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Variational method for field theories on the lattice and the spectrum of the ϕ^4 theory in $1+1$ dimensions

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An approximation technique to construct the low-lying energy eigenstates of any bosonic field theory on the lattice is proposed. It is based on the SLAC blocking method, after performing a finite-spin approximation to the individual degrees of freedom of the problem. General expressions for any polynomial self-interacting theory are given. Numerical results for ϕ^2 and ϕ^4 theories in $1+1$ dimensions are offered; they exhibit a fast convergence rate. The complete low-lying energy spectrum of the ϕ^4 theory in $1+1$ dimensions is calculated.

I. INTRODUCTION

In the past few years, we have seen a marked increase in the study of approximate, nonperturbative methods in quantum field theories. These methods are particularly important for the study of physical issues, such as quark confinement in QCD, computation of hadron masses, etc. Roughly speaking, we may divide these techniques into two large groups: those inspired by semiclassical approximations, and those based in the lattice strategy.[1-3] In this second group, perhaps the most notable results have been those coming from Monte Carlo simulations for the spectrum of gauge theories, using techniques developed by Creutz.[4] Similar Monte Carlo techniques have been used for scalar theories in different space-time dimensionality, with important results in particular for the $\lambda\phi^4$ theory.[5] A severe limitation of Monte Carlo techniques is the determination of excited-state masses for given quantum numbers. This is because Monte Carlo techniques typically look at numerically evaluated propagators at large distances, where they become exponentials in the mass of the lowest-lying state. But higher masses may not be separated unambiguously.

In this paper we develop and apply a technique for determining all the rest-state ($\mathbf{P}=0$) energies of a quantum field theory on the lattice. It is based on the Hamiltonian blocking procedure of Drell, Weinstein, and Yankielowicz[6] which constructs approximate ground and first excited states for lattice theories. The basic philosophy of their method consists in projecting the Hamiltonian of the theory into a truncated space of suitably chosen states; this leads to an effective new Hamiltonian. This process is repeated successively producing a series of effective Hamiltonians whose coupling parameters change at each step. The procedure is iterated until entering a regime which can be handled simply by perturbation theory. The criterion to choose the retained states is implemented at each stage, by dissecting the lattice into small sets of degrees of freedom (blocks), which are diagonalized; the trial space of states for the whole lattice is expanded by the product of the low-lying states of the individual blocks. Depending on the number and nature of the states one retains in each Hamiltonian block, one may be driven to effective Hamiltonians which maintain the original form, or in general to the appearance of an increasing number of new couplings, induced by renormalization. This second possibility may be a source of difficulty and even may render the method impractical. In this paper we wish to remark that a straightforward and efficient way to apply this philosophy to boson theories consists in performing a finite-spin approximation to the individual degrees of freedom. Even in the most unfavorable situations such as the massless free theory, we have observed a fast convergence of the results with the number of retained states per site (size of the spin). Once one sticks to a definite size of the spin, this fixes the largeness of the basic matrices one has to deal with, without any other additional worries such as new couplings, looseness of locality, etc. In addition, the results are improved at will by increasing the size of both spin and block. In this sense the finite-spin strategy looks preferable to other variational methods[7,8]—similar to this spirit of thinning degrees of freedom to those which deal always with the boson degree of freedom, by performing canonical transformations between the block variables and retaining the lowest energy.

The method may be summarized as follows. After diagonalizing the original Schrödinger problem of one (n) degree of freedom, we retain a certain number (e) of its low-lying energy eigenstates. With those e states, we construct a finite truncated trial space for a two ($2n$) degrees-of-freedom problem, which may be diagonalized because all the required information is known from the previous stage. The new diagonalization renders new eigenstates, and we retain again the lowest ones. The process is iterated for successive scales until attaining a convergence for the ground-state energy density, gaps, etc. We never construct effective Hamiltonians, but only matrix elements of the original Hamiltonian—at different scales—in the trial space built from the previous stage.

In a renormalization-group spirit[3,6] one should analyze the evolution of the couplings (matrix elements) generated in the process. Should the number of retained states be arbitrarily large, that would imply a complicated work. Therefore we stick to a direct Rayleigh-Ritz spirit of the approximate method, looking simply at the energy eigenvalues of the theory. The general formalism appears in Sec. II. Sections III and IV are devoted to the numerical analysis to the ϕ^2 and ϕ^4 theories, respectively. Finally our conclusions are exposed in Sec. V.

II. GENERAL FORMALISM IN TWO DIMENSIONS

A general self-interacting bosonic theory with nearest-neighbor interactions is described, on a one-dimensional spatial lattice, by the Hamiltonian

$$H=\sum_{j=-\mathcal{N}}^{\mathcal{N}}[\tfrac{1}{2}p_j^2+V(x_j)-x_jx_{j+1}] \tag{1}$$

with the canonical equal-time commutation relations $[p_{j'},x_j]=-i\delta_{j'j}$. $V(x_j)$ is a polynomial in the coordinate x_j with appropriate restrictions. H represents an infinite set of quantum-mechanical problems in each coordinate x_j, coupled by the x_jx_{j+1} terms.

Without the existence of the gradient terms, all the structure of the spectrum of H would be given once we know the spectrum of

$$h=\tfrac{1}{2}p^2+V(x)\ . \tag{2}$$

Hence, as a first step we diagonalize h as accurately as possible. Let us assume that this is already known and that $|n\rangle$ are their eigenvectors

$$h|n\rangle=E_n|n\rangle\ . \tag{3}$$

In fact we only require the lowest-lying ones, say, up to e,

$$n=1,2,3,4,\ldots,e\ . \tag{4}$$

Let us define the matrix elements of the operator X by x_{ij},

$$x_{ij}=\langle i|X|j\rangle\ . \tag{5}$$

After knowing the low-energy structure of the 2^N-degrees-of-freedom problem, with $N=0$, we continue by describing approximately the case of $N=1$, based on the just-acquired information. Now, the Hamiltonian of the two degrees of freedom is

$$H_1=h_1+h_2-x_1x_2\ . \tag{6}$$

The trial wave functions for this problem will be the set expanded by the states

$$|K\rangle_{12}=A_{IJ}^K|I\rangle_1|J\rangle_2\ , \tag{7}$$

where the summation over indices is understood, and the coefficients A_{IJ}^K satisfy

$$A_{IJ}^KA_{IJ}^{K'}=\delta_{KK'}\ . \tag{8}$$

Then we have

$${}_{12}\langle L|h_1+h_2|K\rangle_{12}=D^{LK}=A_{IJ}^LA_{IJ}^K(E_I+E_J) \tag{9}$$

and

$${}_{12}\langle L|X_1X_2|K\rangle_{12}=F^{LK}=A_{I'J'}^LA_{IJ}^KX_{I'I}X_{J'J}\ , \tag{10}$$

$$X_{I'I}={}_1\langle I'|X_1|I\rangle_1\ , \tag{11}$$

so that

$$\langle L|H_{12}|K\rangle_{12}=D^{LK}-F^{LK} \tag{12}$$

by diagonalizing the former matrix, and retaining the e lowest eigenstates (denoted by $|I\rangle'_{12}$) we would have

$$H_{12}|I\rangle'_{12}=E'_I|I\rangle'_{12} \tag{13}$$

and the new eigenstates are related to $|K\rangle_{12}$ through

$$|I\rangle_{12}=B_{IJ}|J\rangle_{12}\ . \tag{14}$$

With them we form a new set of trial wave functions to describe the set of 2^2 degrees of freedom,

$$|M\rangle_{14}=A_{IJ}^M|I\rangle'_{12}|J\rangle'_{34}\ , \tag{15}$$

$$H_{14}=H_{12}+H_{34}-X_2X_3\ . \tag{16}$$

Using the basis $|M\rangle_{14}$ the matrix representation of H_{14} is done by

$${}_{14}\langle M'|H_{14}|M\rangle_{14}=D^{M'M}-F^{M'M}\ , \tag{17}$$

where

$$D^{M'M}=A_{IJ}^{M'}A_{IJ}^M(E'_I+E'_J)\ , \tag{18}$$

$$F^{M'M}=A_{I'J'}^{M'}A_{IJ}^MX'_{I'I}X'_{J'J}\ , \tag{19}$$

$$X'_{I'I}=B_{I'L'}A_{ST'}^{L'}A_{ST}^LX_{T'T}B_{IL}\ . \tag{20}$$

It is now clear that by successive iterations we shall arrive at the same equations which describe the finite-time evolution of a dynamical system

$$\begin{aligned}X_1(t+1)&=F_1(X_1(t),X_2(t),\ldots,X_n(t))\ ,\\ X_2(t+1)&=F_2(X_1(t),X_2(t),\ldots,X_n(t))\ ,\\ X_n(t+1)&=F_n(X_1(t),X_2(t),\ldots,X_n(t))\ ,\end{aligned} \tag{21}$$

but now the essential physical magnitudes are the eigenvalues of the successive Hamiltonians and the role of the time parameter is played by the number of iterations, that is by the lattice length

$$\begin{aligned}E_1^{(n+1)}&=F_1(E_1^{(n)},E_2^{(n)},\ldots,E_N^{(n)},X_{ij}^{(n)})\ ,\\ E_2^{(n+1)}&=F_2(E_1^{(n)},E_2^{(n)},\ldots,E_N^{(n)},X_{ij}^{(n)})\ ,\\ E_N^{(n+1)}&=F_N(E_1^{(n)},E_2^{(n)},\ldots,E_N^{(n)},X_{ij}^{(n)})\ ,\\ X_{ij}^{(n+1)}&=F'_N(E_1^{(n)},\ldots,E_N^{(n)},X_{ij}^{(n)})\ ,\end{aligned} \tag{22}$$

and we must look for the attractor points of the system. Explicitly,

$$H_{ij}^{(n+1)}=D_{ij}^{(n+1)}-F_{ij}^{(n+1)} \tag{23}$$

with

$$\begin{aligned}D_{ij}^{(n+1)}&=A_{kl}^IA_{kl}^J(E_I^{(n)}+E_J^{(n)})\ ,\\ F_{ij}^{(n+1)}&=A_{k'l'}^IA_{kl}^JX_{k'k}^{(n+1)}X_{L'L}^{(n+1)}\ ,\\ X_{K'K}^{(n+1)}&=B_{K'M'}^{(n)}A_{ST'}^{M'}A_{ST}^MX_{T'T}^{(n)}B_{KM}^{(n)}\ ,\end{aligned} \tag{24}$$

where

$$X^{(1)}_{K'K}=X_{K'K} \ . \tag{25}$$

The successive matrices are quite straightforwardly entered into a computer, and the convergence, for a fixed e, is observed by looking to one or several "observables" like energy density, gaps, etc.

A second convergence process is observed by increasing e, i.e., the number of retained states. Finally the process may be implemented by departing from the initial diagonalization of not one degree of freedom but from a small block. The observation of convergence in this trial of directions for a given observable would be a sign of real accuracy in the prediction.

The application of this method for a theory with a high number of spatial dimensions is, at least conceptually, quite straightforward. After performing the block choice, that in $2+1$ theory may be clusters of four sites per block, and in $3+1$ theory cubes containing eight sites each, and having defined the size of the spin as explained previously, one would diagonalize every block Hamiltonian and perform the truncation of states. In a $2+1$ theory the number of original links which form a new link would be $2(N-1)$ for any step in which we are processing 2^{N+1} sites, and the equivalent for $3+1$ dimensions. But this does not imply the storing of new information in memory but a simple multiplication of the $X_{II'}$ matrix by the number of sites. Aside from this difference all else is similar to the $1+1$ examples analyzed here. The only concept that would more complicate the result is the increasing number of sites per block, which would imply larger matrices to deal with.

III. ϕ^2 THEORY

In the free bosonic theory the potential V in Eq. (1) is

$$V=\tfrac{1}{2}(\mu^2+2)X^2 \ , \tag{26}$$

where μ is the mass parameter. The Hamiltonian of this field theory represents an infinite number of harmonic oscillators coupled by the gradient.

The exact solution of this theory describes a system of noninteracting oscillators of frequency

$$\omega_k=[\mu^2+4\sin^2(\tfrac{1}{2}k)]^{1/2} \ , \quad k=\frac{2\pi}{2\mathcal{N}+1}n \ , \qquad n=0,\pm1,\pm2,\ldots,\pm\mathcal{N} \ , \tag{27}$$

with ground-state energy density

$$\epsilon_0=\frac{1}{2\pi}\int_0^{\pi}dk[\mu^2+4\sin^2(\tfrac{1}{2}k)]^{1/2} \tag{28}$$

and a value for the gap exactly coincident with μ.

To begin the application of the approximate variational procedure, we notice that here all the information (E_n,X_{ij}) about the spectrum of the single-site problem is automatically known.

To illustrate in detail the general expressions of Sec. II, let us particularize them here for $e=2$. Thus the trial space for $N=2$ is

$$\begin{aligned}
&|1\rangle_{12}=|1\rangle_1|1\rangle_2 \ , \\
&|2\rangle_{12}=\frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2-|2\rangle_1|1\rangle_2) \ , \\
&|3\rangle_{12}=\frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2+|2\rangle_1|1\rangle_2) \ , \\
&|4\rangle_{12}=|2\rangle_1|2\rangle_2 \ ,
\end{aligned} \tag{29}$$

where $|1\rangle$ and $|2\rangle$ correspond to the ground state and first excitation of the harmonic oscillator with Hamiltonian

$$\begin{aligned}
&h=\frac{p^2}{2}+\tfrac{1}{2}(\mu^2+2)x^2 \ , \\
&h|1\rangle=E_1|1\rangle \ , \\
&h|2\rangle=E_2|2\rangle \ , \\
&A_{(0)}=(\langle 2|x|1\rangle)^2 \ .
\end{aligned} \tag{30}$$

Hence

$$H_{12}=\begin{bmatrix} 2E_1 & 0 & 0 & -A_{(0)} \\ 0 & E_2+E_1+A_{(0)} & 0 & 0 \\ 0 & 0 & E_2+E_1-A_{(0)} & 0 \\ -A_{(0)} & 0 & 0 & 2E_2 \end{bmatrix} . \tag{31}$$

Its lowest eigenvalues are

$$\begin{aligned}
&\lambda^{(1)}_1=E_2+E_1-[(E_2-E_1)^2+A_{(0)}{}^2]^{1/2} \ , \\
&\lambda^{(1)}_2=E_2+E_1-A_{(0)} \ , \\
&E_1=\lambda^{(0)}_1 \ , \\
&E_2=\lambda^{(0)}_2 \ ,
\end{aligned} \tag{32}$$

corresponding to

$$\begin{aligned}
&|1\rangle_{12}=\alpha_1|1\rangle_1|1\rangle_2+\beta_1|2\rangle_1|2\rangle_2 \ , \\
&|2\rangle_{12}=\frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2+|2\rangle_1|1\rangle_2) \ , \\
&\alpha_1=\frac{1}{[1+(\omega^{(1)})^2]^{1/2}} \ , \quad \beta_1=\omega^{(1)}\alpha_1 \ , \\
&\omega^{(1)}=\frac{1}{A_{(0)}}\{E_1-E_2+[(E_1-E_2)^2+A_{(0)}{}^2]^{1/2}\} \ .
\end{aligned} \tag{33}$$

The Hamiltonian of the 2^2 block is

$$H_{14}=h_{12}+h_{34}-x_2x_3 \tag{34}$$

to be diagonalized in the truncated space expanded by

$$\begin{aligned}
&|1\rangle_{14}=|1\rangle_{12}|1\rangle_{34} \ , \\
&|2\rangle_{14}=\frac{1}{\sqrt{2}}(|1\rangle_{12}|2\rangle_{34}-|2\rangle_{12}|1\rangle_{34}) \ , \\
&|3\rangle_{14}=\frac{1}{\sqrt{2}}(|1\rangle_{12}|2\rangle_{34}+|2\rangle_{12}|1\rangle_{34}) \ , \\
&|4\rangle_{14}=|2\rangle_{12}|2\rangle_{34} \ ,
\end{aligned} \tag{35}$$

the new Hamiltonian matrix is

$$H_{14}=\begin{bmatrix} -2\lambda_1^{(1)} & 0 & 0 & -A_{(1)} \\ 0 & \lambda_1^{(1)}+\lambda_2^{(1)}+A_{(1)} & 0 & 0 \\ 0 & 0 & \lambda_1^{(1)}+\lambda_2^{(1)}+A_{(1)} & 0 \\ -A_{(1)} & 0 & 0 & -2\lambda_2^{(1)} \end{bmatrix} . \tag{36}$$

Therefore, in general we would have

$$\lambda_1^{(n+1)}=\lambda_1^{(n)}+\lambda_2^{(n)}-[(\lambda_1^{(n)}-\lambda_2^{(n)})^2+A_{(n)}{}^2]^{1/2} ,$$
$$\lambda_2^{(n+1)}=\lambda_1^{(n)}+\lambda_2^{(n)}-A_{(n)} ,$$
$$G_{AP}{}^{(n+1)}=[(\lambda_1^{(n)}-\lambda_2^{(n)})^2+A_{(n)}{}^2]^{1/2}-A_{(n)} , \tag{37}$$
$$A_{(n)}=\tfrac{1}{2}A(n-1)\frac{(1-\omega^{(n)})^2}{1+(\omega^{(n)})^2} ,$$
$$\omega^{(n)}=\frac{1}{A_{(n-1)}}\{\lambda_1^{(n)}-\lambda_2^{(n)}+[(\lambda_1^{(n)}-\lambda_2^{(n)})^2+A_{(n-1)}{}^2]^{1/2}\} .$$

And the only information we need, to trigger the iterative relations is

$$\lambda_1^{(0)} , \quad \lambda_2^{(0)} , \quad A_{(0)} . \tag{38}$$

In Tables I and II we collect the numerical values obtained for the ground-state energy density and the mass gap, for different choices of e. In Fig. 1 the mass gap is plotted in the successive approximations, showing the fast rate of convergence of this method. Except for the simplest case of two states retained at each step ($e=2$), in general ($e>2$) we have performed the calculations in the center-of-mass system of the particles ($\mathbf{P}=0$); this is implemented by retaining symmetric states (under exchanges of site coordinates) only. In addition to the advantage of being able to observe the second mass gap clearly, because of the absence of $\mathbf{P}\neq 0$ states, we have the bonus of saving memory and time of CPU in the computer. The obtained results for the second excited states agree with the expectations of having a second mass gap identical to the first one, $E_2-E_0=2(E_1-E_0)$, and the onset of the continuum spectrum after this second excitation.

Negative values for μ in Fig. 1, should be understood in the sense of inserting a minus sign to the mass term of the Hamiltonian ($\mu^2x^2\rightarrow-\mu^2x^2$). The prediction of the critical point in that region is a product of our approximations. Near the critical point ($\mu=0$), the spacing of the energy levels of the primary oscillators tends to zero and that is why our finite-spin approximation is particularly bad there. However, increasing e quickly corrects that defect, as is apparent in Fig. 1. This situation, in which all the energy spacings of the one degree of freedom tend to zero, does not appear in the next application (ϕ^4 theory) and therefore it is reasonable to expect that the point $\mu=0$ of the free theory is the worst possible situation for this method to be applied.

IV. ϕ^4 THEORY

For the potential V [Eq. (1)] in the ϕ^4 theory, we choose

$$V=\tfrac{1}{2}(-\mu^2+2)x^2+\lambda x^4 . \tag{39}$$

In all the numerical analysis we adopt $\lambda=1$, and μ^2 works as the coupling constant of the model. The Hamiltonian (1) of this lattice field theory represents an infinite number of double-well anharmonic oscillators coupled by the gradient. The critical behavior of this system is well known; it possesses a second-order phase transition for a certain value of the parameter $\mu=\mu^*$, and for $\mu>\mu^*$ the theory is spontaneously broken. The first gap of the theory vanishes at the critical point, corresponding to an

TABLE I. Convergence of the ground-state energy density of the free bosonic theory. μ is the mass parameter; i in $\epsilon(i)$ is the number of states of the individual harmonic oscillators, retained in this method.

μ	$\epsilon(2)$	$\epsilon(4)$	$\epsilon(6)$	Exact
0.0	0.6784	0.6703	0.6698	0.6366
0.1	0.6804	0.6726	0.6722	0.6405
0.2	0.6864	0.6793	0.6790	0.6499
0.3	0.6962	0.6901	0.6899	0.6636
0.4	0.7096	0.7045	0.7045	0.6809
0.5	0.7264	0.7223	0.7223	0.7014
0.6	0.7463	0.7431	0.7430	0.7245
0.7	0.7689	0.7665	0.7665	0.7501
0.8	0.7942	0.7923	0.7923	0.7778
0.9	0.8216	0.8202	0.8202	0.8074
1.0	0.8511	0.8501	0.8500	0.8388
2.0	1.2197	1.2196	1.2196	1.2160
3.0	1.6563	1.6563	1.6563	1.6548
4.0	2.1203	2.1203	2.1203	2.1196
5.0	2.5975	2.5975	2.5975	2.5971

TABLE II. Convergence of the mass gap of the free bosonic theory.

μ	$m(2)$	$m(4)$	$m(6)$
0.0	0.6290	0.1586	0.062 36
0.1	0.6351	0.1820	0.113 35
0.2	0.6529	0.2419	0.203 83
0.3	0.6821	0.3239	0.301 29
0.4	0.7217	0.4127	0.400 50
0.5	0.7706	0.5092	0.500 18
0.6	0.8277	0.6051	0.600 12
0.7	0.8919	0.7029	0.700 05
0.8	0.9621	0.8021	0.800 01
0.9	1.0374	0.9026	0.900 00
1.0	1.1168	1.0016	1.000 00
2.0	2.0290	2.0000	2.000 00
3.0	3.0103	3.0000	3.000 00
4.0	4.0047	4.0000	4.000 00
5.0	5.0025	5.0000	5.000 00

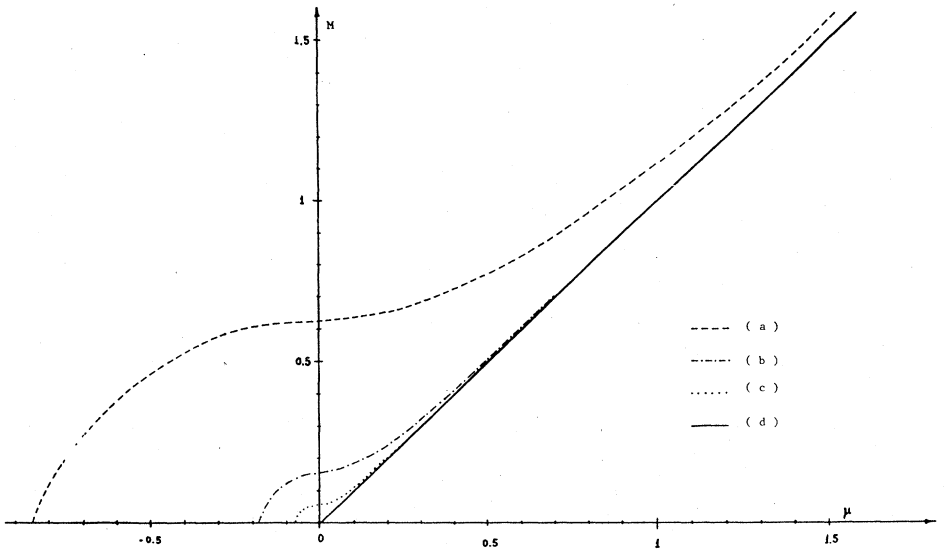

FIG. 1. A plot of the convergence of the mass gap for the free bosonic theory. (a) Two states retained per site. (b) Four. (c) Six. (d) Exact result.

infinite correlation length. With respect to the second gap, it has been established recently that in the region where the theory has only one vacuum state $\mu<\mu^*$ there are no bound states. This means that $M_2=2M_1$; i.e., the second excited state corresponds to having two particles at rest, each one having a mass given by the first gap. It does not mean that we have a free theory, but the absence of bound states reflects the repulsive nature of the interaction between the particles.[9–11]

Previous attempts to study, in an approximate way, the ϕ^4 theory by a blocking Hamiltonian procedure[8] correctly describe several qualitative properties of the theory, but it is difficult to improve systematically the results. With the triad of convergence mentioned previously, it is simple to compute accurately different magnitudes. In particular, the description of the excitations seems one of the best achievements of this method.

The machinery developed in Sec. II is directly applied to this theory, after having diagonalized the Hamiltonian of the double-well oscillator

$$h=\frac{p^2}{2}+\tfrac{1}{2}(-\mu^2+2)+x^4 \tag{40}$$

for different values of μ^2. To obtain in a quick way the eigenvalues and eigenvectors of (40), we apply the method explained in Ref. 12. There it is shown how to use an appropriately scaled harmonic basis to obtain a quick convergence of the double-well problem.

As explained in Sec. III for the free case, we retain only symmetric states, in order to construct spatially homogeneous wave functions.[13]

In Fig. 2 the convergence process for the first mass gap to the left of the critical point is shown. Similarly, the convergence of the ground-state energy density appears in

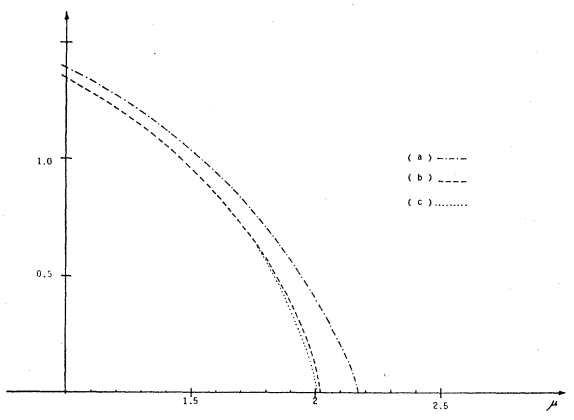

FIG. 2. A plot of the convergence of the mass gap of the ϕ^4 theory, at the "left-hand side" of the critical point (nonmagnetized region). (a) Two states retained per site. (b) Four. (c) Six.

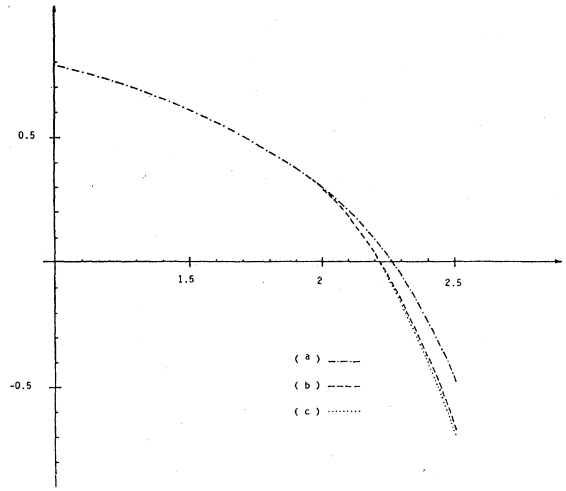

FIG. 3. A plot of the convergence of the ground-state energy density of the theory. (a) Two states retained per site. (b) Four. (c) Six.

TABLE III. Convergence of the ground-state energy density of the theory when increasing the number of retained states from 6 to 12.

μ	$\epsilon(6)$	$\epsilon(8)$	$\epsilon(12)$
1.5	0.614 471	0.614 470	0.614 469
1.8	0.447 220	0.447 210	0.447 213
2.0	0.292 488	0.292 431	0.292 415
2.2	0.023 958	0.022 547	0.021 908
2.4	−0.410 070	−0.413 113	−0.414 818
3.0	−2.924 330	−2.931 38	−2.935 573
3.5	−6.873 962	−6.882 21	−6.884 320
4.5	−22.413 88	−22.417 21	−22.417 560

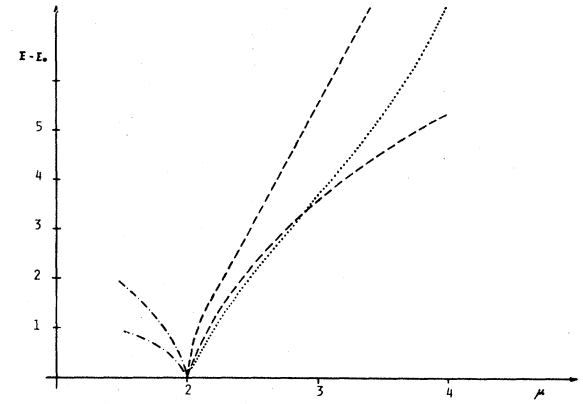

FIG. 4. Low-lying spectrum of the 1 + 1 theory. It exhibits the "freelike" nature of this model in the nonmagnetized regions and the crossing between the kink and particle levels for $\mu \simeq 2.8$.

Fig. 3. In Fig. 4 the final results for the low-lying energy spectrum are plotted.

In Table III we show the convergence rate for the ground-state energy density as we increase the number of retained states from 6 to 12; there is a good convergence for all values of μ so we believe our results are a good approximation for the exact ground-state energy density. In Table IV we collect our results for this theory when we retain 12 states per site. ϵ_0 is our converged prediction for the ground-state energy density and M_i is the difference between the E_i and E_{i-1} eigenvalues of the theory; M_i shows the onset of the spontaneous breaking of the symmetry ($\phi \to -\phi$) for $\mu > \mu^* = 2.003$, i.e., $|2I\rangle$ and $|2I+1\rangle$ ($I=0,1,\ldots$) are degenerate. For $\mu < \mu^*$ we see that the second excitation (at rest) corresponds to having two particles of equal mass and then no bound states appear in the theory so the effective quantum potential must be either repulsive or freelike. Furthermore $M_3 = E_3 - E_2$, $M_4 = E_4 - E_3, \ldots$ vanish which should be understood as the onset of the continuous spectrum of two particles (in the center-of-mass system) due to the relative motion. In order to elucidate the nature of such states in this region we have computed the Bethe-Salpeter wave function of the different excitations, i.e., $\langle 0|\phi|k\rangle$, and in particular the expectation value of the field ϕ on a given state; then we observe that

$$\begin{aligned} &\langle k|\phi|k\rangle = 0 , \\ &\langle 0|\phi|1\rangle = \frac{1}{\sqrt{L}}\phi_0(\mu) , \\ &\langle 0|\phi|k\rangle = 0 \quad (k \neq 1) , \end{aligned} \tag{41}$$

where L is the lattice length, and at the same time

$$\langle 1|\phi|2\rangle = \frac{1}{L}\phi_0(\mu) \tag{42}$$

which corresponds to the interpretation of $|1\rangle$ or $|2\rangle$ as the one- (two-) particle state. In Table V the values for $\langle 0|\phi|0\rangle$ and $\phi_0(\mu)$ are listed.

Things are different for $2.8 \geq \mu \geq \mu^*$ (see Fig. 4) where

TABLE IV. Numerical values of the low-lying spectrum of the 1 + 1 ϕ^4 theory. 12 states per site have been retained.

μ	ϵ_0	M_1	M_2	M_3	M_4	M_5
1.5	0.614 469	0.956 12	0.956 12	0	0	0
1.6	0.565 112	0.844 33	0.844 33	0	0	0
1.8	0.447 213	0.560 72	0.560 72	0	0	0
2.0	0.292 415	0.024 4	0.044 4	0	0	0
2.1	0.177 585	0	0.666 60	0.042 72	0	0.290 58
2.2	0.021 908	0	1.074 5	0.069 32	0	0.467 91
2.4	−0.414 818	0	1.769 40	0.115 59	0	0.769 11
2.6	−1.035 459	0	2.417 88	0.128 48	0	1.080 45
2.8	−1.865 651	0	3.062 40	0.049 71	0	1.481 48
2.9	−2.368 764	0	3.358 77	0	0.034 19	1.696 48
3.0	−2.935 573	0	3.610 45	0	0.113 28	1.861 87
3.2	−4.278 119	0	4.028	0	0.375	2.202 0
3.5	−6.884 320	0	4.519 5	0	0.966	2.743 1
4.0	−13.145 436	0	5.304	0	2.149	3.153
4.5	−22.417 560	0	6.081	0	3.562	2.510
5.0	−35.496 125	0	6.842	0	5.224	1.617
5.5	−53.271 007	0	7.590	0	7.141	0.446
6.0	−76.726 175	0	8.325 0	0	8.325 0	0

TABLE V. Numerical values for the "magnetization" and one-particle amplitudes in the $1+1$ ϕ^4 theory.

μ	$\lvert\langle 0\lvert\phi\rvert 0\rangle\rvert$	Φ_0
1.5	0	0.717
1.8	0	0.928
2.0	0	3.882
2.4	0.929	0.342
2.5	1.012	0.326
3.0	1.354	0.304
3.2	1.475	0.299
3.5	1.651	0.298
4.0	1.929	0.288
6.0	2.970	0.243
10	4.989	0.188

the spectrum looks typically like a degenerate ground state ($M_1=0$) followed by a nondegenerate state $|2\rangle$ ($E_3\neq E_2$), and above these three several bound and degenerate states appear ($E_4=E_3, E_6=E_5, \ldots$). In this case we observe that

$$\langle 2|\phi|k\rangle=0\ ,\quad k=0,1,\ldots,$$

so the field does not connect to the $|2\rangle$ state either with itself or with any other; this enforces the interpretation of these states as a symmetric combination of kinklike and anti-kink-like states (because we have taken only spatially symmetric wave functions). The corresponding antisymmetric combination would be rejected in our algorithm since it does not satisfy the symmetric boundary condition; that is the reason why the $|2\rangle$ state appears as nondegenerate.[14] In what follows we shall call such a state a kink state. Furthermore,

$$\langle 0|\phi|3\rangle=\frac{1}{\sqrt{L}}\phi_3(\mu)\ ,$$
$$\langle 1|\phi|4\rangle=\frac{1}{\sqrt{L}}\phi_3(\mu)\ , \tag{43}$$

whereas

$$\langle 0|\phi|4\rangle=\langle 1|\phi|3\rangle=0 \tag{44}$$

so $|3\rangle$ and $|4\rangle$ are the first particle-like excitations; then if $2.8\geq\mu\geq\mu^*$ the first excitation is a kinklike state whereas for $\mu>2.8$ the first excitation is a particlelike state (this is what is expected from semiclassical analysis[15]).

With respect to the $\langle I|\phi|I\rangle$ expectation value we can fit the results of Table V as

$$\langle I|\phi|I\rangle=\left[\left[\frac{\mu}{\mu^*}\right]^{\alpha}-1\right]^{1/\alpha}\quad(|I\rangle\neq|\text{kink}\rangle)\ ,$$
$$\alpha\simeq 3.22\ . \tag{45}$$

Then for $\mu\geq 4$ we obtain $\langle 0|\phi|0\rangle\simeq\mu/2$ and $M_1\simeq\sqrt{2}\mu$, in agreement with the semiclassical expectations, because the potential wells are very profound.

V. CONCLUSIONS

We have presented a variational method to study lattice field theories. It is inspired by block-spin techniques, but it never constructs effective Hamiltonians. On the contrary, it builds variational wave functions for progressively extended and correlated length scales of the theory.

From the convergence observed in the free case where near $\mu=0$ one should expect the worst possible performance of the method, one deduces that our results for the ϕ^2 theory are within an error of 6% in the ground-state energy density. The nice results also obtained for the ϕ^4 theory are a check of the freelike nature of the model for $\mu<\mu^*$ and a verification of the level crossing for the $\mu>\mu^*$ predicted by semiclassical arguments.

With respect to the method itself we observe a number of clear advantages with respect to other approaches. As mentioned previously, it never leads to the appearance of a new coupling, which is one of the shortcomings of block-spin techniques and furthermore the method can be refined to whatever degree desired, simply by increasing the size of the matrices to be diagonalized.

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