Conformal invariance and string theory in compact space: Bosons

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(Received 19 February 1985)

The area law of the Nambu-Goto string is generalized to include a solid-angle-type term, which is purely topological in nature. Such a term exists and is unique provided the manifold M in which the string lives satisfies certain topological conditions. This generalization may be useful to maintain conformal invariance in case M is compact. Using methods of Polyakov and Friedan we identify the conformal anomaly coefficient with the central charge of the Virasoro algebra of this string theory. As an illustration we choose M to be a compact Lie group and compute the anomaly coefficient following the work of Knizhnik and Zamolodchikov.

I. INTRODUCTION

Superstring theory in 10 dimensions offers an attractive possibility of unifying all known interactions including gravity.[1] For special gauge groups like SO(32) and $E_8 \times E_8$ the theory is free of gauge and gravitational anomalies.[2] This theory is based on a supersymmetric generalization of the Nambu-Goto string where the action is proportional to the area swept by the string.

However, two questions which are not yet well understood relate to the nature and mechanism of the compactification of string theory,[3] and to its short-distance properties. In this paper we consider only Bose strings and present a natural generalization of the area law of string theory. This generalization may be of significance in a discussion of the above two questions. Our work has been motivated by recent studies in QCD current algebra.[4]

In Sec. II we discuss two simple examples that motivate the generalization. In Sec. III we present the model. We discuss the consequences of reparametrization invariance in Sec. IV. In Sec. V we consider a specific example of the model in which the manifold where the string lives is compact, and discuss its conformal invariance and critical dimension. Section VI contains concluding remarks.

II. TWO SIMPLE EXAMPLES

To motivate the generalization consider the motion of a nonrelativistic particle with electric charge $k/2$ in three dimensions. The generalization of the free action consists of adding the interaction with a unit magnetic monopole sitting at the origin. The (Euclidean) action of any closed trajectory C specified by the three functions $x^i(t)$ then becomes

$$S(C)=\oint_C \frac{1}{2}\left[\frac{dx^i}{dt}\right]^2 dt+\frac{ik}{2}\oint_C A_i^{\text{mon}}\frac{dx^i}{dt}dt$$

which by Stokes's theorem can be written as

$$S(C)=\oint_C \frac{1}{2}\left[\frac{dx^i}{dt}\right]^2+\frac{ik}{2}\int_\Sigma F_{ij}dx^i\wedge dx^j\,. \tag{1}$$

Σ is any two-surface whose boundary is C and F_{ij} is the field due to a unit magnetic monopole $F_{ij}=\epsilon_{ijk}x^k/r^3$.

If we write the monopole term in polar coordinates r,θ,ϕ, it becomes $i(k/2)\int_\Sigma \sin\theta\, d\theta\, d\phi$. The integral over Σ is the solid angle subtended by Σ at the origin. This immediately explains why $S(C)$ is independent of local deformations of Σ. Also, since the total solid angle of a closed surface enclosing the origin is 4π, it is clear that if $e^{-S(C)}$ is to be totally independent of Σ, k must be an integer.

For a relativistic particle the first term in (1) is just the length of C. By adding the monopole term we are adding another "geometrical" term, the solid angle subtended by C at the origin. This term also has the obvious property that it is not sensitive *per se* to the length or spatial extent of the trajectory C. It picks out only the "compact dimensions," for it measures only the angular spread of the trajectory.

As another example, consider a two-dimensional Euclidean field theory with the field x taking values in R^4. Consider a configuration under which the image in R^4 of the two-dimensional spacetime is a closed two-surface Σ_2 topologically equivalent to a sphere. The action is

$$S(\Sigma_2)=\int_{\Sigma_2}\frac{1}{2}\frac{\partial x^a}{\partial\xi^\mu}\frac{\partial x^a}{\partial\xi^\mu}d^2\xi+\frac{ik}{\pi}\int_{\Sigma_2}A_{ab}^{\text{mon}}(x)\frac{\partial x^a}{\partial\xi^\mu}\frac{\partial x^b}{\partial\xi^\nu}\epsilon^{\mu\nu}d^2\xi$$

in which the second term is the interaction of the field with the antisymmetric Kalb-Ramond potential[5] $A_{ab}(x)$ of a "monopole." The corresponding monopole field strength is $F_{abc}=\epsilon_{abcd}x^d/R^4$. By Stokes's theorem

$$S(\Sigma_2)=\int_{\Sigma_2}\frac{1}{2}\frac{\partial x^a}{\partial\xi^\mu}\frac{\partial x^a}{\partial\xi^\mu}+\frac{ik}{\pi}\int_{\Sigma_3}F_{abc}dx^a\wedge dx^b\wedge dx^c\,, \tag{2}$$

where Σ_3 is a three-surface in R^4 with boundary Σ_2. The integral over Σ_3 is over the solid angle

$$d\Omega^{(3)}=\sin^2\theta\sin\phi\, d\theta\, d\phi\, d\gamma$$

in polar coordinates R,θ,ϕ,γ. For the same considerations as for (1), $S(\Sigma_2)$ does not depend on the specific choice of Σ_3, and k is quantized. It is worthwhile noting that in terms of the SU(2) matrix $U=(x^4+ix^i\tau_i)/R$, (2) can be written as

$$S(\Sigma_2)=\int_{\Sigma_2}[\tfrac{1}{2}(\partial_\mu R)^2+\tfrac{1}{4}R^2\mathrm{Tr}(\partial_\mu U\partial_\mu U^{-1})]d^2\xi + \frac{ik}{12\pi}\int_{\Sigma_3}\epsilon^{ijk}\mathrm{Tr}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_k UU^{-1})d^3\xi \,.$$

For configurations of constant "radius" R, this is the SU(2)×SU(2) chiral model with the Wess-Zumino term.[4] The new term is sensitive only to the compact dimensions.

III. THE BOSON STRING MODEL

The above idea of coupling the system to a Kalb-Ramond monopole can be carried over to the string theory. Let the string lie in a manifold M. The string variables x^a are then the local coordinates of M. A string world sheet is defined by a map into M from a two-dimensional parameter space σ_2 parametrized by (ξ^1,ξ^2), i.e., by specifying the functions $x^a(\xi)$. If σ_2 has a metric $g_{\mu\nu}(\xi)$, the area action can be written as

$$A_1(g,x)=\frac{1}{4\lambda'^2}\int_{\sigma_2}d^2\xi\sqrt{g}\,g^{\mu\nu}\partial_\mu x^a\partial_\nu x^b G_{ab}(x)\,, \qquad (3)$$

where $G_{ab}(x)$ is the metric of M in terms of the x coordinates. The integral in (3) equals the area of the string world sheet[6] if $g_{\mu\nu}(\xi)$ satisfies the classical equation motion $\delta A_1(g,x)/\delta g^{\mu\nu}(\xi)=0$.

To generalize the solid-angle term we consider vacuum configurations of a closed string. The world sheet Σ_2 (the image of σ_2) is then a closed surface in M. For free strings Σ_2 is topologically the two-dimensional sphere S^2 and for interacting strings it is S^2 with handles. We can always find a three-surface Σ_3 in M of which Σ_2 is the boundary, provided the second Betti number of M is zero.[7] The solid-angle term for the string world sheet Σ_2 is constructed by locating a three-surface Σ_3 of which Σ_2 is the boundary, and integrating a three-form F in M over Σ_3:

$$A_2\sim\int_{\Sigma_3}F\,.$$

The form F must be closed (curl free), i.e., $dF=0$, so that continuous deformations of Σ_3 keeping the boundary Σ_2 fixed do not change the integral. This is necessary because we do not want the dynamics to depend on a specific choice of Σ_3, it should depend only on the string world sheet Σ_2.

We further require that F not be an exact form, i.e., it cannot be expressed as $F=dA$, otherwise Stokes's theorem will enable us to write A_2 as the integral of the two-form A over Σ_2 itself. Since F is closed but not exact, the manifold M must have a topological obstruction that allows for the existence of three-forms that are closed but not exact. This is possible if and only if the third Betti number $\beta_3(M)$ is nonzero.

This means that there may exist different Σ_3's with the same boundary Σ_2, which cannot be continuously deformed into each other. Γ must be such that its integral over different Σ_3's with the same boundary Σ_2 should differ by at most a multiple of 2π. This implies that $F/2\pi$ must be an integral three-form and the existence of such forms is also guaranteed by the nonvanishing of $\beta_3(M)$.

Finally, if we require that F be unique (up to normalization), then $\beta_3(M)$ must be 1. Thus the condition that we should be able to add a unique solid-angle-type term to the string theory results in the constraints

$$\beta_2(M)=0,\quad \beta_3(M)=1 \qquad (4)$$

for the possible compactifications. The solid-angle term in the action is then

$$iA_2=ik\int_{\Sigma_3}F\,, \qquad (5)$$

where k is an integer. It depends (modulo $2\pi i$) only on the string world sheet Σ_2 and not on the specific choice of Σ_3. By construction it is reparametrization invariant under transformations that are continuously connected to the identity. The full string action is the topological term (5) added to the geometrical term (3):

$$A(g,x)=A_1(g,x)+iA_2(x)\,. \qquad (6)$$

To write A_2 in terms of the ξ coordinates, we extend σ_2 to a three-surface σ_3 parametrized by coordinates (ξ^1,ξ^2,ξ^3). If F_{abc} are the components of F in the x coordinates, we have

$$A_2(x)=k\int_{\sigma_3}d^3\xi\,\epsilon^{ijk}\partial_i x^a\partial_j x^b\partial_k x^c F_{abc}(x)\,. \qquad (7)$$

If σ_3 is a manifold with metric $\tilde{g}_{ij}(\xi^1,\xi^2,\xi^3)$, (7) can be put in the form

$$A_2(x)=k\int_{\sigma_3}d^3\xi\sqrt{\tilde{g}}\,e^{ijk}\partial_i x^a\partial_j x^b\partial_k x^c F_{abc}(x)\,,$$

where $e^{ijk}\equiv\epsilon^{ijk}/\sqrt{\tilde{g}}$ is the fully antisymmetric tensor on σ_3. Note that A_2 is independent of the metric, by construction it has nothing to do with geometry. It is a purely topological term. The full reparametrization invariant string action is then

$$A(g,x)=\frac{1}{4\lambda'^2}\int_{\sigma_2}d^2\xi\sqrt{g}\,g^{\mu\nu}\partial_\mu x^a\partial_\nu x^b G_{ab}(x) + ik\int_{\sigma_3}d^3\xi\,\epsilon^{ijk}\partial_i x^a\partial_j x^b\partial_k x^c F_{abc}(x)\,. \qquad (8)$$

Before concluding this section we would like to make an additional remark about the three-form F. In the monopole case the two-form F represented the magnetic field of the monopole. Thus the F that occurs in (1) satisfies the Maxwell equations; i.e., in $R^3-\{0\}$, it is a solution of

$$dF=0\,,$$

$$d*F=0\,.$$

$dF=0$ is just the statement that $\nabla\cdot\mathbf{B}=0$ everywhere except at the site of the monopole. $d*F=0$ is the other Maxwell equation $\nabla\times\mathbf{B}=0$. Thus the Maxwell equations simply state that both F and its dual are closed in $R^3-\{0\}$. This implies that F is a harmonic form, i.e.,

$$\Delta F=0\,,$$

where Δ is the Laplace-Beltrami operator, $\Delta=d*d*+*d*d$.

The three-form F in the string theory also satisfies the same equations. We have already said that F exists and is unique if and only if $\beta_2(M)=0$ and $\beta_3(M)=1$. We wish to add that if, in addition, M is a compact, orientable Riemannian manifold, F can be determined uniquely (up to normalization) by demanding that it be harmonic: $\Delta F=0$. If M does not have a boundary, this is also equivalent to the statement that F satisfies "Maxwell equations" $dF=0$, $d*F=0$. This construction may be considered as a generalization of the concept of magnetic monopole.

These statements are in particular true for compact Lie groups. In this, up to normalization, the three-form F is given by

$$F=F_{abc}d\theta^a\wedge d\theta^b\wedge d\theta^c\ ,$$
$$F_{abc}=\mathrm{Tr}\left[\frac{\partial U}{\partial\theta^a}U^{-1}\frac{\partial U}{\partial\theta^b}U^{-1}\frac{\partial U}{\partial\theta^c}U^{-1}\right]$$

antisymmetrized in abc. Here U is a matrix, an element of the group, and θ^a, $a=1,2,\ldots,\dim M$ are the local coordinates of the group.

IV. WARD IDENTITIES OF REPARAMETRIZATION INVARIANCE, LIOUVILLE ACTION AND CONFORMAL ANOMALY COEFFICIENT

We now discuss the quantization of the model (8). Following Polyakov,[8] the amplitude for string propagation is given by

$$Z=\int \mathscr{D}g\exp\left[-\mu_0{}^2\int d^2\xi\sqrt{g}\right]\int \mathscr{D}_g x\, e^{-A(g,x)}$$
$$\equiv\int \mathscr{D}g\, e^{-S(g)}\ , \tag{9}$$

where $\int \mathscr{D}_g x$ represents integration over all string configurations and $\int \mathscr{D}g$ represents integration over all metrics. The measure $\mathscr{D}_g x$ is defined by a metric in function space that depends only on $\det g$.

If (9) represents the string propagator, it must be reparametrization invariant. This requires $S(g)$ to be reparametrization invariant: if two metrics are obtained from each other by a reparametrization $\xi\to\xi'$, $S(g)$ is the same for both. In particular, $S(g)$ must be invariant under conformal reparametrizations [in which $z\equiv\xi^1+i\xi^2\to f(z)$, $\bar z\equiv\xi^1-i\xi^2\to\bar f(\bar z)$ and f and $\bar f$ are, respectively, analytic and anti-analytic-functions].

Since $S(g)$ is reparametrization invariant one can work locally in the conformally Euclidean gauge $g_{\mu\nu}=e^{\phi}\delta_{\mu\nu}$. Then

$$Z=\int \mathscr{D}_g\phi\exp\left[-\mu_0{}^2\int d^2\xi\sqrt{g}\right]\int \mathscr{D}_g x\,\mathscr{D}_g b\,\mathscr{D}_g c\times\exp[-A(g,x)-A(g,b,c)]$$
$$\equiv\int \mathscr{D}g\phi\, e^{-S(\phi)}\ , \tag{10}$$

where b and c are the Faddeev-Popov fields corresponding to the conformal gauge. We shall argue that $S(\phi)$ is proportional to the Liouville action:

$$S(\phi)=\frac{\lambda}{48\pi}S_L=\frac{\lambda}{48\pi}\int d^2\xi(\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+\mu^2e^{\phi}) \tag{11}$$

and show that the constant λ is both the conformal anomaly coefficient and the coefficient of the c number in the central extension of Virasoro algebra. Our notation is that of Friedan,[9] and we generalize for an arbitrary $A(g,x)$, Friedan's discussion of the Polyakov string.

Liouville action

Consider infinitesimal reparametrizations

$$z\to z+v^z(z,\bar z),\quad \bar z\to\bar z+v^{\bar z}(z,\bar z)\ . \tag{12}$$

These induce a transformation of the metric

$$g^{\alpha\beta}(z,\bar z)\to g^{\alpha\beta}(z,\bar z)+\delta g^{\alpha\beta}(z,\bar z)\ ,$$
$$\delta g^{\alpha\beta}(z,\bar z)=\partial_\rho v^\alpha(z,\bar z)g^{\beta\rho}(z,\bar z)+\partial_\rho v^\beta(z,\bar z)g^{\alpha\rho}(z,\bar z)$$
$$-v^\rho(z,\bar z)\partial_\rho g^{\alpha\beta}(z,\bar z)\ . \tag{13}$$

The $\alpha\beta\rho$ indices each take two values z and $\bar z$. The scalar fields x^a also transform:

$$x^a\to x^a(z,\bar z)+\delta x^a(z,\bar z)\ ,$$
$$\delta x^a(z,\bar z)=-v^\alpha(z,\bar z)\partial_\alpha x^a(z,\bar z)\ . \tag{14}$$

These transformations, being symmetries of the theory, generate Ward identities. A first Ward identity is obtained by computing the change δS in $S(\phi)$ due to the above transformations and setting it equal to zero:

$$\delta S=\int dz\,d\bar z\sqrt{g}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta}\ .$$

Using (13) for $\delta g^{\alpha\beta}$, one finds

$$\delta S=2\int dz\,d\bar z\sqrt{g}\,v^\alpha\nabla^\beta\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}}\right]\ , \tag{15}$$

where ∇^β stands for the covariant derivative. Since $\delta S=0$ for arbitrary v^α this leads to the "Gauss law,"

$$\nabla^z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}\right]+\nabla^{\bar z}\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{z\bar z}}\right]=0\ ,$$
$$\nabla^z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{z\bar z}}\right]+\nabla^{\bar z}\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\bar z\bar z}}\right]=0\ . \tag{16}$$

$\delta g^{z\bar z}$ is a variation of the trace of the metric and δg^{zz} is a traceless variation. This can be seen from

$$g^{\alpha\beta}=\begin{bmatrix}g^{zz} & g^{z\bar z}\\ g^{\bar z z} & g^{\bar z\bar z}\end{bmatrix}$$
$$=\begin{bmatrix}g^{11}-g^{22}+2ig^{12} & g^{11}+g^{22}\\ g^{11}+g^{22} & g^{11}-g^{22}-2ig^{12}\end{bmatrix}\ .$$

g^{11}, g^{12}, and g^{22} are the components of the metric in ξ^1,ξ^2 coordinates. Thus the Ward identity (16) relates the trace anomaly to a traceless variation of S with respect to the metric.

In the conformally Euclidean gauge

$$g_{zz}=g_{\bar z\bar z}=g^{zz}=g^{\bar z\bar z}=0$$

and

$$g_{z\bar z}=g_{\bar z z}=(g^{z\bar z})^{-1}=(g^{\bar z z})^{-1}=\frac{\rho}{2}\equiv\tfrac{1}{2}e^{\phi}\ .$$

The variations of g are being considered from an initial metric of this type. Thus $\delta g^{\bar z\bar z}=-2e^{-\phi}\delta\phi$ is a variation along the gauge slice and δg^{zz} a variation orthogonal to the gauge slice. Equation (16) can be written as

$$\nabla^z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}\right]=\nabla_z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\phi}\right]\ . \tag{17}$$

Since S is a dimensionless scalar, the left-hand side of (17) is a rank-1, dimension-3 tensor. The only such object local in ϕ is $\nabla_z R$ where R is the curvature scalar. Hence

$$\nabla^z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}\right]=\frac{\lambda}{48\pi}\nabla_z R \tag{18}$$

and

$$\nabla_z\left[\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\phi}\right]=\frac{\lambda}{48\pi}\nabla_z R\ . \tag{19}$$

Integrating (19),

$$\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\phi}=\frac{\lambda}{48\pi}(R+\mu^2)\ , \tag{20}$$

where μ is an arbitrary constant. Since $R=-4\rho^{-1}\partial_z\partial_{\bar z}\phi=-\rho^{-1}\partial_\mu\partial_\mu\phi$, (20) integrates to the Liouville action

$$S=\frac{\lambda}{48\pi}\int d^2\xi(\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+\mu^2 e^{\phi})\ . \tag{11}$$

It is important to emphasize that the Liouville action (11) is a direct consequence of locality and reparametrization invariance. The anomaly coefficient λ is yet to be computed.

Conformal anomaly as a Jacobian of local rescalings, and Ward identity in presence of sources

The energy momentum tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu}=-\frac{2\pi}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}[A(g,x)+A(g,b,c)]\ . \tag{21}$$

The vacuum expectation value of T_{zz} is given by

$$\langle T_{zz}\rangle_0=\frac{-2\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}\ . \tag{22}$$

Equation (22) is obtained by differentiating $S(\phi)$ (10) with respect to g^{zz}. In the conformal gauge the measure does not change under a traceless variation δg^{zz}, because it depends only on ϕ. Hence $\delta/\delta g^{zz}$ can be taken inside the functional integral, resulting in (22).

The Ward identity (18) then reads

$$\nabla^z\langle T_{zz}\rangle_0=\frac{-\lambda}{24}\nabla_z R\ . \tag{23}$$

Equation (23) defines an analytic energy momentum tensor T^0_{zz}:

$$T^0_{zz}=T_{zz}-\frac{\lambda}{24}(-\partial_z\phi\,\partial_z\phi+2\,\partial_z\partial_z\phi) \tag{24}$$

which satisfies

$$\partial_{\bar z}\langle T^0_{zz}\rangle_0=0\ . \tag{25}$$

We will now establish (25) in the presence of arbitrary sources. Then T^0_{zz} is analytic as an operator. This operator will be used to define the generators of the Virasoro algebra whose central charge will turn out to be $-\lambda$. In the process we show that the conformal anomaly is related to the Jacobian of a Weyl rescaling transformation.[10]

We only consider sources for the string variables x^a. The sources for the ghost fields can be treated as in Ref. 9. Define the free energy by

$$e^{-F(g,x)}=\int\mathscr{D}_g x\,\mathscr{D}_g b\,\mathscr{D}_g c\times\exp[-A(g,x)-A(g,b,c)-(\chi,x)_g]\ , \tag{26}$$

where χ is an arbitrary source and

$$(\chi,x)_g\equiv\int\frac{dz\,d\bar z}{2\pi}\sqrt{g}\,\chi^a(z,\bar z)x^a(z,\bar z)\ . \tag{27}$$

Make a reparametrization transformation (14) on $x^a(z,\bar z)$. We do not change g, but look upon the change in x as simply a change of variables. In this transformation since the coordinates $z,\bar z$ change without a corresponding change in g, distances on the string world sheet are getting rescaled. Thus this transformation is not a conformal reparametrization but a local Weyl rescaling.

Since $A(g,x)$ is reparametrization invariant, the change in $A(g,x)$ due to this change in x can be expressed in terms of the variation of $A(g,x)$ with respect to the metric: Under (13) and (14)

$$\begin{aligned}\delta A&=\int dz\,d\bar z\sqrt{g}\left[\frac{\delta A}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta}+\frac{1}{\sqrt{g}}\frac{\delta A}{\delta x^a}\delta x^a\right]\\&=\int dz\,d\bar z\sqrt{g}\,v^\alpha\left[2\nabla^\beta\left[\frac{1}{\sqrt{g}}\frac{\delta A}{\delta g^{\alpha\beta}}\right]-\nabla_\alpha x^a\frac{1}{\sqrt{g}}\frac{\delta A}{\delta x^a}\right]\\&=0\ .\end{aligned}$$

Thus if x'^a is related to x^a by (14),

$$\begin{aligned}A(g,x)&=A(g,x')+2\int dz\,d\bar z\sqrt{g}\,v^\alpha\nabla^\beta\left[\frac{1}{\sqrt{g}}\frac{\delta A}{\delta g^{\alpha\beta}}\right]\\&=A(g,x')-\frac{1}{\pi}\int dz\,d\bar z\sqrt{g}\,(v^z\nabla^z T_{zz}+v^{\bar z}\nabla^{\bar z}T_{\bar z\bar z})\\&\equiv A(g,x')+\Delta_1\ .\end{aligned} \tag{28}$$

In the second step we use the property that $\delta A/\delta g^{z\bar z}=0$ in the conformal gauge.

Another term in (26) that will change under this change of variables is $(\chi,x)_g$. It is easily seen that $(\chi,x)_g=(\chi,x')_g+\Delta_2$, with

$$\Delta_2 = \int \frac{dz\,d\bar{z}}{2\pi}\sqrt{g}\,v^\alpha \chi^a \nabla_\alpha x^a \,. \tag{29}$$

Finally, the measure $\mathscr{D}_g x'$ in (26) differs from $\mathscr{D}_g x$ by a Jacobian. $\mathscr{D}_g x = \mathscr{D}_g x' \det B$ where B is the operator $1+v^\alpha \partial_\alpha$. $\det B$ is independent of x and χ. Thus,

$$e^{-F(g,\chi)} = \int \mathscr{D}_g x' \mathscr{D}_g b\, \mathscr{D}_g c \times \exp[-A(g,x') - A(g,b,c) - (\chi,x')_g - \Delta + \ln\det B]\,, \tag{30}$$

where $\Delta = \Delta_1 + \Delta_2$. Expanding $e^{-\Delta+\ln\det B}$ to first order in v^α we get

$$\langle \ln\det B - \Delta\rangle_\chi = 0 \tag{31}$$

for the expectation value in the presence of sources. To first order in v^α, $\ln\det B = \int dz\,d\bar{z}\sqrt{g}\,v^\alpha b_\alpha(z,\bar{z})$ for some b_α. Using (28) and (29) for Δ we have

$$b_z - \frac{1}{2\pi}\chi^a \nabla_z \langle x^a\rangle_\chi + \frac{1}{\pi}\nabla^z \langle T_{zz}\rangle_\chi = 0\,. \tag{32}$$

By setting the source equal to zero it can be seen that the Jacobian term is given by

$$b_z = -\frac{1}{\pi}\nabla^z \langle T_{zz}\rangle_0 = \frac{\lambda}{24\pi}\nabla_z R\,. \tag{33}$$

This establishes the relation between the Jacobian and conformal anomaly.

From (32) and (33) it follows that

$$\nabla^z \langle T_{zz}\rangle_\chi = -\frac{\lambda}{24}\nabla_z R + \frac{1}{2}\chi^a \nabla_z \langle x^a\rangle_\chi\,. \tag{34}$$

This is the Ward identity (23) in the presence of sources. Equation (34) can be rewritten

$$\partial_{\bar{z}} \langle T^0_{zz}\rangle_\chi = \tfrac{1}{2} g_{z\bar{z}} \chi^a \nabla_z \langle x^a\rangle_\chi\,. \tag{35}$$

Thus

$$\partial_{\bar{z}} \langle T^0_{zz}\rangle_\chi = 0 \tag{36}$$

provided the source vanishes in the neighborhood of z. Since the expectation value of T^0_{zz} in the presence of an arbitrary source that vanishes near z is analytic, T^0_{zz} as an operator is analytic. This is the quantum energy momentum tensor.

λ as central charge of Virasoro algebra

The moments, or "Fourier components" of T^0_{zz} are defined by

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T^0_{zz}, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{37}$$

C_0 being a simple closed contour encircling the origin once counterclockwise. We now show following Ref. 9 that L_n satisfies the Virasoro algebra of conformal symmetry, and that $-\lambda$ is the central charge in the algebra.

A second Ward identity can be established by differentiating (34) with respect to g^{zz}. We use the fact that $\langle T_{zz}\rangle_\chi$ is a second-rank tensor, and its expression as a functional integral. In this variation it is important to note that R and the covariant derivative also depend upon the metric. The result is (for a source that vanishes at z and w but is otherwise arbitrary)

$$\nabla^z \langle T_{zz} T_{ww}\rangle_{c,\chi} = \frac{\lambda}{12}\nabla_z^3 1(z,w) + [-2\nabla_z 1(z,w) + 1(z,w)\nabla_w]\langle T_{ww}\rangle_\chi\,. \tag{38}$$

$\langle T_{zz} T_{ww}\rangle_{c,\chi}$ is the connected part of the expectation value of the time-ordered product to T_{zz} and T_{ww} in the presence of sources, and $1(z,w) \equiv (1/\sqrt{g})\delta(z-w)\delta(\bar{z}-\bar{w})$ is the covariant δ function.

From (38) it can be shown that

$$\langle T^0_{zz} T^0_{ww}\rangle_\chi = \frac{-\lambda}{2(z-w)^4} + \left[\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right]\langle T^0_{ww}\rangle_\chi. \tag{39}$$

The operator-product expansion (39) can be considered a solution of the Ward identity (38).

The Virasoro algebra for L_n follows from the operator-product expansion (39). First, we note that the time ordering in (39) can be taken to be with respect to the absolute magnitude of z, lower values of $|z|$ corresponding to earlier times. Then, consider

$$\begin{aligned}\langle [T^0_{zz}, L_m]\rangle_{c,\chi} &= \langle T^0_{zz} L_m - L_m T^0_{zz}\rangle_{c,\chi} \\ &= \left\langle T^0_{zz} \oint_{C_0} \frac{dw}{2\pi i} w^{m+1} T^0_{ww} - \oint_{C_{0z}} \frac{dw}{2\pi i} w^{m+1} T^0_{ww} T^0_{zz}\right\rangle_{c,\chi}\,.\end{aligned}$$

C_{0z} is chosen to be a circle centered at the origin that contains z and C_0 a circle that does not contain z. Then the product of T's is time ordered in both the terms. Thus

$$\langle [T^0_{zz}, L_m]\rangle_{c,\chi} = -\oint_{C_z} \frac{dw}{2\pi i} w^{m+1} \langle T^0_{zz} T^0_{ww}\rangle_{c,\chi}\,, \tag{40}$$

where C_z is a contour that contains z but not the origin. We now use (39) for the integrand in (40) and deform the contour C_z to a small circle around the singularity $w = z$. The contour integration then yields

$$\langle [T^0_{zz}, L_m]\rangle_{c,\chi} = \frac{\lambda}{12} m(m^2-1) z^{m-2} - \partial_z \langle T^0_{zz}\rangle_\chi z^{m+1} - 2\langle T^0_{zz}\rangle_\chi (m+1) z^m\,.$$

This is an operator equation since it holds for arbitrary source. Expanding T^0_{zz} on both sides in powers of z using (37), and comparing coefficients yields the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{(-\lambda)}{12} n(n^2-1)\delta_{n+m,0}\,. \tag{41}$$

This completes the identification of $-\lambda$ as the central charge.

We would like to emphasize that the only input in deriving the results of this section is that S is reparametrization invariant and a local functional of ϕ.

V. A CONFORMALLY INVARIANT MODEL IN COMPACT SPACE

We now consider a specific example of a conformally invariant string theory of the type (8), in which M is a Lie group, and determine for this model the value of λ. The conformal invariance of the corresponding two-dimensional field theory in flat space was first discussed by Witten[11] and the model has been solved for the Green's functions and the central charge of the Virasoro algebra by Knizhnik and Zamolodchikov.[12]

To begin with, let M be a simple, compact Lie group. For compact Lie groups $\beta_2(M)=0$, $\beta_3(M)=1$. Equation (8) takes the form

$$A(g,x)=\frac{1}{4\lambda'^2}\int d^2\xi\sqrt{g}\,g^{\mu\nu}\mathrm{Tr}\partial_\mu U\,\partial_\nu U^{-1} - \frac{ik}{24\pi}\int d^3\xi\,\epsilon^{ijk}\mathrm{Tr}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_k UU^{-1})\ , \tag{42}$$

where $U=e^{\lambda' x^a t_a}\epsilon M$, λ' a dimensionless number, and the anti-Hermitian generators t_a satisfy $\mathrm{Tr}(t_a t_b)=-2\delta_{ab}$, $[t_a,t_b]=f_{abc}t_c$. $k=24\pi k'/N_G$, where k' is the integer appearing in (8) and N_G is a normalization factor that depends on the volume of the smallest three-sphere in G. The solid-angle term is the familiar Wess-Zumino term.

One-loop calculation

By expanding U about a background field U_0: $U=U_0 e^{i\lambda' x^a t_a}$, $S(g)$ defined in (9) can be determined to one loop. It is essentially the ln det of the operator

$$D^{ac}=-\delta^{ac}g^{-1/2}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu)-M^{\mu\nu}f_{abc}R^b_\mu\partial_\nu\ , \tag{43}$$

where

$$M^{\mu\nu}=-g^{\mu\nu}+i\alpha\epsilon^{\mu\nu}/\sqrt{g},\quad \alpha=\frac{k\lambda'^2}{4\pi} \tag{44}$$

and $R^b_\mu=-\frac{1}{2}\mathrm{Tr}(U_0{}^{-1}\partial_\mu U_0 t_b)$. ln det D can be evaluated by the short-time expansion of the heat kernel of D, and yields

$$S(\phi)=\frac{(26-\dim M)}{48\pi}\int d^2\xi(\tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi+\mu^2 e^\phi) + \frac{C_v}{8\pi}(1-\alpha^2)\int d^2\xi\,\mathrm{Tr}\partial_\mu U_0\partial_\mu U_0{}^{-1} \times\ln\left[\frac{\Lambda^2}{\mu'^2}\rho(\xi)\right]\ . \tag{45}$$

Here Λ is the cutoff, μ' the arbitrary scale parameter, C_v is defined by $2C_v\delta^{ad}=f^{abc}f^{dbc}$, and $\rho=e^\phi$.

The first term in $S(\phi)$ is the conformally invariant term $\lambda S_L/48\pi$, with $\lambda=26-\dim M$. The second term, because of $\ln\rho(\xi)$ is *not* conformally invariant for groups with $C_v\neq 0$, *unless* $\alpha=\pm 1$. In a flat two-dimensional spacetime, where $\rho=1$, this corresponds to the vanishing of the β function.[13] Since $\alpha\propto k$ this indicates that if we wish the string theory compactified as above to have reparametrization invariance, the solid-angle term is necessary.

The above one-loop calculation, though instructive, does not reflect the presence of curved space in which the string lives. In the next section we present the exact answer for the anomaly coefficient λ, using the algebraic method of Ref. 12.

Kac-Moody, Virasoro algebras

The model (42) has the property that

$$A(g,VU)=A(g,V)+A(g,U) - \frac{1}{2\lambda'^2}\int d^2\xi\sqrt{g}\,M^{\mu\nu}\mathrm{Tr}(V^{-1}\partial_\mu VU^{-1}\partial_\nu U)\ . \tag{46}$$

The last term contains one contribution from the area term and another from a total divergence part of the solid-angle term. In the conformal gauge and in $z\bar{z}$ coordinates,

$$g_{zz}=g_{\bar{z}\bar{z}}=g^{zz}=g^{\bar{z}\bar{z}}=0\ ,$$

$$g_{z\bar{z}}=g_{\bar{z}z}=(g^{z\bar{z}})^{-1}=(g^{\bar{z}z})^{-1}=\frac{\rho}{2}=\frac{e^\phi}{2}\ ;$$

thus

$$M^{zz}=M^{\bar{z}\bar{z}}=0\ ,$$

$$M^{z\bar{z}}=2\rho^{-1}(-1+\alpha)\ ,$$

$$M^{\bar{z}z}=2\rho^{-1}(-1-\alpha)\ .$$

At $\alpha=\pm 1$ a cancellation occurs between the area and solid-angle contributions to M. Only one component of M survives and it is a two-dimensional projection matrix. This results in a remarkable local symmetry for (42). For example, if $\alpha=1$,

$$M=-4\rho^{-1}\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

and

$$A(g,VUW^{-1})=A(g,U)\ , \tag{47}$$

where V,W are analytic and anti-analytic functions, respectively:

$$V=e^{w^a(z)t_a},\quad W=e^{\bar{w}^a(\bar{z})t_a}\ . \tag{48}$$

The classical equations of motion for the left and right currents $L_\mu\equiv\partial_\mu UU^{-1}$ and $R_\mu\equiv U^{-1}\partial_\mu U$ are $\nabla_\mu(M^{\mu\nu}L_\nu)=\nabla_\nu(M^{\mu\nu}R_\mu)=0$. This follows from the variation of the action (42). At $\alpha=1$ these state that the current $J\equiv-\frac{1}{2}k\,\partial_z U$. U^{-1} is an analytic function and $\bar{J}\equiv-\frac{1}{2}kU^{-1}\partial_{\bar{z}}U$ is an anti-analytic function:

$$\partial_{\bar{z}}J=\partial_z\bar{J}=0\ . \tag{49}$$

Under the transformation (48) J transforms as a left tensor:

$$\delta_w J(z)=[w(z),J(z)]-\tfrac{1}{2}k\,\partial_z w(z)\ . \tag{50}$$

By establishing classical Poisson bracket relations,[11] it can be shown that $J(\bar{J})$ is the generator of left (right) multiplication and satisfies the Kac-Moody algebra with a central extension $k(-k)$. Defining $J \equiv J^a t_a$, this becomes

$$[J^a(z), U(w,\bar{w})] = -\pi i \delta(z-w) t_a U(w,\bar{w}) , \tag{51}$$

$$[J^a(z), J^b(w)] = 2\pi i \delta(z-w) f^{abc} J^c(z) + \pi i k \delta^{ab} \partial_z \delta(z-w) . \tag{52}$$

The assumption that this symmetry is realized at the quantum level is sufficient to guarantee the conformal invariance of string theory, and to determine λ exactly. The method of Ref. 12 applies also to the model (42). The Kac-Moody algebra can be written in terms of the moments of the analytic function $J^a(z)$:

$$J_n^a = \oint_{C_0} \frac{dz}{2\pi i} z^n J^a(z), \quad n = 0, \pm 1, \pm 2, \ldots \tag{53}$$

where C_0 is a simple contour encircling the origin once counterclockwise. In terms of J_n^a, the algebra (52) becomes

$$[J_n^a, J_m^b] = f^{abc} J_{n+m}^c - kn \delta^{ab} \delta_{m+n,0} . \tag{54}$$

The generators of the Virasoro algebra are defined by

$$L_n = \frac{1}{\kappa} \sum_{m=-\infty}^{m=+\infty} :J_m^a J_{n-m}^a: . \tag{55}$$

κ is a constant and the dots denote normal ordering: J_n with negative n sits to the left. Using the Kac-Moody algebra (54) it can be seen that

$$[L_n, J_m^a] = -m J_{n+m}^a \tag{56}$$

provided $\kappa = -2(C_v + k)$. Equation (56) is just the condition that the current $J^a(z)$ is a rank-1 tensor. Finally, L_n satisfies the Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{C'}{12} n(n^2-1) \delta_{n+m,0} \tag{57}$$

with central charge

$$C' = \frac{k \,\dim M}{C_v + k} . \tag{58}$$

This establishes conformal symmetry since the Virasoro algebra is realized in terms of the operators L_n constructed out of the symmetry generators. Note that for $k' = 1$, $c' = \text{rank}$ (6).

In a conformally invariant theory L_n are the moments of the analytic energy momentum tensor T_{zz}. In the string theory the full T_{zz} contains not only a product of J's[11] coming from $A(g,x)$, but also a contribution from the Faddeev-Popov ghost action $A(g,b,c)$. Thus L_n (55) contains another term constructed out of the moments b_n and c_n of the Faddeev-Popov fields. Since $A(g,b,c)$ is quadratic and massless, the conformal invariance is not disturbed. Equations (56) and (57) are still obtained but with $C' = k \dim M/(C_v + k) - 26$. This determines the coefficient λ in (11) to be

$$\lambda = 26 - \frac{k \,\dim M}{C_v + k} . \tag{59}$$

If M is a product of simple factors $M = [\text{U}(1)]^d \times G_1 \times G_2 \times \cdots \times G_n$, (55) contains a sum of terms with a different factor κ_i for each factor. These are determined from the requirement (56) that each J_i transform as a rank-1 tensor under conformal transformations. The radius of the U(1)'s determines a relative scale of compactification between the U(1) and the G_i variables; if it is large the former may be considered the noncompact spacetime variables. The final expression for λ is the sum of the conformal anomalies from all fields—Faddeev-Popov, free and compact fields:

$$\lambda = 26 - \left[d + \sum_i \frac{k_i \dim G_i}{C_v^i + k_i} \right] . \tag{60}$$

In a superstring theory the Liouville action $S_L(\phi)$ is replaced by the super-Liouville action $S(\phi, \chi)$.[15] The ghost contribution to λ in this case is 15. The full λ in this case is likely to be

$$\lambda = 15 - (\text{sum of conformal anomaly coefficients of all fields}) . \tag{61}$$

Weyl invariance and critical dimension

From the above calculations it is clear that as far as free strings are concerned (Σ_2 is topologically a sphere) the string model (42) is conformal invariant and its Virasoro algebra has a central extension with charge λ given by (59). An important question that arises is whether the interacting string theory is conformal invariant with a central extension. The answer is not clear, though there are hints[16] that a consistent interacting string theory is possible only when the central charge is equated to zero. In case this is true, the equation

$$\lambda = \sum (\text{conformal anomalies}) = 0 \tag{62}$$

imposes another restriction on compactification.

This is like the chiral anomaly cancellation condition in interacting gauge theories, where one derives restrictions on fermion representations.

At $\lambda = 0$ the string theory admits Weyl invariance viz., under transformations which transform the metric $g_{\mu\nu} = \delta_{\mu\nu} e^{\phi}$ to the metric $\delta_{\mu\nu}$, because of the absence of the Liouville term. One may note that in (60), if G is chosen to be SO(32) or $E_8 \times E_8$, $k = 1$, and the number of flat dimensions is $d = 10$, then $\lambda = 0$ and the string theory has Weyl invariance.

VI. CONCLUSION

In this paper we have generalized the Nambu-Goto string to include a topological solid-angle term which is allowed by reparametrization invariance. This term may be necessary to maintain conformal invariance when strings live in compact manifolds. In the Kaluza-Klein approach it is a basic question to determine the compact manifold. To determine this manifold or the metric $G_{ab}(x)$ in (8), we may not add an Einstein term to determine this metric by a dynamical principle. Doing so will lead us back to the problems of conventional gravity theory which we are trying to avoid.

What is required is a new principle, a hint of which is already present in this work. It seems that conformal invariance may be this new principle and we may understand the topological relations (4) $\beta_2(M)=0$, $\beta_3(M)=1$ as required by conformal invariance.

In the present work we have used the specific example of a Lie group to illustrate our point. In order to use the algebraic computation method for more general manifolds we need a generalization of current algebra. Even more important is a supersymmetric generalization of the present work. We hope to report on this in the near future.

Note added. While completing this work we became aware of the work of other authors which has since been published:

(i) D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. **54**, 620 (1985); **54**, 1736(E) (1985). We thank A. Dhar for informing us about this work.

(ii) P. Goddard and D. Olive, Nucl. Phys. **B257** [FS14], 226 (1985). These authors have also obtained the Virasoro algebra using Kac-Moody algebra. We thank P. P. Divakaran for bringing this report to our attention.

Note added in proof. (i) We outline the massless spectrum of the closed-string model (8) with spacetime $=R^d\times G$. The string variables are flat coordinates x^μ, $\mu=1,2,\ldots,d$ and group coordinates x^a, $a=1,2,\ldots,\dim G$. Group variables are replaced by the moments J_n^a and $\bar{J}_n^a$ of the currents, as in (53), and flat variables by the oscillators α_n^μ, $\bar{\alpha}_n^\mu$ and momentum p^μ. The constraints $T_{zz}=T_{\bar{z}\bar{z}}=0$ can be used in the light-cone gauge to solve for the mass operator $p_\mu p^\mu$ in terms of the flat transverse oscillators $\alpha_n^i,\bar{\alpha}_n^i$ and the moments $J_n^a,\bar{J}_n^a$. The ground state $|0\rangle$ is defined to be annihilated by these operators for $n\geq 0$. The ground state is tachyonic. The states at the first excited level which satisfy the closed-string constraint are massless. These are $\alpha_{-1}^i\bar{\alpha}_{-1}^i|0\rangle$ (dilaton), $(\alpha_{-1}^i\bar{\alpha}_{-1}^j+\alpha_{-1}^j\bar{\alpha}_{-1}^i)|0\rangle$ (graviton), $(\alpha_{-1}^i\bar{\alpha}_{-1}^j-\alpha_{-1}^j\bar{\alpha}_{-1}^i)|0\rangle$ (Kalb-Ramond particle), $\alpha_{-1}^i\bar{J}_{-1}^a|0\rangle$ and $\bar{\alpha}_{-1}^iJ_{-1}^a|0\rangle$ (left- and right-moving gauge particles), $J_{-1}^a\bar{J}_{-1}^b|0\rangle$ (scalar). If the last two are excluded, this coincides with the massless bosonic sector of the heterotic string. (ii) Strings on group manifolds and the associated current algebra have been discussed long ago in P. Goddard, Nucl. Phys. **B116**, 157 (1976). We thank Peter Goddard for bringing this to our attention. (iii) After the completion of this work we learned that the construction of higher-dimensional monopoles has been done independently by R. I. Nepomechie, Phys. Rev. D **31**, 1921 (1985). The implications of conformal invariance for strings in curved spaces has been independently discussed by E. S. Fradkin and A. A. Tseylin, Lebedev Institute Report No. N261, 1984 (unpublished).

ACKNOWLEDGMENTS

We thank B. Sakita for several useful discussions on string theory. We acknowledge a discussion with M. S. Narasimhan. We are especially grateful to Gautam Mondal and Nitin Nitsure for help in some mathematical questions.

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