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Internal states in the pion static model

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Methods for performing accurate computations of eigenvalues and eigenfunctions of the static-model Hamiltonian with weak and intermediate coupling are considered. The best results are obtained by using coherent-meson-pair states constructed from invariant-pair-free states with up to six p-wave π mesons in a single internal-mode state. The additional states needed for an accurate computation of excited-state energies are exhibited. The best ground-state wave functions are obtained by imposing constraints that are known to be true for the exact ground-state wave function.

I. INTRODUCTION

The static model (SM) of the nucleon-pions system, that is to say, the model system whose Hamiltonian is

$$H_{\rm SM}=\int \omega(k)a_\lambda^\dagger(\mathbf{k})a_\lambda(\mathbf{k})d\mathbf{k} - \frac{f}{m}\tau_\lambda \int \frac{\sigma\cdot\mathbf{k}\tilde{v}^\dagger(k)}{[16\pi^3\omega(k)]^{1/2}}[a_\lambda^\dagger(\mathbf{k})+a_\lambda(\mathbf{k})]d\mathbf{k} \tag{1.1}$$

has been studied for quite a long time;[1] it describes the interaction of p-wave pions with a distributed source with spherically symmetric density $v(r)$. The Fourier transform of $v(r)$ is $\tilde{v}(k)$; both v and $\tilde{v}$ are real, and the summation convention is used for the isospin index λ. In the past this model has been used to interpret low- and medium-energy scattering of p-wave pions by nucleons, as well as static properties of the nucleon dressed by its pion cloud. More recently, with the general acceptance of the picture of the nucleon as consisting of a quark-gluon core region surrounded by a region dominated by pions, the static model has come to be regarded as a possible way of learning about the current form factor $\tilde{v}$ in (1.1) that describes the interaction of the core with the pion field; it is expected that $\tilde{v}(k)$ will provide information about the nature of the core itself.[2] The usefulness of the information obtained will depend critically on the accuracy with which $\tilde{v}(k)$ can be determined from the comparison of theoretical quantities with experimental data; this paper is devoted to the consideration of methods for computing properties of the static model accurately.

Recent work[3] has shown that the most direct way of treating a static-model Hamiltonian is to decompose the field creation and annihilation operators $a_\lambda^\dagger(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ into internal and external parts. In the pion static model, the internal part of each operator involves a single p-wave pion mode with three components. When the choice of the separation into external and internal fields is made so as to minimize the effects of the external field in the ground state of the Hamiltonian, the appropriate orthonormal mode function components have been shown to be

$$\begin{aligned}\phi_i(\mathbf{k})&=\frac{f}{m}\frac{k_i\tilde{v}(k)}{G(16\pi^3\omega^3)^{1/2}}=\phi_i^*(\mathbf{k})\ ,\\ G^2&=\frac{f^2}{4\pi}I_3\ ,\\ I_3&=\frac{1}{3\pi}\int_0^\infty \frac{k^4\tilde{v}^2(k)}{\omega^3(k)}dk\ .\end{aligned} \tag{1.2}$$

Note that I_3 and the internal modes ϕ_i are independent of the coupling strength f and depend only on the shape of the source density $v(r)$. In terms of these modes, the decomposition of the field operator is

$$a_\lambda(\mathbf{k})=A_{\lambda i}\phi_i(\mathbf{k})+a_\lambda^{\rm ex}(\mathbf{k}) \tag{1.3}$$

(summation convention now for the p-wave index i as well), where the nonzero commutation relations of the creation and annihilation operators $A_{\lambda i}^\dagger$ and $A_{\lambda i}$ are

$$[A_{\lambda i},A_{\mu j}^\dagger]=\delta_{\lambda\mu}\delta_{ij}\ , \tag{1.4}$$

and the external field is required to be orthogonal to the internal modes,

$$\int \phi_i^*(\mathbf{k})a_\lambda^{\rm ex}(\mathbf{k})d\mathbf{k}=0\ . \tag{1.5}$$

Under this decomposition of the field operators, the Hamiltonian splits into internal, external, and interaction parts:

$$\begin{aligned}H_{\rm SM}&=H^{\rm in}+H^{\rm ex}+H^{\rm Yu}+H^{\rm Yu\dagger}\ ,\\ H^{\rm in}&=W[A^\dagger\cdot A-G\rho\cdot(A^\dagger+A)]=Wh^{\rm in}\ ,\\ H^{\rm ex}&=\int \omega(k)a_\lambda^{\rm ex\dagger}(\mathbf{k})a_\lambda^{\rm ex}(\mathbf{k})d\mathbf{k}\ ,\\ H^{\rm Yu}&=(A^\dagger-G\rho)_{\lambda i}\int \chi_i(\mathbf{k})a_\lambda^{\rm ex}(\mathbf{k})d\mathbf{k}\ ,\end{aligned} \tag{1.6}$$

where ρ and the dot product are defined by

$$\begin{aligned}\rho_{\lambda i}&=\tau_\lambda\sigma_i\ ,\\ B\cdot C&=B_{\lambda i}C_{\lambda i}\ .\end{aligned} \tag{1.7}$$

The internal Hamiltonian $H^{\rm in}$ contains only the internal-mode operators; it describes the "internal system" that consists of the source plus meson field in the internal modes only. The operator H^{in} has a discrete spectrum that is bounded below and is the result of the interaction of the internal modes with the source. The relative spacing within this spectrum is determined by the intrinsic parameter G, which is the only parameter in h^{in} and is given by (1.2); it is evident that G is a normalized coupling strength. The scale of the spectrum of H^{in} is determined by the parameter

$$W=\int \omega(k)\phi_i{}^2(\mathbf{k})d\mathbf{k}=\langle\omega\rangle \ , \tag{1.8}$$

which, like ϕ_i, is independent of the coupling constant f. It is sometimes convenient to write H^{in} in the form

$$H^{\text{in}}=G^2WH_A=VH_A \ ,$$
$$H_A=\frac{1}{G^2}[A^\dagger\cdot A-G\rho\cdot(A^\dagger+A)]=\frac{1}{G^2}h^{\text{in}} \ , \tag{1.9}$$

since the ground-state energy of H_A is less dependent on G than the ground-state energy of H^{in}.

The term H^{ex} is just the energy of the external meson field without interaction; since the internal field is square integrable, the external part of the field is the only part that can describe pions that have asymptotic plane-wave components corresponding to scattering states. Actual scattering of the external pion field is generated by the residual Yukawa interaction, contained in H^{Yu} and its adjoint, of the external meson field with the internal system. Transitions between internal states that accompany the emission of an external meson have a strength that is proportional to the corresponding matrix element of the internal operator $A-G\rho$. The momentum dependence of the external-meson emission is in the functions $\chi_i(\mathbf{k})$, the part of $\omega(k)\phi_i(\mathbf{k})$ that is orthogonal to $\phi_i(\mathbf{k})$,

$$\chi_i(\mathbf{k})=[\omega(k)-W]\phi_i(\mathbf{k}) \ ; \tag{1.10}$$

like the functions ϕ_i, these functions are independent of the coupling strength f.

The splitting of the Hamiltonian H_{SM} in (1.6) implies a two-step approach to determining its eigenvectors. The first step is the determination of the discrete states of H^{in}. These states lie within the internal subspace, generated by the operators $A^\dagger$ and ρ acting on the bare source states, of the full Hilbert space. The second step is the calculation of the effects of adjoining the rest of the Hilbert space, generated by the operators $a^{\text{ex}\dagger}$ acting on the internal subspace. That this is a reasonable procedure was demonstrated in a previous study,[4] where the ground-state energy in the single-mode approximation was compared with an elaborate computation[5] that involved as many as six pion-field modes. Of course, scattering of pions first appears at the second stage of the procedure. The two-step approach means that the coupling-constant renormalization constant z is to be considered the product of z_{in}, the renormalization constant for H^{in} alone, and z_{ex}, the additional renormalization due to interaction with the external modes. The results of Ref. 4 indicate that $z_{\text{ex}}\simeq 1$, and that z_{in} is responsible for nearly all of the coupling-constant renormalization.

A previous paper[3] has given the formalism for computing the scattering once the eigenvalues and eigenstates of H^{in} are all known. The present work is devoted to the study of H^{in} itself. As noted above, the spectrum of H^{in} is necessary for computing meson scattering; it is also required for the determination of the potential between two static sources of meson field.[6] The techniques used in treating H^{in} will also be useful in considering other static-source problems, such as a system of gluons interacting with a static source consisting of quarks. The spectrum of H^{in} was also treated in Ref. 4; reasonable, but not completely satisfactory accuracy was obtained with the techniques available at that time, the main limitation being that meson states were available only for up to four mesons. A recent paper[7] has classified states with up to six p-wave π mesons. The present work uses the results of Ref. 7 to obtain results for weak and intermediate coupling in the static model that are accurate enough to be used in computations of the source-source pion potential, for example. For strong coupling, the procedures given first by Pauli and Dancoff[8] and developed further by Parmentola[9] can be used.

In addition to energy eigenvalues, the ground-state wave function is important for determining the coupling-constant renormalization. The present work shows that the use of constraints can help to provide a better state vector than is given by simply minimizing the expectation value of the Hamiltonian.

II. PRELIMINARIES FOR h^{in}

It is easy to get a lower bound on the spectrum of h^{in} by rewriting it in the form

$$h^{\text{in}}=(A-G\rho)^\dagger\cdot(A-G\rho)-9G^2 \ , \tag{2.1}$$

where $\rho\cdot\rho=9$ has been used. Thus $-9G^2$ is a lower bound; it is also the correct ground-state energy to order G^2.

From the commutation relation

$$[A,h^{\text{in}}]=A-G\rho \tag{2.2}$$

it follows that the matrix elements of $A-G\rho$ are zero within any eigenvector multiplet of h^{in}:

$$h^{\text{in}}\,|\,i,\mu\rangle=\epsilon_i\,|\,i,\mu\rangle \implies \langle i,\mu\,|\,A-G\rho\,|\,i,\nu\rangle=0 \ , \tag{2.3}$$

where μ and ν run over the indices of the $(2T+1)(2J+1)$ substates of the (T,J) multiplet i. Note that $A-G\rho$ is the internal operator associated with emission of an external meson, so that (2.3) also expresses the fact that no eigenstate of H^{in} can emit or absorb an external meson without simultaneously making a transition to an eigenstate of H^{in} belonging to a different eigenvalue of H^{in}. This is the criterion that was used to determine the internal-meson mode functions in the first place; it ensures that the meson field around any internal state is correct through first order in the external meson field.

The coupling-constant renormalization constant z_{in}, which depends on G, is defined in terms of the matrix elements of the operator ρ that involve only the ground state $|\,g\rangle$ of h^{in}. Let $|\,g_0\rangle$ be the bare source state; then z_{in} is given by

$$\langle g \mid \rho \mid g \rangle = z_{\rm in} \langle g_0 \mid \rho \mid g_0 \rangle = z_{\rm in} \rho \ , \tag{2.4}$$

and the renormalized coupling constants G_R and f_R are defined by

$$G_R = z_{\rm in} G \ , \quad f_R = z_{\rm in} z_{\rm ex} f \simeq z_{\rm in} f \ , \tag{2.5}$$

where the fact that $z_{\rm ex}$ is close to unity has been used. Since it follows from (1.2) that

$$G_R{}^2 = I_3 z_{\rm ex}{}^{-2} \frac{f_R{}^2}{4\pi} \ , \tag{2.6}$$

and $f_R{}^2/4\pi$ is known to be 0.08, the coupling constant G_R is reasonably well determined once the source function v is assumed to have a particular shape. For this reason it is often useful to give the various quantities derived from $h^{\rm in}$ as functions of G_R rather than as functions of G.

III. METHODOLOGY FOR $h^{\rm in}$

The term N state will be used to denote a state of the source plus internal-mode pions that has isospin and spin each $\frac{1}{2}$; a generic N state consists of components each of which has a definite number n of internal-mode pions and definite isospin T and angular momentum L for the pion part of the N state. Since[10] there are no internal-mode pion states with (TL) equal to (01) or (10), the allowed pion components of N states have (TL) equal to (00) and (11). The internal-mode operator $A \cdot A$ is invariant under isospin and space rotations; it destroys an invariant pair of internal-mode pions. An invariant-pair-free (IPF) state of the internal-mode pions is one that satisfies

$$A \cdot A \mid \ \rangle = 0 \ . \tag{3.1}$$

Reference 7 lists all the IPF states with up to six pions for a single p-wave pion mode and shows how to construct states that contain invariant pairs from the IPF states.

From the tables in Ref. 7 it can be seen that there are 12 IPF N states and 23 N states with up to and including six internal-mode pions. The parentage tables in the same reference make possible the construction and diagonalization of the 23×23 matrix of $h^{\rm in}$ in the subspace of these 23 explicit-pair (EP) N states; the lowest eigenvalue, which will be denoted EEP, is an upper bound and a good approximation to the lowest eigenvalue of $h^{\rm in}$.

An alternate procedure uses coherent-meson-pair (CMP) states.[4] For each n-pion IPF state $\mid n,\alpha \rangle$, there is a one-parameter family of CMP states $\mid n,\alpha,y \rangle$ that satisfies

$$A \cdot A \mid n,\alpha,y \rangle = y \mid n,\alpha,y \rangle \ ; \tag{3.2}$$

these states are given explicitly by

$$\mid n,\alpha,y \rangle = g_{9+2n}(yA^\dagger \cdot A^\dagger) \mid n,\alpha \rangle \ , \quad g_\nu(x) \equiv \sum_{m=0}^{\infty} \frac{(\nu-2)!!}{2^m m!(\nu+2m-2)!!} x^\nu \ . \tag{3.3}$$

Now the 12×12 matrix of the Hamiltonian can be com-

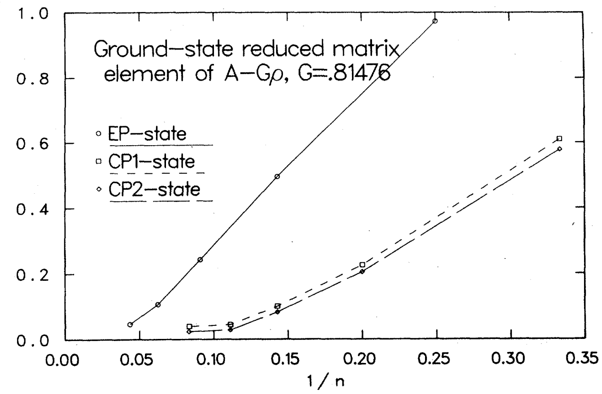

FIG. 2. Ground-state reduced matrix element of $A - G\rho$ for the state computed with explicit pairs (EP state), with one pair-coherence parameter (CP1 state), and with two pair-coherence parameters (CP2 state) when the normalized coupling constant G is 0.814 76. The value of the reduced matrix element is plotted against the number of states n used in the computation.

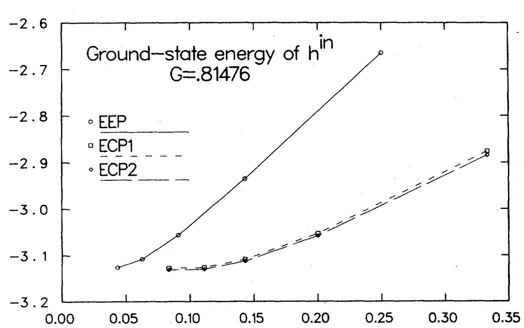

FIG. 1. Ground-state energy computed with explicit pairs (EEP), with one pair-coherence parameter (ECP1), and with two pair-coherence parameters (ECP2) when the normalized coupling constant G is 0.814 76. The energy is plotted against the number of states n used in the computation.

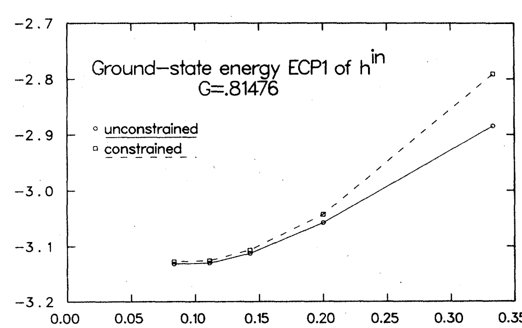

FIG. 3. Comparison of ground-state energies computed with and without enforcing the constraint that the expectation value of $A - G\rho$ be zero. In both cases one pair-coherence parameter was used. The energy is plotted against the number of states n used in the computation.

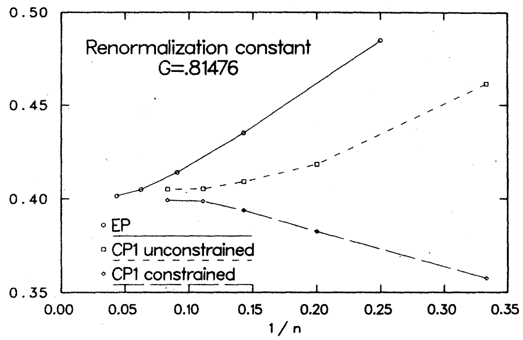

FIG. 4. Comparison of renormalization constants computed with and without enforcing the constraint that the expectation value of $A-G\rho$ be zero. In both cases two pair-coherence parameters were used. The renormalization constant is plotted against the number of states n used in the computation.

puted for the CMP states, with each state having its own pair-coherence parameter y, and diagonalized to give the eigenvalue spectrum. Finally, the coherence parameters can be chosen so as to minimize the lowest eigenvalue of this matrix. It turns out that it is sufficient to use only one, the same for all the IPF states, or two, one for the (00) states and one for the (11) states, coherence parameters; the values of the lowest eigenvalues at the respective minima will be denoted ECP1 and ECP2.

Figure 1 shows EEP, ECP1, and ECP2 plotted against $1/n$ for $G=0.81476$, which is the value of G that was used in Refs. 4 and 5; n is the number of states used in each calculation:

Maximum number of mesons	0	1	2	3	4	5	6
n for IPF states	1	2	3	5	7	9	12
n for total states	1	2	4	7	11	16	23

(3.4)

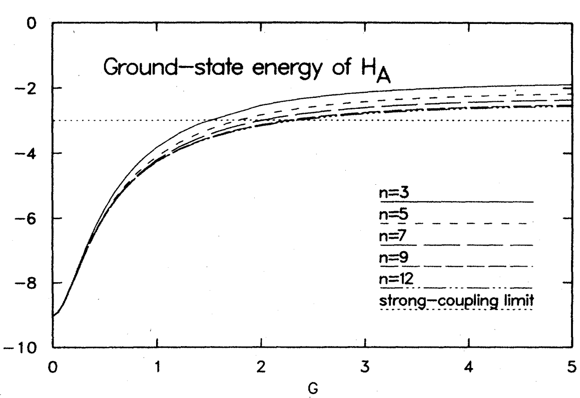

FIG. 5. Ground state of H^{in} computed with two coherence parameters without constraint on the matrix element of $A-G\rho$. The number of coherent-pair states used in each computation is n. The line at -3 is the strong-coupling-limit value of the ground-state energy.

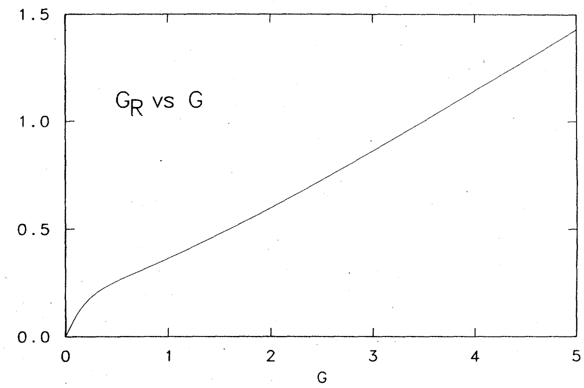

FIG. 6. The renormalized coupling G_R as a function of the unrenormalized coupling G.

From Fig. 1 it is clear that the coherent-pair calculation produces better results with fewer states than the calculation with explicit pairs.

A check on the accuracy of the state vectors can be obtained from the matrix elements of $A-G\rho$, which, according to (2.3), are zero within any eigenvector multiplet of H^{in}. All of the ground-N-state matrix elements of $A-G\rho$ can be expressed in terms of a single reduced matrix element by dividing each one by the corresponding bare source matrix elements of the source operator $\rho/3$. Figure 2 shows a plot of this reduced matrix element for the ground state for the same three cases that were plotted in Fig. 1. Again the superiority of the coherent-pair method is evident.

Since the matrix elements of $A-G\rho$ are crucial for the scattering calculation, in particular, the vanishing of matrix elements of $A-G\rho$ within an exact eigenvector multiplet, a computational technique that ensures the vanishing of the matrix elements of $A-G\rho$ within the ground-N-state multiplet was also investigated. The basis for this technique is the observation that (2.3) means that the ground state of H^{in} is the state that minimizes the expectation value of H^{in} *subject to the constraint that the expectation value of $A-G\rho$ vanish.* The minimization with the constraint is performed by using the standard Lagrange

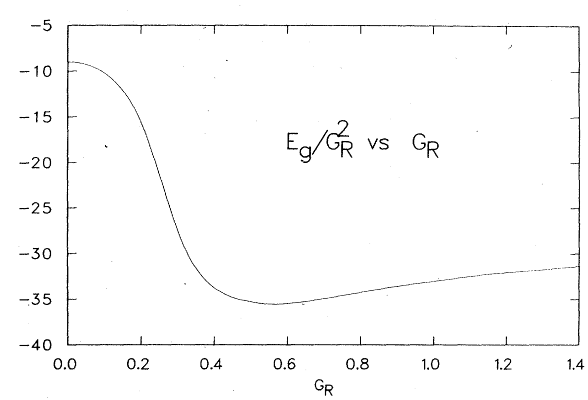

FIG. 7. Plot of $E_g/G_R{}^2$, where E_g is the ground-state energy of h^{in}, and G_R is the renormalized coupling.

multiplier method, which requires many evaluations of the lowest eigenvalue of corresponding 12×12 or 23×23 matrices and therefore was only applied to the coherent-pair (12×12) subspace. Figure 3 shows the unconstrained energy ECP1 and its corresponding constrained value; the constraint does not drastically affect the computed energy while it, of course, significantly improves the computed expectation value of $A-G\rho$. The indicated improvement in the state vector that is obtained by using the constraint is also apparent in Fig. 4, which shows the renormalization constant determined by the various methods. The value obtained with the constraint differs from the unconstrained one; the explicit-pair calculation favors the constrained value of the renormalization constant.

The result of these various comparisons is that (1) the best value of the energy is the one obtained with two coherence parameters in an unconstrained computation, and (2) the best values of parameters that depend on the ground-state wave function, such as the renormalization constant or the expectation value of the meson number operator, are obtained by using two coherence parameters in a calculation in which the matrix elements of $A-G\rho$ are constrained to be zero within the ground-state multiplet.

In order to indicate how the calculations converge with the number of pions, Fig. 5 shows the ground-state energy of H_A plotted against G for various values of n, the maximum number of π mesons in the state-vector components; the energies were computed with two coherent-pair parameters and without constraining the wave function.

Excited states of the $h^{\rm in}$ do contain invariant pairs, so that a computation that uses just the CMP states based on IPF states does not give good convergence for the excitation energy of the first excited state of $h^{\rm in}$. Various ways of adding states with invariant pairs can be imagined; a simple and effective method for adding such states is based on the observation that the "derivative" CMP states

$$|\,n,\alpha,y,d\,\rangle \equiv \frac{1}{c_{n\alpha}(y)}\,\frac{d}{dy}\,|\,n,\alpha,y\,\rangle$$

are easily seen to be orthogonal to all the states $|\,m,\beta,y\,\rangle$ and are normalized if the $c_{n\alpha}(y)$ are chosen suitably. Moreover, the derivative states $|\,n,\alpha,y,d\,\rangle$ are not IPF states, so that they are a simple set of states that can be added to the variational subspace in order to introduce the possibility of invariant-pair excitation. With the techniques described in Ref. 4, it is straightforward to work out the matrix elements that involve these derivative states. When the derivative CMP states are included in the computation, there are two CMP states for each IPF state, and the sizes of all matrices are doubled.

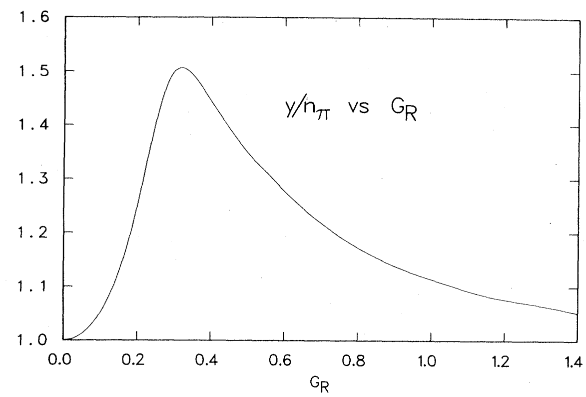

FIG. 9. The ratio of the pair-coherence parameter y to the expectation value of the pion number operator in the ground state of $H^{\rm in}$, plotted against the renormalized coupling G_R.

IV. RESULTS FOR $h^{\rm in}$

The dependence of the renormalized coupling constant G_R on the unrenormalized one G is shown in Fig. 6, while Fig. 7 displays $E_g/G_R{}^2$, where E_g is the ground-state energy of $h^{\rm in}$. The G_R dependence of the expectation number of the pion number operator in the ground state of $H^{\rm in}$ is shown in Fig. 8, and Fig. 9 shows how the ratio of the pair-coherence parameter y to the pion number expectation varies with G_R. The derivative CMP states are used in the computation of the excitation energy of the first excited N state; Fig. 10 shows the results.

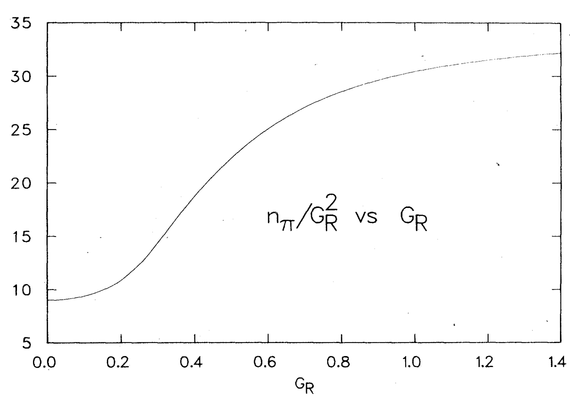

FIG. 8. The ratio of the expectation value n_π of the pion number operator in the ground state of $H^{\rm in}$ to $G_R{}^2$ plotted against the renormalized coupling G_R.

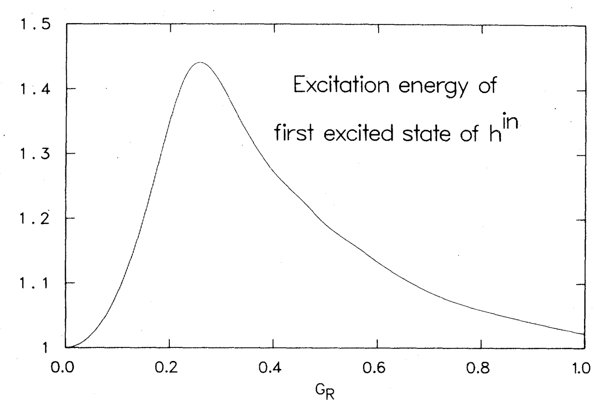

FIG. 10. The excitation energy of the first excited N state of $h^{\rm in}$.

V. SUMMARY

Various techniques for computing approximations to the eigenvalues and eigenfunctions of the internal Hamiltonian h^{in} of the pion static model in weak and intermediate coupling have been considered. The best methods utilize coherent-meson-pair states and, when needed, also derivative CMP states. The ground-state energy can be computed accurately by an unconstrained matrix diagonalization, while the best ground-state wave function is obtained in a matrix eigenvalue minimization over vectors constrained to satisfy a condition known to be fulfilled by the exact ground-state wave function. The computation of excitation energies requires the enlargement of the matrix subspace; an appropriate set of derivative CMP states has been shown to give adequate results.

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