How to describe weak-interaction mixing and maximal CP violation?

H. Fritzsch
Sektion Physik der Universität München, München, Germany
and Max-Planck-Institut für Physik und Astrophysik, Werner Heisenberg Institut für Physik, München, Germany
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It is argued that nature exhibits a very specific kind of weak-interaction mixing among quarks, and that accordingly a specific description of this mixing is warranted. We propose and discuss a parametrization of the weak-interaction mixing among the three quark flavors, which in several respects is more appropriate in describing the observed mixing pattern than the standard form. Furthermore, CP violation can be described in a simple way, and the notion of maximal CP violation finds a simple geometrical interpretation.

It is a well known, yet unexplained feature of the weak interaction among the quarks that the quark states for which the charged weak currents are diagonal are not eigenstates of the strong interactions; i.e., they are mixtures of different quark "flavors." In QCD the quark eigenstates of the strong interactions are simply those states in which the quark mass matrix is diagonal. If all quarks of a given charge were degenerate in mass (e.g., of zero mass), the phenomenon of weak-interaction mixing would not exist. In this sense the quark-mixing parameters of the weak interactions can be viewed as elements of the quark mass matrix. Any attempt to gain insight into the physical structure of the quark masses is at the same time an attempt to obtain information about the mixing parameters. Therefore both the quark masses and the weak-interaction mixing parameters must depend on each other. To establish such a dependence and to find its mathematical expression is one of the most important and interesting tasks in theoretical particle physics at present. The most economical way would be to express the quark-mixing parameters solely in terms of quark masses. In the past several attempts have been made in this direction (see, e.g., Refs. 1—3) which are in agreement with the pattern of the weak mixing parameters observed, especially with the new information gained recently (for a recent discussion, see, e.g., Ref. 4).

In general, the charged weak currents can be written as $\bar{U}\gamma_{\mu L}D'$, where $\gamma_{\mu L}$ denote the left-handed projections of γ_μ:

$$\gamma_{\mu L}=\tfrac{1}{2}\gamma_\mu(1+\gamma_5)\ ,$$

and $U=(u,c,t)$, $D'=(d',s',b')$. The quarks u, c, and t are mass eigenstates, while the quarks d', s', and b' are mixtures of mass eigenstates:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \qquad (1)$$

We shall assume that only six quarks are present. In this case the mixing matrix V must be unitary.

Taking into account the constraints imposed by unitarity, the experimental data including the new results on the lifetime and branching fractions of b-flavored particles give (one-standard-deviation range, we use the results of Ref. 4, only the absolute values of the mixing elements are given)

$$\begin{aligned} &|V_{ud}|=0.9723\text{—}0.9737,\quad |V_{us}|=0.228\text{—}0.234,\quad |V_{ub}|=0.000\text{—}0.004\ ,\\ &|V_{cd}|=0.228\text{—}0.234,\quad |V_{cs}|=0.974\text{—}0.9727,\quad |V_{cb}|=0.039\text{—}0.051\ ,\\ &|V_{td}|=0.005\text{—}0.015,\quad |V_{ts}|=0.038\text{—}0.050,\quad |V_{tb}|=0.9987\text{—}0.9993\ . \end{aligned} \qquad (2)$$

As clearly shown by these results, the weak-interaction mixing seems to be of a rather specific nature. Especially noteworthy is the smallness of the matrix elements V_{ub} and V_{td}, as well as the fact that the matrix element V_{cb} is relatively small compared to V_{us}: $|V_{cb}/V_{us}|\approx 0.17$ —0.22. [The latter is implied by the relatively large lifetime of the b-flavored particles (see, e.g., Ref. 5).] The smallness of V_{ub} follows from the experimental constraints on the branching ratios $\Gamma(b\to u)/\Gamma(b\to c)$ which lead to the requirement

$$|V_{ub}/V_{cb}|<0.12\ \ (90\%\ \text{C.L.})\ . \qquad (3)$$

In particular, we should like to stress that the ratio Eq. (3) is significantly less than $\tan\theta\approx 0.23$ (θ is the Cabibbo angle). This will turn out to be of relevance below.

Nothing is known directly about the matrix element V_{td}. However, the unitarity requirements force V_{td} to be larger than 0.005, as denoted above.

In any case the experimental constraints on the quark mixing indicate that the mixing proceeds mainly between neighboring families, while the mixing between the first and third family is very small. Within a specific ansatz about the quark mass matrices this hierarchical pattern of mixing has been anticipated in Ref. 1. (For a generalization to more than three families, see Ref. 6.)

Independent of theoretical considerations about the origin of the remarkable observed mixing pattern, the fact that the transition element V_{ub} is not only much smaller than V_{us} or V_{cb}, but also smaller than V_{td} should be considered thoughtfully when choosing the best way to parametrize the matrix V. A number of different ways have been discussed in the literature, following the original parametrization of Kobayashi and Maskawa (KM).[7] The purpose of this paper is to give simple and convincing arguments in favor of a particular parametrization, which in the context of CP violation has been discussed previously by Chau and Keung,[8] and which is a slight modification of the one given by Maiani a number of years ago.[9]

In view of the fact that V_{ub} is very small, one should consider the case $V_{ub}=0$ as a very good starting point for a parametrization, and the small value of V_{ub} (plus the corresponding contributions to the other elements implied by unitarity) should be introduced later as a correction. Suppose one starts with a transition matrix

$$V=\begin{bmatrix} V_{ud} & V_{us} & 0 \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} . \tag{4}$$

In general, all matrix elements are complex numbers. However, one can easily see that by suitable phase rotations of the quark fields all matrix elements can be made real. For example, a change of phase $u \rightarrow ue^{i\alpha_u}$, $d \rightarrow de^{i\alpha_d}$ (α_u and α_d are arbitrary angles) can be absorbed by the matrix V if one carries out the substitution $V_{ud} \rightarrow V_{ud}e^{i(\alpha_d-\alpha_u)}$. In case of six quarks five such relative phases are available, which can be used to absorb the phases of, e.g., the elements V_{ud}, V_{us}, V_{cs}, V_{cb}, and V_{tb}. Since the second and the third column must be orthogonal, V_{ts} must be real as well. Likewise, the first and second row must be orthogonal. Thus V_{cd} must be real, and therefore the remaining matrix element V_{td} must also be real. In the limiting case $V_{ub}=0$ all phases can be absorbed, and no CP violation exists.

The orthogonal matrix V can be parametrized by rotation angles only in one way:

$$V=\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\tau & s_\tau \\ 0 & -s_\tau & c_\tau \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta c_\tau & c_\theta c_\tau & s_\tau \\ s_\theta s_\tau & -c_\theta s_\tau & c_\tau \end{bmatrix} \tag{5}$$

(s_θ stands for $\sin\theta$, c_θ for $\cos\theta$ etc.).

The angle θ (Cabibbo angle) describes the mixing in the d-s system, while the angle τ describes the s-b mixing. According to experiment one has $\theta=13.2°$–$13.5°$, $\tau=2.2°$–$2.9°$. The mixing can be viewed as a two-step process. We start out from the three orthogonal axes defined by the mass eigenstates (d,s,b). First a rotation by θ is carried out in the (d,s) plane (about the b axis, angle θ) followed by a rotation about the new d' axis (angle τ). After these two steps we arrive at the weak-interaction eigenstates (d',s',b') (see Fig. 1). It is worth emphasizing that after the mixing the d' quark is still located in the plane spanned by the mass eigenstates d and s, while both the s' and the b' quarks are outside the s-b plane. Presumably this signifies an important feature of the physical mechanism causing the weak-interaction mixing. (This, of course, is related to the order of the two-step mixing process.) The Cabibbo mixing is first, followed by the s-b mixing. If it were the other way around, V_{ub} would differ from zero ($V_{ub}=s_\theta s_\tau$), and V_{ts} would be zero. This possibility, however, is excluded by experiment, since one would expect $|V_{ub}/V_{cb}| \approx \tan\theta \approx 0.23$, in disagreement with Eq. (3).

As a final step we introduce a nonzero (complex) value of V_{ub} in such a way that the unitarity of the mixing matrix is preserved. A number of possibilities exist; however, one is particularly singled out as follows. Introducing a small value of V_{ub} means tilting the d' quark very slightly outside the d-s plane. During such a process the mixing angles θ and τ have to be changed slightly such that the unitarity constraints are obeyed. One can easily see that these changes are minimized if one describes the introduction of a nonzero value of V_{ub} by a slight rotation including a phase rotation about the s' axis right after the Cabibbo rotation. In this case the d' axis is lifted vertically above the d-s plane, and the value of θ remains unchanged. Thus we are led to the following parametrization:

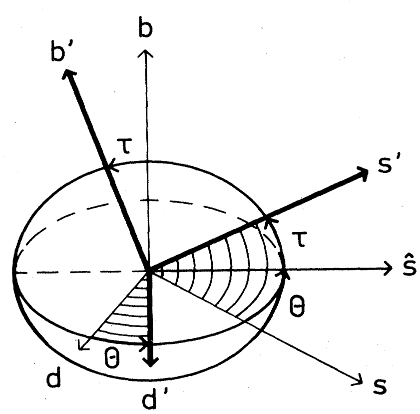

FIG. 1. The mixing of the quarks, observed in nature, seems to proceed in two consecutive steps: (1) The Cabibbo rotation in the d-s plane (angle $\theta \approx 13°$). (2) A rotation by an angle $\tau \approx 2°$–$3°$ in the $\hat{s}$-b plane. CP violation is caused by a very small rotation, tilting the d' quark slightly outside the d-s plane by an angle $\sigma \lesssim 0.2°$, not shown here.

$$V=\begin{bmatrix}1&0&0\\0&c_\tau&s_\tau\\0&-s_\tau&c_\tau\end{bmatrix}\begin{bmatrix}c_\sigma&0&s_\sigma e^{-i\delta}\\0&1&0\\-s_\sigma e^{+i\delta}&0&c_\sigma\end{bmatrix}\begin{bmatrix}c_\theta&s_\theta&0\\-s_\theta&c_\theta&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}c_\sigma c_\theta & c_\sigma s_\theta & s_\sigma e^{i\delta}\\ -c_\tau s_\theta-c_\theta s_\tau s_\sigma e^{i\delta} & c_\tau c_\theta-s_\theta s_\tau s_\sigma e^{i\delta} & c_\sigma s_\tau\\ s_\theta s_\tau-c_\tau c_\theta s_\sigma e^{i\delta} & -c_\tau s_\theta s_\sigma e^{i\delta}-c_\theta s_\tau & c_\sigma c_\tau\end{bmatrix}. \quad (6)$$

Here s_σ is the third rotation angle, which is responsible for a nonzero value of V_{ub}, and which must be very small. The experimental data require $s_\sigma < 0.004$, i.e., $\sigma < 0.23°$.

Because of the small values of σ and τ we can set $\cos\sigma=1$ and $\cos\tau=1$ (accuracy $\sim 0.1\%$), in which case the mixing matrix takes the form

$$V=\begin{bmatrix}c_\theta & s_\theta & s_\sigma e^{-i\delta}\\ -s_\theta-c_\theta s_\tau s_\sigma e^{i\delta} & c_\theta-s_\theta s_\tau s_\sigma e^{i\delta} & s_\tau\\ s_\theta s_\tau-c_\theta s_\sigma e^{i\delta} & -c_\theta s_\tau-s_\sigma s_\theta e^{i\delta} & 1\end{bmatrix}. \quad (7)$$

A parametrization of this kind has been discussed previously by Chau and Keung in the context of *CP* violation.[8] It is interesting that here the *CP*-violating phase factors $e^{i\delta}$ are all multiplied with factors of order 10^{-3}. Using the chain of arguments discussed above we see that the parametrization Eq. (6) emerges uniquely as the natural candidate for parametrizing the mixing matrix V. For example, the parametrization introduced by Kobayashi and Maskawa has, although mathematically equivalent to Eq. (6), serious drawbacks. It can be written as a product of three rotations:

$$V(\text{KM})=\begin{bmatrix}1&0&0\\0&c_2&s_2\\0&-s_2&c_2\end{bmatrix}\begin{bmatrix}c_1&s_1&0\\-s_1&c_1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&e^{i\delta}\end{bmatrix}\begin{bmatrix}1&0&0\\0&c_3&s_3\\0&-s_3&c_3\end{bmatrix}$$

$$=\begin{bmatrix}c_1 & s_1c_2 & s_1s_3\\ -s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta'} & c_1c_2s_3+s_2c_3e^{i\delta'}\\ s_1s_2 & +c_1s_2c_3+c_2s_3e^{i\delta'} & c_1s_2s_3+c_2c_3e^{i\delta'}\end{bmatrix}. \quad (8)$$

(Here the c_i, s_i stand for $\cos\theta_i$, $\sin\theta_i$. *CP* violation is described by the phase δ'.)

The relations between the KM angles $\theta_1,\theta_2,\theta_3$ and the angles θ,τ,σ in Eq. (6) are given by

$$c_1=c_\sigma c_\theta,\quad c_3s_1=c_\sigma s_\theta,\quad s_1s_3=s_\sigma,$$
$$c_\sigma s_\tau=(c_1{}^2c_2{}^2c_3{}^2+c_3{}^2s_2{}^2+2c_1c_2c_3s_2s_3\cos\delta')^{1/2}. \quad (9)$$

In the limit $V_{ub}=0$ one has, of course, $\sigma=0$ and $\theta_3=0$, $\theta=\theta_1$, $\tau=\theta_2$, and the representations (6) and (8) are identical. Otherwise the relations (9) involve the KM phase δ'. Approximately (within $\sim 1\%$) one finds

$$\theta\cong\theta_1,\quad \theta_3\theta_1\cong\sigma,$$
$$(\theta_3{}^2+\theta_2{}^2+2\theta_2\theta_3\cos\delta')^{1/2}\cong\tau. \quad (10)$$

The Kobayashi-Maskawa representation describes first a rotation by θ_3 about the axis defined by the mass eigenstate of the d quark, followed by the Cabibbo rotation about the new b' axis, and finally a rotation by θ_2 about the new d' axis.

As one can readily see, the special case $V_{ub}=0$ is realized by setting $\theta_3=0$, in which case the mixing matrix, after absorbing the phase $e^{i\delta}$, is identical to the one given in Eq. (5). As we have emphasized, the nonzero value of V_{ub} as well as *CP* violation arise, if the d'-quark axis is tilted slightly (by an angle of at most 0.2°) out of the d-s plane. Within the Kobayashi-Maskawa representation this is described by a rotation about the d axis by an angle θ_3, which must be less than 1.2° ($s_3 < 0.02$), followed by the Cabibbo rotation, in such a way that the d' axis is driven away from the d-s plane by an angle of order s_1s_3.

This procedure expresses the experimental facts, in particular, the surprisingly small value of V_{ub}, in a rather fussy way, while the parametrization (6) gives much more insight. The latter may, in fact, reflect directly, via the sequence of rotations discussed above, the underlying physical mechanism, which is responsible for the generation of the weak-interaction mixing. We expect that it will emerge naturally in a future theory of the quark mass matrix. Furthermore, the description of *CP* violation is much more natural in the representation (7). In the limit $V_{ub}\to 0$, i.e., $\sigma\to 0$, the *CP*-violating phase is automatically turned off, while in the KM representation (8) the phase δ' remains present and has to be absorbed subsequently by a suitable phase rotation of the quark field.

For this reason we suggest that the parametrization of Kobayashi and Maskawa should be abandoned, and that the matrix (6) should be used instead.

We conclude this note with two comments.

(a) A specific form of the quark mixing matrix has been introduced recently by Wolfenstein.[10] In this parametrization, which is assumed to be a perturbative expansion in a parameter λ, the Cabibbo angle θ is interpreted to be of $O(\lambda)$, the b-s mixing angle τ is taken to be of $O(\lambda^2)$, and the angle σ of $O(\lambda^3)$. Although numerically in agreement with the observed value of τ and with the experimental limit on σ, we doubt whether it is a useful way to parametrize the weak-interaction mixing. In all attempts to relate the mixing angles to the quark mass parameters, no perturbative expansion in $\theta\sim\lambda$ seems to emerge. Instead the small value of V_{ub} finds its explanation in the very small value of $(m_u/m_c)^{1/2}$ (Refs. 1 and 2). The fact that the angle τ is numerically close to $\theta^2 \sim \lambda^2$, may well be an accident. For this reason we doubt that Wolfenstein's parametrization is a useful way of describing the weak-interaction mixing. Of course, the situation will change, if arguments can be given that the relation $\tau \cong \theta^2$ is more than just a numerical coincidence.

(b) The phase parameter δ in Eq. (6) causes *CP* violation. In particular the amplitudes of the *CP*-violating processes in the case of strange- or charmed-particle decays are proportional to the factor $A^{CP} = \text{Im}(V_{us} V_{cd} V_{cs}^{\dagger} V_{ud}^{\dagger})$ (see e.g., Ref. 11). In the KM parametrization Eq. (8) one finds

$$A^{CP} = -\sin\delta' c_1 c_2 c_3 s_1{}^2 s_2 s_3 \ , \tag{11}$$

and in the parametrization Eq. (6)

$$F^{CP} = -\sin\delta c_\theta c_\tau c_\sigma{}^2 s_\theta s_\tau s_\sigma \ . \tag{12}$$

One finds

$$\frac{\sin\delta'}{\sin\delta} \cong \frac{s_\tau}{s_2} \ . \tag{13}$$

For arbitrary values of the mixing angles the ratio (13) can, of course, assume arbitrary values. However, in the limit $V_{ub} \to 0$ ($\theta_3 \to 0, \sigma \to 0$) the phases δ and δ' must be identical. Nevertheless, in reality δ and δ' may differ substantially.

It is interesting to note that in a very good approximation A^{CP} can be expressed in terms of the matrix elements of V as follows:

$$A^{CP} = -\sin\delta \,|\, V_{ud} V_{us} V_{ub} V_{cb} \,| \ . \tag{14}$$

(This holds within an accuracy of 1%.) Since the factor multiplying $\sin\delta$ is a measurable quantity and as such independent of phase conventions, it is apparent that A^{CP} is maximal if $\delta = \pm\pi/2$. Thus we define maximal *CP* violation as follows: *CP violation is maximal, if the phase* δ *of the complex rotation in the d'-b plane introduced in Eq. (6) is maximal.* Using the representation of the quark mixing matrix given in Eq. (6), our definition of maximal *CP* violation is very similar to the one given recently in Ref. 12, although our definition seems simpler and more transparent in its physical content.

It remains to be seen whether the *CP* violation observed in reality is indeed maximal in the sense described above. If this turns out to be the case, we suspect that both maximal *CP* violation and the specific structure of the quark-mixing pattern discussed here, especially the relation $|V_{ub}/V_{cb}| \ll 1$, are intimately related to each other.

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