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Quantum relativistic oscillator. III. Contraction between the algebras of SO(3,2) and the three-dimensional harmonic oscillator

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This paper concerns a relativistic model of a particle with internal structure: the quantum relativistic oscillator. We distinguish the nonrelativistic limit of the overall motion from the nonrelativistic limit of the internal structure. The latter limit is given in terms of a contraction between the algebra of SO(3,2) and that of the three-dimensional harmonic oscillator. We analyze this contraction process in detail and show that the quantum relativistic oscillator is the appropriate relativistic generalization of harmonic internal motion.

I. INTRODUCTION AND SUMMARY

The relativistic quantum mechanics of extended objects is of considerable current importance, both theoretically and experimentally, primarily for determining and interpreting possible models for the spectrum of hadrons. The relativistic rotator (rigid string or rigid stick),[1] the symmetric top,[2] the relativistic oscillator,[3] and the relativistic string[4] are examples that have received more or less extensive treatment. A unification of these models, at least in part, can be achieved by the use of the spectrum-generating-group approach,[5] in particular by the relativistic version using the group SO(3,2),[6] which is itself an extension of the structure of the first relativistic elementary-particle equation, the Dirac equation.

The present paper is concerned with the *quantum relativistic oscillator,* a structure which may be defined in two very different ways. One procedure is to give the Hamiltonian along with the Poincaré generators and the generators of the SO(3,2) spectrum-generating group [see Eqs. (4.1)—(4.5), below]. The quantum relativistic oscillator is then obtained from a special class of SO(3,2) irreducible representations (irreps) [see Eqs. (3.11)—(3.12b), below]. Alternatively, the quantum relativistic oscillator may be defined as a restriction of the quantized relativistic string to its lowest two modes: $n=0$ (center-of-mass translation) and $n=1$ (lowest vibrational mode).

It is by no means obvious that these two definitions of the quantum relativistic oscillator are in any way related, nor is it obvious that these definitions achieve the desired relativistic analog to the three-dimensional nonrelativistic oscillator. We shall demonstrate that there exists a general class of irreps of SO(3,2) that indeed fulfills our requirements.

In considering the nonrelativistic limit of these models one must keep in mind that, roughly speaking, two limits of motion are involved. The first kind of motion is an overall motion of the center of mass. The second kind is internal, setting up the mass spectrum of the particles. There is an analogy with the Dirac equation where the internal motion corresponds to the *Zitterbewegung.*

The nonrelativistic limit in the sense of limiting oneself to particle states which have a slow (relative to the speed of light) motion of the center of mass is easily obtained. These particles have the same mass spectrum as the particles of the relativistic theory. This is discussed in Sec. II.

The nonrelativistic limit of the internal motion is the main topic of the present paper. The relativistic SO(3,2) models considered, unlike the Dirac equation, allow an interesting nonrelativistic limit for the internal motion. In this limit one obtains the three-dimensional nonrelativistic oscillator with its spectrum for the internal motion. To be precise, in the rest frame of the particle we carry out an "Inönü-Wigner" contraction[7] for the SO(3,2) irreps. That it is, in fact, sufficient to do just this in the rest frame will be clear from the discussion in Sec. II.

Section III gives a brief résumé of the SO(3,2) irreps used in the contraction process. Section IV contains the details of the contraction process by which we obtain the nonrelativistic limit of the internal motion. We shall in particular carry out in detail the contraction process by which this class of irreps goes over into the nonrelativistic three-dimensional harmonic oscillator, including the oscillator Hamiltonian for the internal motion. The states belonging to an irreducible representation of the spectrum-generating group SO(3,2) are uniquely labeled in the rest frame by the representations of the $SO(3)\times SO(2)$ subgroup generated by $\mathbf{S}$ (the spin in the rest frame) and the operator Γ_0 (see Sec. III). These two operators determine the spin and a new "vibrational" quantum number, respectively, of the set of Poincaré irreps belonging to the given SO(3,2) irrep. In the contraction limit we find that the spin is related to the orbital motion of the (three-dimensional) nonrelativistic oscillator, while the label sup-

plied by Γ_0 contracts into the principal quantum number of the nonrelativistic oscillator. For a somewhat different approach, using a similar contraction, but not distinguishing between internal and overall motion, see Ref. 8. The final section, Sec. V, is devoted to conclusions.

II. NONRELATIVISTIC LIMIT OF THE OVERALL MOTION OF THE MODEL

The nonrelativistic limit of a relativistic point particle with rest mass m is, in physics, defined by taking the relativistic theory and considering only states which move slowly: $\mathbf{p}^2 \ll m^2$. For such states the Newtonian or Schrödinger theory is a good approximation, provided one uses as the Newtonian mass the rest mass of the relativistic particle. Formally one may let $c \to \infty$, or $m \to \infty$, but this is not really what is the essence of the limit, c is fixed (taken to be 1 in this paper) and Newtonian ballistics works well using as mass the rest mass of the projectiles.

When taking the nonrelativistic limit of the Dirac equation there is a complication due to the *Zitterbewegung* [this *Zitterbewegung* is analogous to the internal motion of the SO(3,2) model of this paper]. The Dirac equation may be cast in a manifestly covariant form as a constraint on a four-component function $\psi(x)$ of a four-vector x

$$(p^\nu \gamma_\nu + m)\psi(x) = 0 \ . \tag{2.1}$$

The nonrelativistic limit of this equation may be discussed by studying solutions to (2.1),[9] or more conveniently, by introducing "Newton-Wigner" coordinates[10] via a "Foldy-Wouthuysen" transformation.[11] These coordinates correspond to an average position, averaged over the *Zitterbewegung*. In these coordinates the Dirac equation takes on a form which is equivalent to (2.1), but not as manifestly covariant. The wave function in these coordinates involves the average position $\mathbf{X}$ and t: $\phi(\mathbf{X},t)$, the generators for Poincaré transformations, which for (2.1) are the familiar P_μ, $P_\mu \times X_\nu + \frac{1}{4}\{\gamma_\mu,\gamma_\nu\}$, are[12]

$$\begin{aligned}
&\mathbf{P}\phi(\mathbf{X},t) = -i\frac{\partial}{\partial \mathbf{X}}\phi(\mathbf{X},t) \ , \\
&P^0\phi(\mathbf{X},t) = \rho_3(\mathbf{P}^2+m^2)^{1/2}\phi(\mathbf{X},t) \ , \\
&\mathbf{J}\phi(\mathbf{X},t) = (\mathbf{X}\times\mathbf{P} + \tfrac{1}{2}\boldsymbol{\sigma})\phi(\mathbf{X},t) \ , \\
&\mathbf{K}\phi(\mathbf{X},t) = \rho_3\left[\tfrac{1}{2}\{P_0,\mathbf{X}\} + \frac{\boldsymbol{\sigma}\times\mathbf{P}}{2(P_0+m)}\right]\phi(\mathbf{X},t) \ .
\end{aligned} \tag{2.2}$$

The nonrelativistic approximation to (2.2) is straightforward using $\mathbf{P}^2/m^2 \ll 1$, and leads to the familiar expressions. The nonrelativistic Hamiltonian is shifted by the constant m and special attention must be paid to the limit of the boosts $\mathbf{K}$.[7]

Actually, the form (2.2) of (2.1) is directly related to Wigner's unitary irreps of the Poincaré group corresponding to massive particles.[10,11] The doubling due to ρ_3 in (2.2) drops out and one has for any spin s

$$\begin{aligned}
&\mathbf{P}\phi_i(\mathbf{X},t) = -i\frac{\partial}{\partial \mathbf{X}}\phi_i(\mathbf{X},t) \ , \\
&P^0\phi_i(\mathbf{X},t) = (\mathbf{P}^2+m^2)^{1/2}\phi_i(\mathbf{X},t) \ , \\
&\mathbf{J}\phi_i(\mathbf{X},t) = \mathbf{X}\times\mathbf{P}\phi_i(\mathbf{X},t) + \tfrac{1}{2}\boldsymbol{\sigma}_{ij}\phi_i(\mathbf{X},t) \ , \\
&\mathbf{K}\phi_i(\mathbf{X},t) = \tfrac{1}{2}(P_0\mathbf{X}+\mathbf{X}P_0)\phi_i(\mathbf{X},t) \\
&\qquad + \tfrac{1}{2}(P_0+m)^{-1}\boldsymbol{\sigma}_{ij}\times\mathbf{P}\phi_i(\mathbf{X},t) \ ,
\end{aligned} \tag{2.3}$$

where $i,j = 1,2,\ldots,2s+1$ and the $\boldsymbol{\sigma}$ are the spin-s generators of the group SU(2). That (2.3) is a unitary irrep of the Poincaré group is easily checked, using the appropriate inner product

$$(\phi,\phi') = \int d\mathbf{X} \sum_i \phi_i^*(\mathbf{X},t)\phi_i'(\mathbf{X},t) \ . \tag{2.4}$$

In analogy to the Dirac equation the wave equations of the general type with which we are concerned in this paper may be given in a manifestly covariant form as a constraint

$$[P^\mu P_\mu - f(\hat{P}^\mu \Gamma_\mu)]\psi(x) = 0 \ . \tag{2.5}$$

The operators Γ_μ are the generators $S_{5\mu}$ of a unitary representation of SO(3,2), whereas $\psi(x)$ takes its value in the Hilbert space of that representation. This is discussed in greater detail in the next section. The remaining six generators $S_{\mu\nu}$, $\mu = 0,1,2,3$ of SO(3,2) appear (in analogy to the $\frac{1}{2}[\gamma_\mu,\gamma_\nu]$) as the generators of the Poincaré group acting on $\psi(x)$ of (2.5)

$$\begin{aligned}
&P_\mu\psi(x) = -i\frac{\partial}{\partial X^\mu}\psi(x) \ , \\
&M_{\mu\nu}\psi(x) = (P_\mu X_\nu - P_\nu X_\mu + S_{\mu\nu})\psi(x) \ .
\end{aligned} \tag{2.6}$$

The operator $\hat{P}^\mu$ in (2.5) is the four-velocity operator,[13] formally $\hat{P}^\mu = P^\mu/(P^\nu P_\nu)^{1/2}$, and the function f in (2.5) is more or less arbitrary and determines the mass spectrum.

Just as for the Dirac equation, there exist coordinates $\mathbf{X},t$ for which the covariance is not manifest, but which provide an equivalent representation. In this equivalent representation the states are represented by a wave function $\phi(\mathbf{X},t)$ with values for fixed $\mathbf{X},t$ in the Hilbert space of the unitary representation (unirep) of SO(3,2) mentioned in connection with (2.5). The inner product is given by

$$(\phi,\phi') = \int d\mathbf{X}(\phi(\mathbf{X},t),\phi'(\mathbf{X},t)) \ , \tag{2.7}$$

where the brackets on the right-hand side (RHS) refer to the inner product in the Hilbert space of the SO(3,2) unirep. The generators for the Poincaré transformation in this representation are

$$\begin{aligned}
&\mathbf{P}\phi(\mathbf{X},t) = -i\frac{\partial}{\partial \mathbf{X}}\phi(\mathbf{X},t) \ , \\
&P^0\phi(\mathbf{X},t) = [\mathbf{P}^2 + f(\Gamma_0)]^{1/2}\phi(\mathbf{X},t) \ , \\
&\mathbf{J}\phi(\mathbf{X},t) = (\mathbf{X}\times\mathbf{P} + \mathbf{S})\phi(\mathbf{X},t) \ , \\
&\mathbf{K}\phi(\mathbf{X},t) = \{\tfrac{1}{2}(\mathbf{X}P^0 + P^0\mathbf{X}) \\
&\qquad + [P_0 + f(\Gamma_0)^{1/2}]^{-1}\mathbf{S}\times\mathbf{P}\}\phi(\mathbf{X},t) \ .
\end{aligned} \tag{2.8}$$

Here $(\mathbf{S})_i=\epsilon_{ijk}S_{jk}$, with S_{jk} from SO(3,2).

One may check in a straightforward way that (2.7) and (2.8) give a unirep of the Poincaré group with a mass spectrum given by the operator

$$M=f(\Gamma_0)^{1/2}\ . \tag{2.9}$$

When doing this check one notices that all one really needs is an internal Hilbert space with self-adjoint operators M and $\mathbf{S}$ defined on it which satisfy

$$[M,\mathbf{S}]=0,\quad [S_i,S_j]=i\epsilon_{ijk}S_k\ . \tag{2.10}$$

This then leads to the question: Why does one need the other operators of the SO(3,2) structure? The answer to this question appears to be that it is desirable for introducing interactions to have a simple manifestly covariant form of the equations as in (2.5). The existence of such a simple covariant form is guaranteed via the Foldy-Wouthuysen transformation which connects $\phi(\mathbf{X},t)$ of (2.7) and (2.8) to $\psi(x)$ of (2.5). This transformation involves the S_{0i} and thus the entire SO(3,2) structure is involved.

The nonrelativistic limit of (2.5) is obtained most easily in the representations (2.7), (2.8), and (2.9). By neglecting higher-order terms in $\mathbf{P}^2/M^2$, where M^2 is an operator defined by (2.9), one obtains for the first four generators of (2.8)

$$\begin{aligned}&\mathbf{P}\phi(\mathbf{X},t)=-i\frac{\partial}{\partial\mathbf{X}}\phi(\mathbf{X},t)\ ,\\ &P^0\phi(\mathbf{X},t)\to\left[M+\frac{\mathbf{P}^2}{2M}\right]\phi(\mathbf{X},t)\ ,\\ &\mathbf{J}\phi(\mathbf{X},t)=(\mathbf{X}\times\mathbf{P}+\mathbf{S})\phi(\mathbf{X},t)\ ,\end{aligned} \tag{2.11}$$

where $M=f(\Gamma_0)^{1/2}$. The limit for the boosts in (2.8) requires more attention,[7] but is not interesting for us, as we are only interested in small velocities.

The nonrelativistic limit (2.11) shows a Schrödinger theory with mass given by the operator $f(\Gamma_0)^{1/2}$, and with a shift of the Hamiltonian by the operator $f(\Gamma_0)^{1/2}$. In other words, the mass in the nonrelativistic theory is represented by an operator which is identical to the rest mass operator of the relativistic theory (2.5). To some this might be obvious by using the rest frame of the relativistic particle. The internal motion, which produces the mass spectrum has been left alone and is still relativistic. This is entirely analogous to what one does when discussing the nonrelativistic limit of the Dirac electron. The next sections are concerned with the nonrelativistic limit of the internal motion of the object.

III. BRIEF RÉSUMÉ OF SO(3,2) IRREPS TO BE USED

We wish to collect here the relevant information[14] on the particular class of SO(3,2) irreps to be used in the relativistic oscillator construction. The SO(3,2) group is generated by two types of operators: a four-vector, Γ_μ (the analog to the Dirac four-vector γ_μ) and the "spin" operators, $S_{\mu\nu}=-S_{\nu\mu}$, which generate a Lorentz group (these operators are the analog to the Dirac spin operators $\frac{1}{2}[\gamma_\mu,\gamma_\nu]$). It is useful to denote Γ_μ as $S_{5\mu}$ and augment the indices μ ($\mu=0,1,2,3$) by adjoining 5 and denoting by A the indices 0,1,2,3,5. In this notation these ten (Hermitian) operators $S_{AB}=-S_{BA}$ obey the commutation relations

$$\begin{aligned}[S_{AB},S_{CD}]=-i(&g_{AC}S_{BD}+g_{BD}S_{AC}\\ &-g_{AD}S_{BC}-g_{BC}S_{AD})\ ,\end{aligned} \tag{3.1}$$

$$g_{AB}=\mathrm{diag}(1,-1,-1,-1,1),\quad A,B=01235\ . \tag{3.2}$$

The SO(3,2) group has two invariant operators constructed from the generators:

(a) the quadratic ("Casimir") operator

$$C_2\equiv\tfrac{1}{2}S_{AB}S^{AB}\ , \tag{3.3}$$

(b) the fourth-order invariant operator

$$C_4\equiv -W^AW_A\ , \tag{3.4}$$

where

$$W^A=\tfrac{1}{8}\epsilon^{ABCDE}S_{BC}S_{DE}\ . \tag{3.5}$$

For compact groups the eigenvalues of the invariant operators (n in number, where n is the rank of the group) uniquely specify the irreps, but for noncompact groups the invariant operators may not be sufficient to distinguish the irreps. In particular, for SO(3,2) one needs, in general, a third (invariant) label: the minimum (or maximum) eigenvalue of Γ_0 depending on whether Γ_0 is bounded from below (or from above). We are interested only in the multiplicity-free irreps of SO(3,2)—the so-called "singleton" irreps.

There are two subgroup chains which are convenient for discussing the irreps of SO(3,2). One chain uses the Lorentz subgroup

$$\mathrm{SO}(3,2)\supset\mathrm{SO}(3,1)\supset\mathrm{SO}(3)\ , \tag{3.6}$$

while the other uses the maximal compact subgroup

$$\mathrm{SO}(3,2)\supset\mathrm{SO}(3)\times\mathrm{SO}(2)\ , \tag{3.7}$$

the latter two groups being generated by S_{ij} ($i,j=1,2,3$) and by Γ_0.

The Majorana representations of SO(3,2)—termed the "remarkable" representations by Dirac[15]—obey, in addition, the special relation

$$\{\Gamma_\mu,\Gamma^\xi\}+\{S_{\mu\nu},S^{\xi\nu}\}=-\delta_\mu^\xi\ , \tag{3.8}$$

which has the consequence that the invariant operators have the value

$$C_2(\mathrm{Majorana})=-\tfrac{5}{4}\ , \tag{3.9a}$$

$$C_4(\mathrm{Majorana})=0\ . \tag{3.9b}$$

Moreover, Eq. (3.8) also implies that the eigenvalue of Γ_0 and the spin $[S^2\to j(j+1)]$ are related by

$$\Gamma_0\to j+\tfrac{1}{2}\ . \tag{3.10}$$

The spinorial relativistic rotator[16] is an algebraic realization of the generators of SO(3,2) using two pairs of boson operators; the special relation, Eq. (3.8), is automatically realized as well. The spinorial relativistic rotator

thus necessarily realizes the Majorana representations, and, in fact, contains both the half-integer—spin representation $j=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and the integer-spin representation $j=0,1,2,\ldots$, each with positive eigenvalues for Γ_0 (and hence positive values for the mass). There are four Majorana irreps, the two mentioned above and a similar pair having negative eigenvalues for Γ_0. [The special condition, Eq. (3.8), has the effect of forcing the length of d_μ to be constant in time, that is, $(d/d\tau)(d_\mu d^\mu)=0$. (The four-vector d_μ is the difference vector $d_\mu=Y_\mu-Q_\mu$, where Y_μ is the center-of-mass position and Q_μ the "charge" position.[17]) This result shows that *there is no vibration, only rotation, in the spinorial relativistic rotator.*]

Since we are interested in the models for which vibration is possible—that is, the quantum relativistic oscillator—we must go beyond the Majorana irreps. A criterion for these more general SO(3,2) irreps is that they *permit contraction with respect to the SO(3) subgroup.*[7] The contraction of some SO(3,2) representations with respect to SO(3) has been discussed in Ref. 14 and the contraction of compact orthogonal groups onto harmonic oscillator symmetry has been studied in Ref. 18. For the relativistic oscillator we are using yet another subclass of SO(3,2) representations.

For the particular class of SO(3,2) singleton irreps which we shall study, the Casimir invariant has the eigenvalue $C_2\to -R$, where R can take a continuous set of real values and the invariant C_4 has the value

$$-C_4\to j_{\min}(j_{\min}+1)[R+(j_{\min}-1)(j_{\min}+2)]\ , \tag{3.11}$$

where $j_{\min}$ is the minimum spin contained in the irrep.

A special case of this class occurs for $j_{\min}=0$. From Eq. (3.11) one sees that C_4 vanishes. Moreover, the minimum eigenvalue for Γ_0 in the irrep is related to the Casimir invariant R. Let us denote by γ_0 the minimum eigenvalue of Γ_0. Then we have

$$\gamma_0\geq\tfrac{1}{2}\ , \tag{3.12a}$$

and one finds the relation

$$C_2\to R=\tfrac{9}{4}-(\gamma_0-\tfrac{3}{2})^2\ . \tag{3.12b}$$

We shall be concerned in the following with the special irreps for which $j_{\min}=0$. It is helpful to give explicitly the (j,γ) content of such an irrep. Figure 1 displays this information.[14]

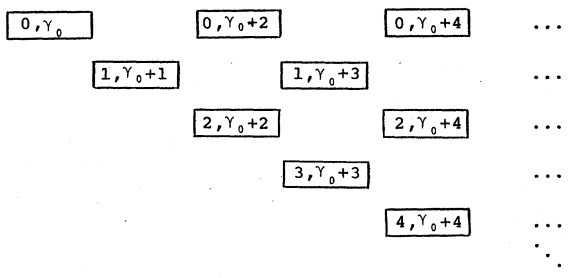

FIG. 1. Reduction of the SO(3,2) irrep labeled by the invariants $C_2=\frac{9}{4}-(\gamma_0-\frac{3}{2})^2$ and $C_4=0$, with respect to the SO(3)×SO(2) subgroup, labeled by (j,γ). Here $j_{\min}=0$ and $\gamma_{\min}\equiv\gamma_0\geq\frac{1}{2}$.

IV. THE HARMONIC OSCILLATOR ALGEBRA AS A CONTRACTION OF SO(3,2)

For the quantum relativistic oscillator we use the SO(3,2) irreps given in Sec. III. Its reduction with respect to the SO(3)×SO(2) subgroup generated by S_{ij} and Γ_0 has been given in Fig. 1. The spectrum of $\mathbf{S}^2$ and $\Gamma_0-\gamma_0$ is identical with the spectrum of angular momentum and principal quantum number of the three dimensional oscillator.

Let us now consider the contraction. For this, we introduce new variables

$$\tilde{\xi}_i\equiv\epsilon S_{i0},\quad \tilde{\pi}_i\equiv\epsilon\Gamma_i,\quad \text{and } \tilde{u}\equiv\epsilon^2\Gamma_0\ , \tag{4.1}$$

where ϵ is the contraction parameter, a *numerical parameter.* The contraction is to be carried out by letting $\epsilon\to 0$ and S_{j0}, Γ_i, and Γ_0 become large, but keeping the operators $\tilde{\xi}_i$, $\tilde{\pi}_i$, and $\tilde{u}$ finite, i.e., we shall contract through a series of representations.

In the limit we obtain the operators

$$\xi_i\equiv\lim_{\epsilon\to 0}\tilde{\xi}_i,\quad \pi_i\equiv\lim_{\epsilon\to 0}\tilde{\pi}_i,\quad \text{and } u\equiv\lim_{\epsilon\to 0}\tilde{u}\ . \tag{4.2}$$

We introduce these definitions into the commutation relations (3.1).

The indices AB divide into four groups, $(5i)$, (50), $(i0)$, and (ij), and accordingly the commutation relations separate into the following nine cases:

$(5i,5j)$:

$$[\tilde{\pi}_i,\tilde{\pi}_j]=-i\epsilon^2 S_{ij}\Longrightarrow[\pi_i,\pi_j]=0\ . \tag{4.3a}$$

$(5i,50)$:

$$[\tilde{\pi}_i,\tilde{u}]=-i\epsilon^2\tilde{\xi}_i\Longrightarrow[\pi_i,u]=0\ . \tag{4.3b}$$

$(5i,0j)$:

$$[\tilde{\pi}_i,\tilde{\xi}_j]=-i\delta_{ij}\tilde{u}\Longrightarrow[\pi_i,\xi_j]=-i\delta_{ij}u\ . \tag{4.3c}$$

$(5i,jk)$:

$$[\tilde{\pi}_i,S_{jk}]=-i(\delta_{ij}\tilde{\pi}_k-\delta_{ik}\tilde{\pi}_j)$$
$$\Longrightarrow[\pi_i,S_j]=i\epsilon_{ijk}\pi_k\ . \tag{4.3d}$$

$(50,i0)$:

$$[\tilde{u},\tilde{\xi}_i]=-i\epsilon^2\tilde{\pi}_i\Longrightarrow[u,\xi_i]=0\ . \tag{4.3e}$$

$(50,ij)$:

$$[\tilde{u},S_{ij}]=0\Longrightarrow[u,S_{ij}]=0\ . \tag{4.3f}$$

$(i0,j0)$:

$$[\tilde{\xi}_i,\tilde{\xi}_j]=-i\epsilon^2 S_{ij}\Longrightarrow[\xi_i,\xi_j]=0\ . \tag{4.3g}$$

$(i0,jk)$:

$$[\tilde{\xi}_i,S_{jk}]=-i(\delta_{ij}\tilde{\xi}_k-\delta_{ik}\tilde{\xi}_j)$$
$$\Longrightarrow[\xi_i,S_j]=i\epsilon_{ijk}\xi_k\ . \tag{4.3h}$$

(ij,kl):

$$[S_i,S_j]=i\epsilon_{ijk}S_k\ . \tag{4.3i}$$

The content of these contracted relations is easily seen to be the following:

The operator u functions as a multiple of the unit operator commuting with everything;

The operators ξ and π each have commuting components and obey the Heisenberg commutation relation (with u as the unit operator);

Under the angular momentum operators $\mathbf{S}$, the operators ξ and π transform as three-vectors.

Before we can interpret these results more fully, however, we must study the behavior of the invariant operators under this contraction. For the Casimir operator C_2 we have the result

$$C_2 \equiv \tfrac{1}{2} S_{AB} S^{AB} = \Gamma_\mu \Gamma^\mu + \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} , \tag{4.4}$$

which, using the substitution Eq. (3.8) becomes

$$C_2 = \frac{\tilde{u}^2}{\epsilon^4} - \frac{\tilde{\pi}\cdot\tilde{\pi}}{\epsilon^2} - \frac{\tilde{\xi}\cdot\tilde{\xi}}{\epsilon^2} + \mathbf{S}\cdot\mathbf{S} . \tag{4.5}$$

Hence in the contraction limit $\epsilon^4 C_2$ will become

$$\epsilon^4 C_2 \mid_{\epsilon\to 0} = u^2 \tag{4.6}$$

with u functioning as a multiple of the unit element. Thus we see that C_2 must increase in such a way that the eigenvalue R multiplied by ϵ^4 stays finite in the limit. The SO(3,2) representations given by Eqs. (3.11) and (3.12) do indeed have the property that this limit can be realized: one takes $\gamma_0 \to \infty$ with $\epsilon^2(\gamma_0) \to$ const. If we fix the scale so that $\epsilon^2 \gamma_0 \to 1$, then we achieve the result that in the limit $\epsilon^2 \Gamma_0 \to \mathbb{1}$, that is, we obtain $u = \mathbb{1}$.

Consider next the invariant C_4. This, from Eq. (3.4), has the form

$$-C_4 \equiv W^A W_A = W^\mu W_\mu + (\tfrac{1}{8} \epsilon_{\mu\nu\rho\sigma} S^{\mu\nu} S^{\rho\sigma})^2 , \tag{4.7}$$

where $W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho} \Gamma_\sigma$. Using the substitutions, Eq. (4.1), we find

$$\text{(a)}\ W_0 = \frac{1}{\epsilon} \mathbf{S}\cdot\tilde{\pi} , \tag{4.8}$$

$$\text{(b)}\ \mathbf{W} = \frac{1}{\epsilon^2} (\tilde{u}\mathbf{S} - \tilde{\xi}\times\tilde{\pi}) , \tag{4.9}$$

$$\text{(c)}\ \tfrac{1}{8} \epsilon_{\mu\nu\rho\sigma} S^{\mu\nu} S^{\rho\sigma} = \xi\cdot\mathbf{S} . \tag{4.10}$$

Thus C_4 is dominated by the term $\mathbf{W}\cdot\mathbf{W}$ and we obtain in the contraction limit (replacing u by the unit operator)

$$\epsilon^4 C_4 \mid_{\epsilon\to 0} = (\mathbf{S} - \xi\times\pi)^2 . \tag{4.11}$$

For the SO(3,2) irreps in which we are interested, the minimum spin is zero and, in consequence [see Eq. (3.11)], C_4 is also zero. It follows that we must have

$$\mathbf{S} = \xi\times\pi . \tag{4.12}$$

For this special class of representations ($j_{\min} = 0$), we see that the orbital angular momentum of the oscillator motion becomes the "spin angular momentum" of the states in the irrep.

More generally, for the SO(3,2) irreps that have $j_{\min} \neq 0$, we find [from Eqs. (3.11) and (4.6)] that

$$\epsilon^4 C_4 \to j_{\min}(j_{\min} + 1) . \tag{4.13}$$

Using the limit found in Eq. (4.11), we see that we have obtained the result

$$\mathbf{S} = \mathbf{j}_{\text{int}} + \boldsymbol{l} , \tag{4.14}$$

where we have defined the orbital angular momentum of the oscillator to be $\boldsymbol{l}$, with

$$\boldsymbol{l} \equiv \xi\times\pi , \tag{4.15}$$

and the "intrinsic" spin j_{int} having the eigenvalue

$$\mathbf{j}_{\text{int}}\cdot\mathbf{j}_{\text{int}} \to j_{\min}(j_{\min} + 1) . \tag{4.16}$$

The interpretation of the limit for the invariant C_4 is then clear: $\mathbf{S}$ is the total angular momentum (in the rest frame) of the extended object. In the contraction limit, this operator is seen to consist of two parts, an intrinsic spin j_{int} [whose value is the minimum value of total angular momentum belonging to the multiplicity-free SO(3,2) irrep] and an orbital angular momentum $\boldsymbol{l}$ (whose value is determined by the orbital angular momentum state of the three-dimensional oscillator). For the special case $j_{\min} = 0$, we see that, in accord with the earlier result, the angular momentum is entirely orbital. Let us remark that this situation also agrees with the results for the relativistic string if we limit the oscillators to only the lowest oscillator mode. Then the spin is once again generated by the orbital motion of the oscillator. SO(3,2) representations with $j_{\min} \neq 0$ correspond to the lowest mode of the "spinning string."

We see that for the contracted group, Γ_0 appears nowhere on the right-hand side of a commutator relation. Hence we are free[7] to modify Γ_0 into Γ_0' subtracting from it a term $\approx 1/\epsilon^2$ in order to obtain an operator which we can use in the limit as giving the mass in the rest frame. To do this we need to calculate Γ_0 to second order.

For the evaluation of Γ_0 to second order, we note that the contraction limit, as determined by the commutation relations, makes no statement as to higher orders: the only possible relation must accordingly be representation dependent. (This is true also for the nonrelativistic limit.) Thus we see that we must use the invariant operators to obtain the required relations, and the Casimir invariant C_2 must suffice, since we have already used $C_4 = 0$.

From Eq. (4.4), we see that C_2 and Γ_0 are related by

$$C_2 = {\Gamma_0}^2 - \boldsymbol{\Gamma}\cdot\boldsymbol{\Gamma} + S_{0i} S^{0i} + \mathbf{S}\cdot\mathbf{S} , \tag{4.17}$$

and the eigenvalue for C_2 is given by

$$C_2 = -\tfrac{9}{4} + (\gamma_0 - \tfrac{3}{2})^2 . \tag{4.18}$$

We have already determined that $\epsilon^2 \gamma_0 \to 1$.

Introducing the variables of Eq. (4.1), we see that

$$\Gamma_0 \to \left[-\frac{9}{4} + \left[\frac{1}{\epsilon^2} - \frac{3}{2} \right]^2 + \frac{\tilde{\pi}^2}{\epsilon^2} + \frac{\tilde{\xi}^2}{\epsilon^2} - \mathbf{S}\cdot\mathbf{S} \right]^{1/2}$$

$$= \frac{1}{\epsilon^2} + \tfrac{1}{2}(\pi^2 + \xi^2 - 3) + O(\epsilon^2) . \tag{4.19}$$

We shall be interested in models where

$$P_\mu P^\mu = f(\hat{P}_\mu \Gamma^\mu) = \frac{1}{\alpha} \hat{P}_\mu \Gamma^\mu \tag{4.20}$$

and in the rest frame

$$P_0P^0=\frac{1}{\alpha}\Gamma_0=M_0{}^2\Gamma_0\ , \tag{4.21}$$

where $M_0{}^2=1/\alpha$. We see that the expansion of $P_0P^0/M_0{}^2$ must be the same as the expansion of Γ_0 given in (4.19). The effective Hamiltonian H is defined by the second-order term in the expansion of P_0

$$P_0\rightarrow\frac{M_0}{\epsilon}+\epsilon H+O(\epsilon^3)\ . \tag{4.22}$$

By identification of the second-order terms in (4.21) we finally obtain

$$H=\tfrac{1}{2}M_0[\tfrac{1}{2}(\pi^2+\xi^2-3)] \tag{4.23}$$

which is the Hamiltonian for the internal motion.

V. NONRELATIVISTIC LIMIT OF THE INTERNAL MOTION OF THE QUANTUM RELATIVISTIC OSCILLATOR

In the models of the general class described by (2.5) there are two choices involved. The first choice is that of a unitary representation, SO(3,2), which gives the Γ_μ and $S_{\mu\nu}$. The second choice is that of the function f. Depending on these choices one obtains the quantum relativistic oscillator,[19] the quantum relativistic rotator,[20] or possibly other structures.[21]

For the quantum relativistic oscillator we use the irreps given in Sec. III. For the function f in (2.5) we pick $(1/\alpha')\hat{P}^\mu\Gamma_\mu$, where α' is the slope parameter $1/\alpha'$,

$$\left[P^\mu P_\mu-\frac{1}{\alpha'}\hat{P}^\mu\Gamma_\mu\right]\psi(x)=0\ . \tag{5.1}$$

The nonrelativistic limit, the contraction of the group SO(3,2) of Sec. IV could be carried out straightforwardly in the form (2.8), which is entirely equivalent to (2.5). In (2.8) the only operators of SO(3,2) which enter are the mass operator $[(1/\alpha')\Gamma_0]^{1/2}$ and the spin operators $(\mathbf{S})_i=\epsilon_{ijk}S_{jk}$. In the form (2.8) the contraction is thus obvious when one obtains an internal group spanned by the S_{ij} and the "nonrelativistic Hamiltonian" of (4.21). The model obtained in this way is fully relativistic; it reduces to a family of unitary irreps of the Poincaré group. Some might claim this model to be the "relativistic harmonic oscillator." However, this model has a serious disadvantage. The contraction makes the transformation between the two equivalent forms (2.5) and (2.8), which contain the S_{0i} of SO(3,2), singular and hence these (2.8) after the contraction no longer have the explicitly covariant form (2.5). Thus it is not clear how to introduce interaction.

To summarize, two nonrelativistic limits exist. One may take, in a relativistically invariant way, the nonrelativistic limit of the internal motion. The model thus obtained is described by a version of (2.8), but does not possess a manifestly covariant form (2.5). What makes much more sense is first to take the nonrelativistic limit of the overall motion. This leads to a model of a Schrödinger particle, still with the mass spectrum of the original relativistic theory. This is just looking at the particle "in its rest frame," leaving the internal motion relativistic. Next, one may take the nonrelativistic limit to the internal motion using the contraction described in Sec. IV.

VI. CONCLUDING REMARKS

We have shown in the preceding section that the quantum relativistic oscillator, as defined by a special class of representations of the SO(3,2) spectrum generator, under contraction goes into the nonrelativistic three-dimensional harmonic oscillator, kinematically and dynamically. Alternatively one may regard this as defining a consistent relativistic generalization of the nonrelativistic (NR) harmonic oscillator. From this point of view, the result is of interest in the extension of the nuclear shell model (based on harmonic oscillators in large part) to the relativistic domain. It is well known that the (NR) three-dimensional oscillator has the symmetry group SU(3), and the construction for the relativistic extension does, in fact, preserve this symmetry: for the Hamiltonian, Eq. (5.1), the rest frame (mass)^2 is a linear function of the eigenvalue $\Gamma_0\rightarrow\gamma$. Each such eigenvalue is degenerate in spin, and it is readily seen that this degeneracy—given by the columns of boxes of Fig. 1—accords with that of the SU(3) irreps $[p00]$, where $p=\gamma-\gamma_0$, using the nuclear physics group chain SU(3)$\supset$SO(3).

Let us remark that the results of the contraction limit are of interest in understanding the kinematics of "relativistic relative coordinates." We found that in the contraction limit, the internal motion of the oscillator is in the observables ξ and π. What are the relativistic observables corresponding to these operators?

It is not hard to see that the corresponding relativistic four-vector

$$d_\mu\equiv S_{\mu\nu}P^\nu(P_\mu P^\mu)^{-1}\ , \tag{6.1}$$

which, in the contraction limit, using Eqs. (4.1) and (4.21), becomes[8]

$$d_\mu\rightarrow\begin{cases}0, & \mu=0\\ \xi_i\dfrac{1}{M_0}, & \mu=1,2,3\ .\end{cases} \tag{6.2}$$

The coordinate d_μ is a "relative coordinate," the difference between the "charge position" X_μ and the "center-of-mass position" Y_μ. (The designations in quotes are taken from analogous quantities for the Dirac electron.)

It is somewhat more difficult to find the relativistic analog to the observable π, but it can be verified that the corresponding four-vector is the operator

$$\dot{d}_\mu\equiv M_0[\Gamma_\mu-P_\mu(P_\nu\Gamma^\nu/P_\nu P^\nu)](P_\alpha P^\alpha)^{-1/2}\ . \tag{6.3}$$

In the contraction limit, using Eqs. (4.1) and (4.21) again, we find that

$$\dot{d}_\mu\rightarrow\begin{cases}0, & \mu=0\\ \pi-\dfrac{\mathbf{P}}{M_0}, & \mu=1,2,3\ .\end{cases} \tag{6.4}$$

Thus in the rest frame we recover the momentum operator π.

The operators d_μ and $\dot{d}_\mu$ have recently been proposed as a new relativistic generalization of the Heisenberg commutation relations.[22] The point we wish to make here is that the contraction limit indicates that these observables are in fact the proper relativistic generalization of the internal coordinates of the oscillator. Because the components of the operators d_μ (and similarly for $\dot{d}_\mu$) do not commute among themselves, unlike the ξ (and π) there may be difficulties in interpreting these observables as relativistic coordinates, but nonetheless the contraction limit definitely shows them to be the proper relativistic generalization for the oscillator. After submission of the present paper, other recent approaches to the relativistic oscillator were brought to our attention.[23]

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