Gauge fixing the SU(N) lattice-gauge-field Hamiltonian

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We exactly gauge fix the Hamiltonian for the SU(N) lattice gauge field and eliminate the redundant gauge degrees of freedom. The gauge-fixed lattice Hamiltonian, in particular for the Coulomb gauge, has many new terms in addition to the ones obtained in the continuum formulation.

I. INTRODUCTION

The Hamiltonian for QCD (quantum chromodynamics) has been widely studied using the lattice and continuum formulations. In a remarkable paper by Drell,[1] a derivation was given of the running coupling constant of QCD using the continuum Hamiltonian; this calculation used weak-field perturbation theory and the Coulomb gauge. The mathematical treatment of gauge fixing the Yang-Mills Hamiltonian goes back to Schwinger;[2] the more recent paper by Christ and Lee[3] gives a clear and complete treatment of gauge fixing the continuum gauge-field Hamiltonian.

The continuum Hamiltonian has until now been given no regulation which preserves gauge invariance; for the one-loop calculation carried out by Drell[1] and Lee,[3] a momentum cutoff is sufficient to ensure renormalizability. However, for two loops and higher it is known that a momentum cutoff violates gauge invariance and renders the theory nonrenormalizable; for the action formulation it is known that dimensional regularization of the Feynman diagrams[4] is sufficient to renormalize the action. For the Hamiltonian, there is no analog of dimensional regularization and hence it is not clear how to regulate the continuum QCD Hamiltonian to all orders.

The lattice Hamiltonian[5,6] is regulated to all orders and could be used for calculations involving two loops or higher. If we want to analyze the lattice Hamiltonian using the weak-coupling approximation, it is necessary to fix a gauge, for example, the Coulomb gauge. Gauge fixing the action of the lattice gauge theory has been solved,[7] and in this paper we extend gauge fixing to the lattice Hamiltonian. Gauge fixing essentially involves only the lattice gauge field and the quarks enter only through the quark-color-charge operator. So we will essentially study only the gauge field and introduce the quark fields when necessary.

Gauge fixing the lattice Hamiltonian is very similar in spirit to gauge fixing the continuum Hamiltonian; this similarity can be clearly seen in the action formulation.[7,8] For the Hamiltonian we will basically follow the treatment given by Christ and Lee.[3] There are, however, significant differences between the lattice and continuum Hamiltonians both for the kinetic operator and the potential term. The lattice gauge field is defined using finite-group elements of SU(N) as the fundamental degrees of freedom whereas the continuum uses only the infinitesimal elements of SU(N). This difference will introduce a lot of extra complications. Given an appropriate generalized interpretation of the basic symbols, it will turn out, however, that the form of the gauge-fixed continuum and lattice Hamiltonians are very similar.

In Sec. II we discuss the Hamiltonian and give a construction of the chromoelectric field operator. We then discuss Gauss's law for the system. In Sec. III we perform a change of variable and eliminate the redundant gauge degrees of freedom. In Sec. IV we evaluate Gauss's law for the new variables and find that the constrained variables decouple exactly from the Gauss's constraint. In Sec. V we evaluate the gauge-fixed lattice Hamiltonian, discuss operator ordering, and introduce the quark-charge operator. In Sec. VI we discuss the main feature of our results.

II. DEFINITIONS

Consider a d-dimensional Euclidean spatial lattice with spacing a; let U_{ni}, $i=1,2,\ldots,d$, be the SU(N) link degree of freedom from lattice site n to $n+\hat{\imath}$ ($\hat{\imath}$ is the unit lattice vector in the ith direction) and let $\psi_n,\bar{\psi}_n$ be the lattice quark field. The Hamiltonian for SU(N) lattice gauge field in the temporal axial gauge is given by[5,6]

$$H=H_{\rm YM}[U]+H_F[\bar{\psi},\psi,U]\ , \tag{2.1a}$$

where

$$H_{\rm YM}=-\frac{g^2}{2a}\sum_{n,i}\nabla^2(U_{ni}) -\frac{1}{ag^2}\sum_{n,ij}{\rm Tr}(U_{ni}U_{n+\hat{\imath},j}U^{\dagger}_{n+\hat{\jmath},i}U^{\dagger}_{nj}) \tag{2.1b}$$

and H_F is the quark-gauge-field part. Note ∇^2 is the SU(N) Laplace-Beltrami operator. The Hamiltonian acts only on gauge-invariant wave functionals Φ. Gauge transformation is given by

$$U_{ni}\rightarrow U_{ni}(\varphi)\equiv\varphi_n U_{ni}\varphi^{\dagger}_{n+\hat{\imath}} \tag{2.2}$$

and the wave functionals Φ are invariant under (2.2), that is,

$$\Phi[U]=\Phi[U(\varphi)]\ . \tag{2.3}$$

By performing an infinitesimal gauge transformation (and introducing the quark field) we have from (2.3) Gauss's law[6]

$$\left[\sum_i [E_a^R(U_{ni})-E_a^L(U_{n-\hat{\imath},i})]-\rho_{na}\right]|\Phi\rangle=0 . \qquad (2.4)$$

The operators E_a^R and E_a^L are first-order Hermitian differential operators with the commutation equation[6]

$$[E_a^L,E_b^L]=iC_{abc}E_c^L , \qquad (2.5a)$$

$$[E_a^R,E_b^R]=-iC_{abc}E_c^R , \qquad (2.5b)$$

$$E_a^R(U)=R_{ab}(U)E_b^L(U), \quad R_{ab}(U)=\mathrm{Tr}(X_a U X_b U^\dagger) , \qquad (2.5c)$$

$$[E_a^R,E_b^L]=0 , \qquad (2.5d)$$

where R_{ab} is the adjoint representation, X_a the generators and C_{abc} the structure constants of SU(N).

The operator $\rho_{na}(\bar{\psi},\psi,U)$ is the lattice quark-color-charge operator[6] and satisfies

$$[\rho_{na},\rho_{mb}]=iC_{abc}\rho_{nc}\delta_{nm} . \qquad (2.5e)$$

From (2.4) and (2.5c) we have

$$0=\left[\sum_i [R_{ab}(U_{ni})E_b^L(U_{ni})-E_a^L(U_{n-\hat{\imath},i})]-\rho_{na}\right]|\Phi\rangle \qquad (2.6a)$$

$$\equiv\left[\sum_{m,i} D_{nmi}^{ab}E_b^L(U_{mi})-\rho_{na}\right]|\Phi\rangle , \qquad (2.6b)$$

where D_{nmi}^{ab} is the lattice covariant *backward* derivative. Let $|n,a\rangle$ be a ket vector of lattice site n and non-Abelian index a; then, from (2.6) we have the real matrix D_i given by

$$D_{nmi}^{ab}=\langle n,a|D_i|m,b\rangle \qquad (2.7a)$$

$$=R_{ab}(U_{ni})\delta_{nm}-\delta_{ab}\delta_{n-\hat{\imath},m} . \qquad (2.7b)$$

We see from above that D_i performs a finite rotation R_{ab} on the ket vector and then displaces it in the backward direction.

We write the Hamiltonian as sum of the kinetic and potential energy, that is

$$H=K(U)+P(\bar{\psi},\psi,U) , \qquad (2.8)$$

where

$$K=-\frac{g^2}{2a}\sum_{n,i}\nabla^2(U_{ni}) \qquad (2.9)$$

and P is the rest of (2.1a). It is known that[9]

$$-\nabla^2(U)=\sum_a E_a^L(U)E_a^L(U) . \qquad (2.10)$$

In light of Gauss's law and (2.10) we identify $E_a^L(U_{ni})$ as the chromoelectric operator of the gauge field corresponding to the link variable U_{ni}. Choose canonical coordinates B_{ni}^a such that

$$U_{ni}=\exp(iB_{ni}^a X_a) . \qquad (2.11)$$

Then we have, suppressing the lattice and vector indices and summing on repeated non-Abelian indices

$$E_a^{L(R)}(U)=e_{ab}^{L(R)}(U)\frac{\partial}{i\partial B^b} \qquad (2.12a)$$

$$\equiv\frac{\delta}{i\delta_{L(R)}B^a} . \qquad (2.12b)$$

Note

$$e_{ab}^L(U)=e_{ba}^R(U) . \qquad (2.13)$$

Explicit expressions for $e_{ab}^{L(R)}$ are given in (3.8).

III. GAUGE FIXING

We can see from Gauss's law that all the U_{ni}'s are not required to describe the gauge-invariant wave functional Φ. We gauge transform U_{ni} to a new set of variables V_{ni} which are constrained; the constrainted variables V_{ni} will decouple from Gauss's law.

Consider the change of variables from $\{U_{ni}\}$ to $\{\varphi_n,V_{ni}\}$, with $\{V_{ni}\}$ having one constraint for each n. That is,

$$\psi_n=\varphi_n\zeta_n, \quad \bar{\psi}_n=\bar{\zeta}_n\varphi_n^\dagger , \qquad (3.1a)$$

$$U_{ni}=\varphi_n V_{ni}\varphi_{n+\hat{\imath}}^\dagger , \qquad (3.1b)$$

and choosing the Coulomb gauge for the lattice gives

$$\chi_n^a(V)\equiv\mathrm{Im}\sum_i \mathrm{Tr}X_a(V_{ni}-V_{n-\hat{\imath},i})=0 . \qquad (3.1c)$$

In canonical coordinates we have

$$V_{ni}=\exp(iA_{ni}^a X_a), \quad \varphi_n=\exp(i\phi_n^a X_a) . \qquad (3.2)$$

For small variation A^a+dA^a, we have

$$V(A+dA)=V(A)\left[1+V^\dagger(A)\frac{\partial V(A)}{\partial A^a}dA^a\right] \qquad (3.3)$$

$$=V(A)[1+iX_a f_{ab}^R(A)dA^b] , \qquad (3.4)$$

where

$$f_{ab}^R=-i\,\mathrm{Tr}\left[V^\dagger\frac{\partial V}{\partial A^a}X_b\right]=f_{ba}^L . \qquad (3.5)$$

Define

$$\delta_{L(R)}A^a=f_{ab}^{L(R)}(A)dA^b \qquad (3.6)$$

then

$$V(A+dA)=V(A)(1+iX_a\delta_R A^a) \qquad (3.7a)$$

$$=(1+iX_a\delta_L A^a)V(A) . \qquad (3.7b)$$

It can be shown that

$$e_{aa}^{L(R)}f_{ab}^{L(R)}=\delta_{ab} \qquad (3.8)$$

and hence matrix e can be determined from (3.5). Under the change of variables (3.1) from U_{ni} to V_{ni}, the potential energy P in (2.8) can be expressed as a function of only V_{ni}. For the kinetic energy K we need the expression for $E_a^L(U)$. Note, using the chain rule and formula (3.8),

$$\frac{\partial}{\partial B_{mj}^{b}}=\sum_{n,i}\frac{\partial A_{ni}^{a}}{\partial B_{mj}^{b}}\frac{\partial}{\partial A_{ni}^{a}}+\sum_{n}\frac{\partial \phi_{n}^{a}}{\partial B_{mj}^{b}}\frac{\partial}{\partial \phi_{n}^{a}} \tag{3.9}$$

$$=\sum_{n,i} f_{a\alpha}^{R}(A_{ni})\frac{\partial A_{ni}^{\alpha}}{\partial B_{mj}^{b}} e_{\alpha\beta}^{L}(A_{ni})\frac{\partial}{\partial A_{ni}^{\beta}}+\cdots . \tag{3.10}$$

Therefore, from (2.12) and (3.10)

$$E_{b}^{L}(U_{mj})=\frac{\delta}{i\delta_{L}B_{mj}^{b}}=\sum_{n,i}\frac{\delta_{R}A_{ni}^{a}}{\delta_{L}B_{mj}^{b}}\frac{\delta}{i\delta_{L}A_{ni}^{a}}+\sum_{n}\frac{\delta_{R}\phi_{n}^{a}}{\delta_{L}B_{mj}^{b}}\frac{\delta}{i\delta_{L}\phi_{n}^{a}} . \tag{3.11}$$

We now evaluate the coefficient functions of the above equation. The constraint [Eq. (3.1c)] is valid under variations of A_{ni}^{a} to $A_{ni}^{a}+dA_{ni}^{a}$, i.e.,

$$0=\chi_{n}^{a}(A) \tag{3.12}$$

$$=\chi_{n}^{a}(A+dA) . \tag{3.13}$$

Hence, from (3.12) and (3.13)

$$\sum_{m,i}\Gamma_{nmi}^{ab}(A)\delta_{R}A_{mi}^{b}=0 , \tag{3.14}$$

where for constraint (3.1b) we have

$$\Gamma_{nmi}^{ab}=\langle n,a \mid \Gamma_{i} \mid m,b\rangle \tag{3.15a}$$

$$=\delta\chi_{n}^{a}/\delta_{L}A_{mi}^{b} \tag{3.15b}$$

$$=\omega_{ni}^{ab}\delta_{nm}-\omega_{n-\hat{\imath},i}^{ab}\delta_{n-\hat{\imath},m} , \tag{3.15c}$$

where from (3.1c)

$$\omega_{ni}^{ab}=\mathrm{Tr}(X_{a}V_{ni}X_{b}+X_{b}V_{ni}^{\dagger}X_{a}) . \tag{3.16}$$

The constraint (3.14) on δA_{ni}^{a} determines $\delta\varphi/\delta B$. Consider from (3.1b), the following variation:

$$V_{ni}(A+dA)=\varphi_{n}^{\dagger}(\phi+d\phi)U_{ni}(B+dB)\varphi_{n+\hat{\imath}}(\phi+d\phi) , \tag{3.17}$$

which yields from (3.7a)

$$\delta_{R}A_{ni}^{a}=\delta_{R}\phi_{n+\hat{\imath}}^{a}-R_{ab}(V_{ni}^{\dagger})\delta_{R}\phi_{n}^{b}+R_{ab}(\varphi_{n+\hat{\imath}}^{\dagger})\delta_{R}B_{ni}^{b} \tag{3.18}$$

$$\equiv\sum_{m}\mathscr{D}_{nmi}^{ab}\delta_{R}\phi_{m}^{b}+R_{ab}(\varphi_{n+\hat{\imath}}^{\dagger})\delta_{R}B_{ni}^{b} . \tag{3.19}$$

From (3.18) and (3.19), we have the lattice covariant *forward* derivative operator $\mathscr{D}_{i}$ given by

$$\mathscr{D}_{nmi}^{ab}=\langle n,a \mid \mathscr{D}_{i} \mid m,b\rangle \tag{3.20}$$

$$=\delta_{ab}\delta_{n+\hat{\imath},m}-R_{ab}(V_{ni}^{\dagger})\delta_{nm} . \tag{3.21}$$

From (3.14) and (3.19), we have

$$\sum_{m,i,b}\langle n,a \mid \Gamma_{i}\mathscr{D}_{i} \mid m,b\rangle\delta_{R}\phi_{m}^{b}+\sum_{m,i,b}\langle n,a \mid \Gamma_{i}R_{i}^{T} \mid m,b\rangle\delta_{R}B_{mi}^{b}=0 , \tag{3.22}$$

where T stands for transpose and

$$\langle n,a \mid R_{i} \mid m,b\rangle=\delta_{nm}R_{ab}(\varphi_{n+\hat{\imath}}) . \tag{3.23}$$

Hence, from (3.22) we have

$$\frac{\delta_{R}\phi_{n}^{a}}{\delta_{L}B_{mj}^{b}}=-\left\langle n,a\left|\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}R_{j}^{T}\right|m,b\right\rangle , \tag{3.24}$$

where $(\Gamma\cdot\mathscr{D})^{-1}$ is the inverse of operator $\sum_{i}\Gamma_{i}\mathscr{D}_{i}$. We also have from (3.19) and (3.24)

$$\frac{\delta_{R}A_{ni}^{a}}{\delta_{L}B_{mj}^{b}}=-\left\langle n,a\left|\left[\mathscr{D}_{i}\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}R_{j}^{T}-R_{j}^{T}\delta_{ij}\right]\right|m,b\right\rangle . \tag{3.25}$$

Hence, from (3.11), (3.24), and (3.25)

$$\frac{\delta}{\delta_{L}B_{mj}^{b}}=\sum_{n,i}\left\langle n,a\left|\left[\delta_{ij}-\mathscr{D}_{i}\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}\right]R_{j}^{T}\right|m,b\right\rangle\frac{\delta}{\delta_{L}A_{ni}^{a}}-\sum_{n}\left\langle n,a\left|\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}R_{j}^{T}\right|m,b\right\rangle\frac{\delta}{\delta_{L}\phi_{n}^{a}} . \tag{3.26}$$

Equation (3.26) provides the solution for expressing the unconstrained chromoelectric operator $\delta/\delta_{L}B$ in terms of the new constrained operator $\delta/\delta_{L}A$ and the gauge transformation $\delta/\delta_{L}\phi$. In essence, this solves the problem of gauge fixing the lattice Hamiltonian. Note that from (3.25) we have the identity

$$\sum_{n,i}\langle l,c \mid \Gamma_{i} \mid n,a\rangle\frac{\delta_{R}A_{ni}^{a}}{\delta_{L}B_{mj}^{b}}=0 \tag{3.27}$$

as expected. We have from (3.14)

$$\sum_{m,i}\langle n,a \mid \Gamma_{i} \mid m,b\rangle\frac{\delta}{\delta_{L}A_{mi}^{b}}=0 . \tag{3.28}$$

Hence, from (2.12) and (3.28)

$$\left[\frac{\delta}{\delta_{L}A_{ni}^{a}},A_{mj}^{b}\right]=\left[\delta_{mm}\delta_{ij}\delta_{ac}-\left\langle n,a\left|\Gamma_{i}^{T}\frac{1}{\Gamma\cdot\Gamma^{T}}\Gamma_{j}\right|m,c\right\rangle\right]e_{cb}^{L}(A_{mj}) . \tag{3.29}$$

IV. GAUSS'S LAW

We check that constrained variables V_{ni} decouple from Gauss's law. Recall from (2.7) and (3.26), we have

$$\sum_{m,j}\langle l,c \mid D_{j} \mid m,b\rangle\frac{\delta}{\delta_{L}B_{mj}^{b}}=\sum_{n,ij}\left\langle n,a\left|\left[\delta_{ij}-\mathscr{D}_{i}\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}\right]R_{j}^{T}D_{j}^{T}\right|l,c\right\rangle\frac{\delta}{\delta_{L}A_{ni}^{a}}-\sum_{n,ij}\left\langle n,a\left|\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_{j}R_{j}^{T}D_{j}^{T}\right|l,c\right\rangle\frac{\delta}{\delta_{L}\phi_{n}^{a}} . \tag{4.1}$$

From the definitions of D_i and $\mathscr{D}_i$ given in (2.7) and (3.21), respectively, we have the crucial operator identity

$$R_j^T D_j^T = \mathscr{D}_j R^T , \tag{4.2}$$

where

$$\langle n,a \mid R \mid m,b \rangle = \delta_{nm} R_{ab}(\varphi_n) . \tag{4.3}$$

Hence, from (4.2) we see that the first term in (4.1) is zero and we have

$$\sum_{m,j} D_{lmj}^{cb} \frac{\delta}{\delta_L B_{mj}^b} = -R_{cb}(\varphi_l) \frac{\delta}{\delta_L \phi_l^b} \tag{4.4}$$

$$= -\frac{\delta}{\delta_R \phi_l^c} . \tag{4.5}$$

We see that V_{ni} has decoupled from Gauss's constraint, and we have from (2.6) and (4.5)

$$\left[\frac{\delta}{i\delta_R \phi_n^a} + \rho_{na} \right] | \Phi \rangle = 0 . \tag{4.6}$$

Solving (4.6), we have from (3.1) (Ref. 6)

$$\Phi(\bar{\psi},\psi,U) = \exp\left[-i \sum_n \rho_{na} \phi_n^a \right] \Phi(\bar{\xi},\xi,V) , \tag{4.7}$$

since, using (2.5e)

$$\frac{\delta}{i\delta_R \phi^a} \exp(i\phi^\alpha \rho_\alpha) = \rho_a \exp(i\phi^\alpha \rho_\alpha) . \tag{4.8}$$

The change of variables from $\{ U_{ni} \}$ to $\{ V_{ni}, \varphi_n \}$ has a Jacobian given by the Faddeev-Popov determinant, and can be shown to be equal to[7]

$$\mathscr{J}^{-1}[V] = \prod_n \int d\varphi_n \prod_{n,a} \delta[\chi_n^a(\varphi_n V_{ni} \varphi_{n+\hat{i}}^\dagger)] . \tag{4.9}$$

For weak coupling, $\mathscr{J}[V]$ has been evaluated to $O(A^2)$ in Ref. 7. Hence we have (suppressing the fermion variables) for some gauge-invariant operator G and gauge-invariant state $| \Phi \rangle$, from (3.1) and (4.7)

$$\langle \Phi | G | \Phi \rangle = \prod_{n,i} \int dU_{ni} \Phi^*[U] G[U,\delta/\delta U] \Phi[U] \tag{4.10}$$

$$= \prod_{n,i} \int dV_{ni} \prod_{n,a} \delta[\chi_n^a(V)] \left[\Phi^*[V] \mathscr{J}^{1/2}[V] \exp\left[i \sum_n \phi_n^a \rho_{na} \right] \right]$$
$$\times (\mathscr{J}^{1/2}[V] \hat{G}[V,\delta/\delta V] \mathscr{J}^{-1/2}[V]) \left[\exp\left[-i \sum_n \phi_n^a \rho_{na} \right] \mathscr{J}^{1/2}[V] \Phi[V] \right] . \tag{4.11}$$

Hence, the effective wave functional with no Jacobian is[3,10,11]

$$\tilde{\Phi}[V] = \mathscr{J}^{1/2}[V] \Phi[V] \tag{4.12}$$

and the effective operator is

$$\tilde{G} = \mathscr{J}^{1/2}[V] \exp\left[i \sum_n \phi_n^a \rho_{na} \right] \times G \exp\left[-i \sum_n \phi_n^a \rho_{na} \right] \mathscr{J}^{-1/2}[V] , \tag{4.13}$$

such that

$$\langle \Phi | G | \Phi \rangle = \langle \tilde{\Phi} | \tilde{G} | \tilde{\Phi} \rangle . \tag{4.14}$$

V. GAUGE-FIXED LATTICE HAMILTONIAN

We need to evaluate the kinetic operator given from (2.9), (2.10), and (2.12b) as (summing on all repeated indices)

$$K = \frac{\delta}{i\delta_L B_{ni}^a} \frac{\delta}{i\delta_L B_{ni}^a} . \tag{5.1}$$

Let us symbolically write the transformation (3.26) as

$$\frac{\delta}{\delta_L B_p} = L_{pq} \frac{\delta}{\delta_L C_q} . \tag{5.2}$$

Then from (5.1) and (5.2)

$$K = L_{pq} \frac{\delta}{i\delta_L C_q} \left[L_{pq'} \frac{\delta}{i\delta_L C_{q'}} \right] \tag{5.3}$$

$$= \frac{1}{L} \frac{\delta}{i\delta_L C_q} \left[L L_{qp}^T L_{pq'} \frac{\delta}{i\delta_L C_{q'}} \right] , \tag{5.4}$$

where

$$L = \det || L_{ab} || . \tag{5.5}$$

For the transformation given by (3.26) we have

$$L = \mathscr{J}[V] \tag{5.6}$$

and the Jacobian $\mathscr{J}$ is given by (4.9). The choice of operator ordering given by (5.4) allows for further simplifications. Recall that from (3.28) that $\delta/\delta_L A_{ni}^a$ is "transverse;" using this equation and Eq. (5.4), we have

$$K=\frac{1}{\mathscr{J}}\frac{\delta}{i\delta_L A^a_{ni}}\left[\mathscr{J}\frac{\delta}{i\delta_L A^a_{ni}}\right]$$
$$+\frac{1}{\mathscr{J}}\left[\frac{\delta}{i\delta_L\phi^a_n}+\frac{\delta}{i\delta_L A^{a'}_{n'i}}\mathscr{D}^{a'a}_{n'ni}\right]\mathscr{J}\left\langle n,a\left|\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma_j\Gamma^T_j\frac{1}{\mathscr{D}^T\cdot\Gamma^T}\right|m,b\right\rangle\left[\frac{\delta}{i\delta_L\phi^b_m}+\mathscr{D}^{Tbb'}_{mm'k}\frac{\delta}{i\delta_L A^{b'}_{m'k}}\right]. \tag{5.7}$$

The effective Hamiltonian, using (4.13), is given by

$$\widetilde{H}=\mathscr{J}^{1/2}e^{i\phi^a_n\rho_{na}}He^{-i\phi^a_n\rho_{na}}\mathscr{J}^{-1/2}. \tag{5.8}$$

Note that

$$e^{i\phi^a_n\rho_{na}}\frac{\delta}{i\delta_L\phi^b_m}e^{-i\phi^a_n\rho_{na}}=-\rho_{mb}. \tag{5.9}$$

We hence have the final expression for the gauge-fixed lattice Hamiltonian given by

$$\widetilde{H}=+\frac{g^2}{2a}\left[\mathscr{J}^{-1/2}\frac{\delta}{i\delta_L A^a_{ni}}\left[\mathscr{J}\frac{\delta}{i\delta_L A^a_{ni}}\mathscr{J}^{-1/2}\right]\right.$$
$$\left.+\mathscr{J}^{-1/2}\left[\frac{\delta}{i\delta_L A^{a'}_{n'i}}\mathscr{D}^{a'a}_{n'ni}-\rho_{na}\right]\mathscr{J}\left\langle n,a\left|\frac{1}{\Gamma\cdot\mathscr{D}}\Gamma\cdot\Gamma^T\frac{1}{\mathscr{D}^T\cdot\Gamma^T}\right|m,b\right\rangle\left[\mathscr{D}^{Tbb'}_{mm'k}\frac{\delta}{i\delta_L A^{b'}_{m'k}}-\rho_{mb}\right]\mathscr{J}^{-1/2}\right]$$
$$+P(\bar{\xi},\xi,V). \tag{5.10}$$

The wave functionals depend on only the constrained variables V_{ni}, i.e.,

$$\widetilde{\Phi}=\widetilde{\Phi}(\bar{\xi},\xi,V). \tag{5.11}$$

Recall we have from (3.29) the commutation equation

$$\left[\frac{\delta}{\delta_L A^a_{ni}},A^b_{mj}\right]=\left[\delta_{nm}\delta_{ij}\delta_{ac}-\left\langle n,a\mid\Gamma^T_i\frac{1}{\Gamma\cdot\Gamma^T}\Gamma_j\mid m,c\right\rangle\right]e^L_{cb}(A_{mj}). \tag{5.12}$$

Equations (5.10)–(5.12) completely define the gauge-fixed Hamiltonian for the SU(N) lattice gauge field. The redundant gauge degrees of freedom $\{\varphi_n\}$ have completely decoupled from the system, as expected. The expression for $\widetilde{H}$ in (5.10) is exact, and is equally valid for strong and weak couplings. Comparing (5.1) and (5.7), we see that the coordinates $\{U_{ni}\}$ are analogous to Cartesian coordinates for the gauge field whereas coordinates $\{V_{ni}\}$ are analogous to curvilinear coordinates.[3]

The quark-color charge ρ_{na} has the instantaneous nonlocal non-Abelian lattice Coulomb potential $(\Gamma\cdot\mathscr{D})^{-1}\Gamma\cdot\Gamma^T(\mathscr{D}^T\cdot\Gamma^T)^{-1}$. As pointed out by Gribov,[12,13] in the continuum theory the operator $\Gamma\cdot\mathscr{D}$ develops a zero eigenvalue for strong gauge-field configurations $A^a_{ni}\gg 0$, and which is due to the existence of multiple gauge-equivalent transverse-gauge-field configurations. For the lattice, presumably the same phenomena exists, and hence the gauge-fixed lattice Hamiltonian is at least valid for weak-gauge-field configurations.[3]

One can also choose the spatial axial gauge for the lattice, but this still leaves a residual gauge invariance which is difficult to impose.[14]

VI. SUMMARY

We exactly gauge fixed the non-Abelian lattice Hamiltonian, and obtained a theory which is regularized to all orders and hence the eigenenergies and eigenfunctionals can be renormalized order by order using weak-coupling perturbation theory.[15] The gauge-fixed form is particularly suited for weak-coupling perturbation theory. We can also study the Gribov problem on the lattice using the gauge-fixed lattice Hamiltonian.

The gauge-fixed (Coulomb) lattice Hamiltonian can be used to study nonperturbative[11] properties of the gauge field. In particular we have obtained the non-Abelian Coulomb potential regularized to all orders, and it should contain information as to how the theory confines quarks.[1]

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