. Any particular, but fixed, order of the ω_{ij}'s in (4) may be used. It is easy to see[3] that the effect of the transformation (2) is to leave (4) form invariant with the ϕ_{ij} replaced as

$$\phi_{ij}\rightarrow\phi'_{ij}=\phi_{ij}+\alpha_i-\alpha_j \ . \tag{5}$$

The invariant phases I_{ijk}, each required to span three generations, are then

$$I_{ijk}=\phi_{ij}+\phi_{jk}-\phi_{ik} \ \ (i<j<k) \ . \tag{6}$$

In the standard three-generation case, U is parametrized by the three mixing angles $\theta_{12},\theta_{23},\theta_{13}$ and the single invariant phase $I_{123}=\phi_{12}+\phi_{23}-\phi_{13}$.

When $N=4$, U is parametrized by $4\times\frac{3}{2}=6$ mixing angles but there are $4\times3\times2/(3\times2)=4$ invariant phases. This is one too many since the total number of independent parameters minus the number of mixing angles is $(4-1)^2-6=3$. There is one linear relation among the four invariant phases which may be expressed as the vanishing of the "coboundary" operator δ on I:

$$(\delta I)_{ijkl}\equiv F^{(4)}_{ijkl}=0 \ \ (i<j<k<l) \ . \tag{7}$$

Explicitly

$$(\delta I)_{1234}=I_{234}-I_{134}+I_{124}-I_{123} \ . \tag{8}$$

Using (6), (8) is seen to vanish; this is the condition for I_{ijk} to be a 2-cocycle. [Actually (8) must vanish since $I=\delta\phi$.]

Now consider the five-generation case. There are $5\times4\times3/(3\times2)=10=C_{53}$ invariant phases, where $C_{NK}\equiv N!/[K!(N-K)!]$. However, there are $C_{54}=5$ relations of the type

$$F^{(4)}_{ijkl}=0\ .$$

But these five relations are not independent; there is one ($C_{55}=1$) linear relation among them:

$$F^{(5)}\equiv\delta F^{(4)}=\delta^2 I=0\ , \qquad (9)$$

since $\delta^2\equiv 0$. Specifically, following the pattern in (8) of sequentially omitting an index and alternating signs, one has

$$(\delta F^{(4)})_{12345}=F^{(4)}_{2345}-F^{(4)}_{1345}+F^{(4)}_{1245}-F^{(4)}_{1235}+F^{(4)}_{1234}=0\ . \qquad (10)$$

Thus there are four independent relations among the ten invariant phases. This yields six independent invariant phases—the correct number for five generations.

The pattern is now clear. For N generations, using the relation $\delta^2=0$ repeatedly, the number of independent invariant phases is seen to be

$$\sum_{K=3}^{N}(-1)^{K+1}C_{NK}\ . \qquad (11)$$

Using

$$\sum_{K=0}^{N}(-1)^{K+1}C_{NK}=0\ ,$$

(11) may be rewritten as

$$C_{N0}-C_{N1}+C_{N2}=\tfrac{1}{2}(N-1)(N-2)\ , \qquad (12)$$

which is exactly the right number $[(N-1)^2-N(N-1)/2]$.

In practice it is often desirable to choose a U in (4) with just the number $(N-1)(N-2)/2$ of phases ϕ_{ij} not equal to zero. These nonzero phases cannot be chosen arbitrarily. For example, in the four-generation case we cannot choose ϕ_{14}, ϕ_{24}, and ϕ_{34} to be the nonvanishing ones since this would put the *arbitrary* invariant I_{123} equal to zero.

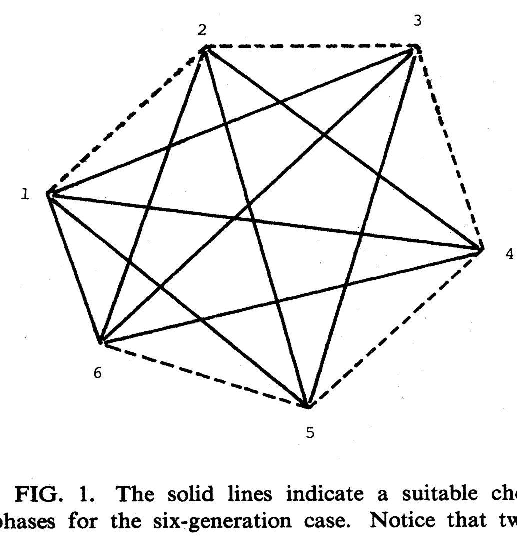

FIG. 1. The solid lines indicate a suitable choice of the phases for the six-generation case. Notice that two triangles have no sides which are edges of the polygon. This is the motivation for choosing all the diagonals, since each triangle must have at least one diagonal as a side.

We must choose phases in such a way that no invariant phase is equal to zero. This can be done with a simple pictorial approach. First, note that the invariant phase may be represented as a triangle joining three vertices. Second, note that a polygon with N vertices has $N(N-3)/2$ diagonals. This is one less than the number of independent CP phases. Now if we draw an N-sided polygon with all its diagonals as illustrated for six generations in Fig. 1 and assign each line to one ϕ_{ij}, our task is to choose $(N-1)(N-2)/2$ lines in such a way that no triangle is unrepresented. Clearly, all triangles will be represented if *all* the diagonals are chosen. This leaves us with one less line than the correct number of independent phases. Therefore, we must add one line on the perimeter of the polygon; for definiteness it is chosen to be the one joining the points 1 and N. Thus in the three-generation case we would choose ϕ_{13}. For four generations we would add ϕ_{14} and ϕ_{24}. For five generations we would add ϕ_{15}, ϕ_{25}, and ϕ_{35}, etc. Obviously, all the indices may be permuted in any way.

A related question is to choose a suitable independent set of invariant phases. This may be done recursively. For three generations we have I_{123}. At four generations we add the two triangles I_{124} and I_{234} generated by the new vertex. At five generations we add the three new triangles I_{125}, I_{235}, and I_{345}, and so on.

We end with some brief remarks.

(1) The existence of CP violation in the theory requires at least one invariant phase I_{ijk} to differ from zero. Note, however [see (3)], that I_{ijk} will appear in weak amplitudes in the combination $S_{ij}S_{ik}S_{jk}\exp(iI_{ijk})$, $i<j<k$. Thus the three associated mixing angles also should not vanish for a particular invariant phase to be effective in producing CP violations.

(2) Although any possible way[4] of parametrizing U is theoretically on the same footing, certain parametrizations have practical advantages. In Ref. 2, it was noted that the three-generation choice $U=\omega_{23}\omega_{12}\omega_{13}$ is representative of a class for which, to good accuracy, the three mixing angles θ_{12}, θ_{23}, and θ_{13} coincide with the *experimental* transition amplitudes $|U_{us}|$, $|U_{cb}|$, and $|U_{ub}|$. Furthermore, the CP-violating amplitudes are proportional[5] to $|U_{us}|\,|U_{cb}|\,|U_{ub}|\sin(I_{123})$, so $|I_{123}|=\pi/2$ gives a convenient practical measure of maximal CP violation. A similar parametrization for four generations has been presented.[6] It has also been noted that in some theoretical models for the KM matrix, the invariant phase associated with the present order of the ω_{ij}'s taking on the value $\pi/2$ corresponds[7] to maximal CP violation. Incidentally, a natural definition of maximal CP violation for the N-generation case would require each invariant phase to take on the value $\pi/2\times$(odd integer). In the model of Ref. 7 applied to four generations the three independent invariant phases I_{123}, I_{124}, and I_{234} were seen to be maximal. I_{134} is also seen to be maximal by using the constraint Eq. (6).

It should be remarked, as discussed in more detail in Ref. 2, that taking an invariant phase to be $\pi/2$ as the criterion for defining maximal CP violation, clearly depends on the separation of the KM parameters into "angles" (CP conserving) and "phases" (CP violating). This separation in turn depends on the particular parametrization scheme chosen. The choice discussed in the preceding paragraph seems to us the most reasonable one, although various other points of view have been expressed in the literature.[4]

(3) We now point out that the invariant phases defined in (6) are unchanged under the full rephasing transformation (1). In other words, in addition to invariance under the transformation (2) which leaves the identity invariant we consider transformations which alter the identity. Specifically, replace (2) by

$$U \to BPUP^\dagger, \quad B = \mathrm{diag}(e^{i\beta_1}, e^{i\beta_2}, \ldots) \ . \tag{2'}$$

The phases in B were previously[2] denoted as the "trivial" ones. For the present purpose we must parametrize U in such a way that the "diagonal" generators (members of the Cartan subalgebra) are included,

$$U = \mathrm{diag}(e^{i\phi_{11}}, e^{i\phi_{22}}, \ldots) \prod_{i<j} \omega_{ij} \ . \tag{4'}$$

For the (realistic) case of small mixing angles, the N phases ϕ_{ii} are approximately the phases of the diagonal elements of U. Now notice that U as written in (4′) is form invariant under the transformation (2′) if we replace

$$\phi_{ij} \to \phi_{ij} + \alpha_i - \alpha_j, \quad i \neq j \ ; \tag{13a}$$

$$\phi_{ii} \to \phi_{ii} + \beta_i \ . \tag{13b}$$

Equation (13a) is the same as (5) which then implies that the I_{ijk} are invariant under the full transformation (2′). It is interesting to also consider the behavior of the phases of the elements of U itself under the transformation (2′). Defining

$$U \equiv |U_{ij}| e^{i\chi_{ij}} \ ,$$

we have

$$\chi_{ij} \to \chi_{ij} + \alpha_i - \alpha_j + \beta_i \ . \tag{14}$$

Evidently, suitable invariants in this case are

$$\tilde{I}_{ijk}(\chi) = \chi_{ij} + \chi_{jk} - \chi_{ik} - \chi_{jj} \quad (i<j<k) \ . \tag{15}$$

Now let us specialize to three generations. For a parametrization (4′) like

$$U = \mathrm{diag}(e^{i\phi_{11}}, e^{i\phi_{22}}, e^{i\phi_{33}}) \omega_{23} \omega_{12} \omega_{13} \ ,$$

we find to good accuracy (see Ref. 2)

$$\chi_{ij} \approx \phi_{ij} + \phi_{ii} \quad (i<j) \ ,$$
$$\chi_{ii} \approx \phi_{ii} \ . \tag{16}$$

Then the invariant in (15) is simply our invariant phase

$$\tilde{I}_{ijk}(\chi) \approx I_{ijk}(\phi) \quad (i<j<k) \ . \tag{17}$$

[Equation (17) will also hold when there are more than three generations if the mixing angles are suitably small and a suitable order (see Ref. 2) of the ω_{ij}'s is used.]

We can now answer, at least to practical accuracy, Greenberg's question[4] of how to relate a suitable combination of invariant quartic structures in U and $U^\dagger$ to a convenient object like the invariant phase parametrizing U. Using (17) yields

$$\arg(U_{12} U^\dagger_{22} U_{23} U^\dagger_{31}) = \tilde{I}_{123}(\chi) \approx I_{123}(\phi) \ , \tag{18}$$

wherein we have identified the relevant quartic invariant. There are, for three generations, eight other different CP-violating quartic invariants of the form

$$\arg(U_{ab} U^\dagger_{bc} U_{cd} U^\dagger_{da}) \ ,$$

but the one in (18) is sufficient. The generalization to $N>3$ is immediate:

$$\arg(U_{ij} U^\dagger_{jj} U_{jk} U^\dagger_{ki}) \approx I_{ijk}(\phi) \quad (i<j<k) \ .$$

It should be stressed that both the invariant phases I_{ijk} and the more complicated objects involving $UU^\dagger UU^\dagger$ are fully invariant under *all* rephasings. The former have the advantages of a simpler "geometrical" interpretation and fewer redundancies; also they correspond to a direct parametrization of U using the generators of the $U(N)$ group. Any parametrization using the $U(N)$ generators has the same transformation laws (13a) and (13b). Note that the N^2 U_{ij} matrix elements themselves really do not give a true parametrization of the theory since they are not independent of each other.

(4) The parametrization (4) would hold for the leptonic analog of the KM matrix if the neutrinos were massive Dirac particles. For Majorana neutrinos the situation is more complicated, as discussed in the first of Ref. 3. The simplest case is when the mixing matrix is square rather than rectangular (no "right-handed" neutrinos present). Then the transformation (2) no longer leaves physical predictions invariant. Hence, the invariant phases are not sufficient. Rather, every CP phase ϕ_{ij} $(i<j)$ must be retained and CP violation begins at the two-generation level.

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