Bifurcation of the type-II solutions of the Yang-Mills equations with static sources

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We show that the analytic type-II solutions of the Yang-Mills field equations obtained previously do possess a bifurcation point.

Recently, there has been some discussion on the stability and bifurcation property of the solutions of the Yang-Mills (YM) field equations with external static sources.[1-8] For the external source specified in the Abelian-gauge frame with vanishing Kronecker index, the bifurcation picture has been clarified.[8] Thus, when the bifurcation parameter e is equal to zero,[8] only the spherically symmetric Abelian Coulomb solution exists and is stable. As soon as $e>0$, the external source density picks up an axially symmetric term and bifurcation takes place. Three branches of solutions emanate from the bifurcation point; two of them, the non-Abelian magnetic dipole solutions, are stable and degenerate in their energy value, while the remaining unstable branch is the Abelian Coulomb solution continued from $e<0$. The bifurcation picture is pitchforklike and the onset of bifurcation is associated with a decrease in symmetry of the gauge-field configuration from the spherical symmetry ($e=0$) to the axial symmetry ($e>0$).

When the external static source is specified in the radial frame with Kronecker index equal to one, it was found that there exists a class of solution, called the type-II solution,[3] which requires a minimum nonzero source strength for its sustenance. Unlike the Abelian-Coulomb-solution case,[8] these solutions come in two branches and are spherically symmetric. Although all the numerical type-II solutions[3,4] indicate bifurcation, the analytic expressions reported in Ref. 5 did not seem to have a bifurcation point. The analytic type-II solutions given by Ref. 6 do have a bifurcation point, but these solutions necessitate the presence of a nonvanishing external source current, $j_i^a(x)\neq 0$. Furthermore, in their analysis,[6] both the energy and the total charge are gauge dependent. In this Brief Report we wish to reconsider the analytic solution of Ref. 5 and show that it can indeed possess a bifurcation point. We shall follow the notations of Ref. 5.

The SU(2) YM equations in the presence of an external static source are

$$(D_\mu F^{\mu\nu})_a = j_a^\nu = \delta_0^\nu \rho_a \ , \tag{1a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\mu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c \ . \tag{1b}$$

Employing the ansatz[3]

$$A_0^a = n^a f(x)/(gr), \quad n^a = x^a/r \ , \tag{2a}$$

$$A_l^a = \epsilon_{laj} n^j [a(x)-1]/(gr) \ , \tag{2b}$$

$$\rho_a(\mathbf{r}) = n_a q(x)/(g r_0^3) \ , \tag{2c}$$

where g is the gauge-field coupling constant, r_0 is a parameter of the dimension of length, and $x=r/r_0$, the YM equations reduce to the following coupled nonlinear equations:

$$-f'' + 2a^2 f/x^2 = xq \ , \tag{3a}$$

$$-a'' + (a^2-1-f^2)a/x^2 = 0 \ . \tag{3b}$$

The prime means differentiation with respect to x. The total energy ξ and the gauge-invariant total charge Q are, respectively, given by

$$\xi = \frac{4\pi}{g^2 r_0}\int_0^\infty dx \left[(a')^2 + \frac{1}{2x^2}(a^2-1)^2 + \tfrac{1}{2}(f')^2 + \frac{1}{x^2} f^2 a^2 \right] , \tag{4}$$

$$Q = \int d^3r [\rho_a(\mathbf{r})\rho_a(\mathbf{r})]^{1/2} = \frac{4\pi}{g}\int_0^\infty dx x^2 |q(x)| \ . \tag{5}$$

The analytic type-II solutions obtained in Ref. 5 are

$$a(x) = \tanh U \ , \tag{6a}$$

$$f(x) = \left[x^2 \left(2U'^2 - \frac{U''}{\tanh U} \right) - 1 \right]^{1/2} \operatorname{sech} U \ , \tag{6b}$$

where

$$U(x) = \sum_{l=1}^{N} b_l (x^{-l} - x^{l+1}), \quad N = 1,2,3,\ \ldots \ . \tag{6c}$$

For $N=1$, the solutions do not seem to indicate a bifurcation point. However, for $N=2$, we find after some tedious search that the analytic solutions (6) do exhibit a bifurcation point: the total energy ξ and the total external charge Q reach their respective minimum value when $b_1=0.630\,00$ $b_2=0.000\,07$. The solutions $a(x)$ and $f(x)$ with b_1 and b_2 near the bifurcation value and their corresponding external charge-density distributions are as shown in Figs. 1(a), 1(b), and 1(c). In Fig. 2, we display the change of the total energy with the total external charge as the parameters b_1 and b_2 are varied. At the bifurcation point, $\xi(g^2 r_0/4\pi) = 11.940\,057$ and $Q(g/4\pi) = 18.271\,114$. As a matter of interest we plot ξ vs Q in Fig. 3 for a fixed $b_1=1.200$ but varying b_2; there is no bifurcation for this set of b_1 and b_2 values. Note that the crossover point corresponds to two different b_2 values. In Table I we list the values of b_1 and b_2 for which ξ and Q assume their respective minimum values, where we have defined $\beta = b_1/b_2$. We also provide

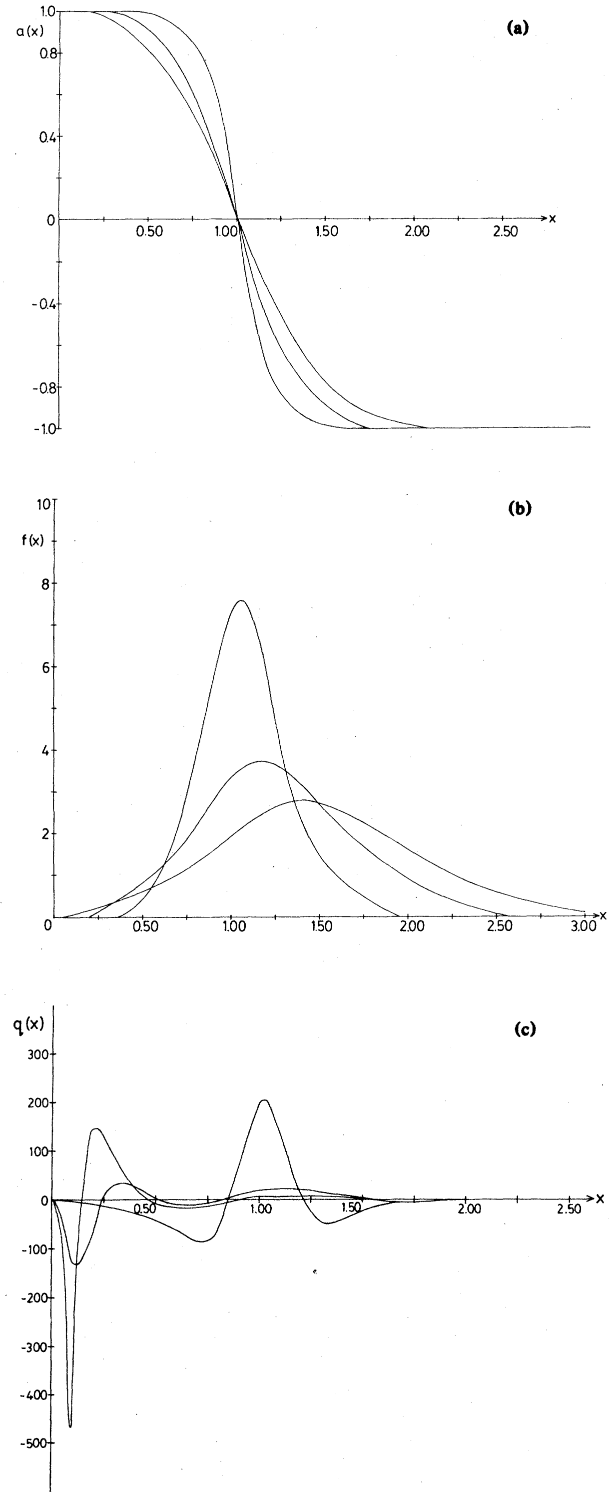

FIG. 1. (a) The function $a(x)$ with $b_1=\beta b_2$, $\beta=9000$. Starting from the curve with the lowest value of $a'(1)$, these correspond to $b_2=0.000\,07$, $0.000\,10$, and $0.000\,20$. (b) Profiles of the function $f(x)$ with $b_1=\beta b_2$, $\beta=9000$. Starting from the curve with the lowest peak value, these correspond to $b_2=0.000\,07$, $0.000\,10$, and $0.000\,20$. (c) Profiles of the external source density $q(x)$ with $b_1=\beta b_2$, $\beta=9000$. Starting from the curve with the lowest negative value, these correspond to $b_2=0.000\,07$, $0.000\,10$, and $0.000\,20$.

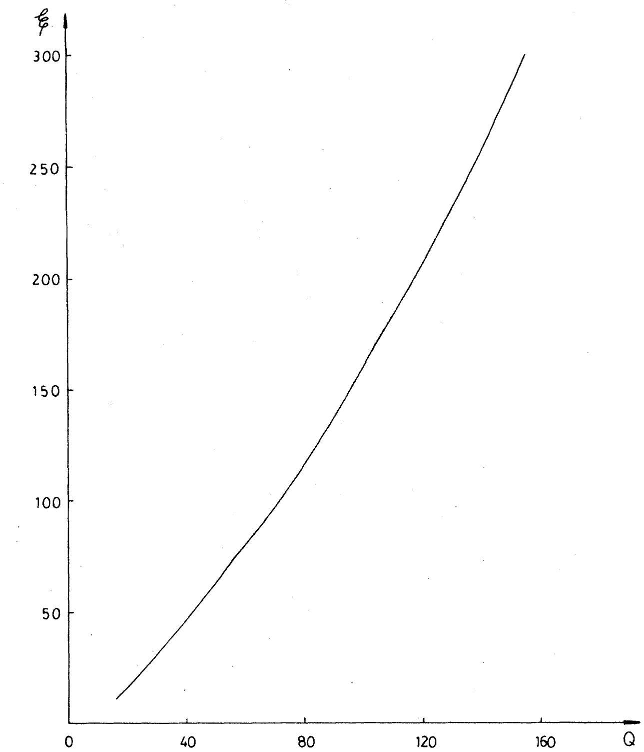

FIG. 2. Variation of the total energy ξ vs the total external charge Q for $b_1=\beta b_2$, $\beta=9000$.

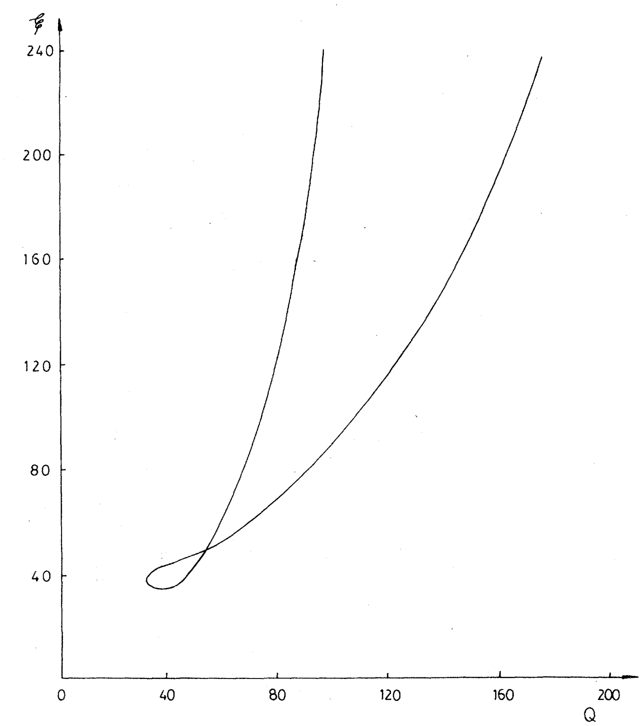

FIG. 3. Variation of the total energy ξ vs the total external charge Q for a fixed $b_1=1.200$.

TABLE I. Values of b_2 and $b_1=\beta b_2$ for which the total energy ξ and the total charge Q assume their respective minimum values.

Value of β	Values of b_2 for which ξ is minimum	Values of b_2 for which Q is minimum	Values of b_2 below which $f(x)$ is imaginary
35	0.016 00	0.022 00	0.016 00
70	0.008 00	0.010 00	0.008 00
110	0.005 00	0.006 00	0.005 00
250	0.002 30	0.002 70	0.002 10
500	0.001 10	0.001 30	0.001 10
800	0.000 70	0.000 80	0.000 70
1200	0.000 47	0.000 49	0.000 45
3000	0.000 19	0.000 21	0.000 19
5000	0.000 11	0.000 13	0.000 11
9000	0.000 07	0.000 07	0.000 07

the values of b_1 and b_2 for which ξ and Q assume their respective minimum values, where we have defined $\beta=b_1/b_2$. We also provide the values of b_1 and b_2 below which the function $f(x)$ becomes imaginary, that is, below which the solutions are not acceptable. Note that in the search for a bifurcation point, we vary the parameter b_2 for each fixed value of β, rather than vary b_2 for each fixed value of b_1. In other words, we vary b_2 and b_1 simultaneously such that their ratio is fixed each time.

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