Phases of higher-order terms in the topological expansion

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We give here the general rules for determining the relative phases of higher-order terms in the topological expansion. We treat separately topological amplitudes and their discontinuities. Phases associated with the various discontinuities of a given topological amplitude are found to be $(-1)^n$, where n is the total number of quark loops in the intermediate state. Further rules are also given for determining the relative phases for the different topological contributions to the same process. Based on these results, the effects of higher-order terms can now be studied.

I. INTRODUCTION

Topological particle theory[1] gives a fully relativistic theory of hadrons as well as a theory of electroweak processes. The hadron sector of the theory which we discuss here gives Regge-behaved amplitudes[2] whose quark degrees of freedom are handled differently than in the standard QCD model but all the desirable properties of earlier dual resonance models are present. The strong interactions are represented in the simplest approximation by planar zero-entropy amplitudes which are to be determined by self-consistency or bootstrap equations[1,3] in which the mass of all stable hadrons have the same degenerate value m_0.

All distinct zero-entropy amplitudes corresponding to the particular process being considered must be summed as a first approximation in the topological expansion. Topologically the zero-entropy amplitudes, being planar, have a single boundary on which the particles lie and no handles. Subsequent corrections consist of including amplitudes with higher numbers of boundaries and handles and with chiral and color switches.[4] Various complexity indices can be introduced to catalog and rank the levels of approximation.[5] Once the zero-entropy amplitudes have been determined by the bootstrap conditions, essentially all higher-order corrections are calculable from these amplitudes.

A critical problem in this whole program is the determination of the relative phases of the various terms to be added in the topological expansion. Recently it was shown how the relative phases for the zero-entropy terms can be determined by the requirements of self-consistency.[6,7] These results are summarized and reviewed in Sec. II. The present paper shows how to extend these results by giving simple rules to determine the relative phases of terms in the topological expansion with arbitrary numbers of boundaries and handles. We discuss here explicitly only the "naked" terms in the expansion having no chiral or color switches. The further generalizations required to include these latter cases must eventually involve off-shell considerations[4] with which we have not yet dealt. In addition to the simple rules for relative phases of the naked terms in the topological expansion, we shall also indicate how each topology separately has the correct spin and statistics properties.[7,8]

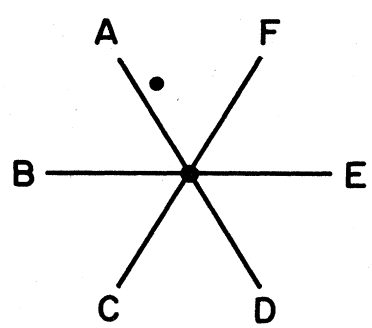

FIG. 1. Zero-entropy amplitude.

II. PHASES OF ZERO-ENTROPY AMPLITUDES

We begin here by reviewing the essential features of the phases of zero-entropy amplitudes. An example of a zero-entropy momentum graph is shown in Fig. 1, where the labels $A,B,C,\ldots$ designate the momenta, spin states, and all other quantum numbers of each particle on the boundary, and the dot represents the starting point in the order of variables.[6,7] It will frequently be convenient for us to regard $A,B,C,\ldots$ in diagrams of the form of Fig. 1 as each being a cluster of particles which possesses a net baryon number ± 1. Assume now that we work with such clusters. Then if there exists another distinct zero-entropy amplitude for the same process with external particles permuted, then these are to be summed in the topological expansion with dots placed such that the labels following the dots (say, in counterclockwise order) are even permu-

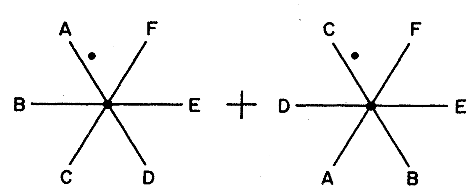

FIG. 2. Dot positions in zero-entropy amplitudes.

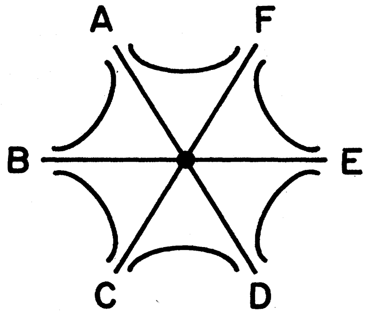

FIG. 3. Thickened momentum graph.

tations of one another. An example is shown in Fig. 2. It should be noted that there may be other contributions to the process in Fig. 2 in which particles whose baryon number is zero are in different clusters.

Momentum graphs such as Fig. 1 can be "thickened" to reveal the quark structure of the particles. This is indicated in Fig. 3. Each line on the boundary of Fig. 3 corresponds to a quark or a diquark line. This graph surface can be given an orientation and we specify this by the convention that all single-quark lines on the boundary have arrows which run clockwise and all diquark lines have arrows which run counterclockwise. If any of the particles in Fig. 3 are actually clusters, further quark and/or diquark lines can be added to fully delineate these particles.

The surface in Fig. 3 may also be embellished by a choice of two orientations for the patch adjacent to each quark line. These two possibilities correspond to the quark being so-called orthoquark—whose sign variables are associated with two-component dotted spinors—and paraquarks—whose spin variables are associated with undotted spinors. (To orient the surface next to each quark in a diquark pair the surface edge must be split but this will not concern us here.[5]) In this paper we shall not consider complications arising from differing path orientations. For simplicity we shall assume that all patches have the same orientation so we can omit an orientation label. Let us, then, take all quarks to be orthoquarks. A careful study of the requirements of self-consistency on the phases of zero-entropy amplitudes[6] shows that terms such as those in Fig. 2 possess phases of the form

$$\tau(-1)^{N_{\text{in}}(m)+N_{\text{in}}(\bar{B})}\Gamma_R \ . \tag{2.1}$$

In (2.1) τ is a phase which depends upon the order of variables (i.e., the placement of the dot in the zero-entropy amplitude):

$$\begin{aligned} \tau &= +1 \quad \text{if the dot lies above a diquark line in the thickened graph ,} \\ \tau &= -1 \quad \text{if the dot lies above a single-quark line in the thickened graph ;} \end{aligned} \tag{2.2}$$

also we have

$$\begin{aligned} N_{\text{in}}(m) &= \text{number of incoming mesons ,} \\ N_{\text{in}}(\bar{B}) &= \text{number of incoming antibaryons ;} \end{aligned} \tag{2.3}$$

finally Γ_R is a residual phase (discussed in detail in Ref. 6) which depends in general on the patch structure and on the particle types. In our discussion Γ_R will play no role because we assume all quarks are orthoquarks and because its dependence on particle types T is of the form

$$\exp\{i\theta_T[N_{\text{in}}(T)-N_{\text{out}}(T)]\}$$

making Γ_R a common factor of all terms even when unitarity-type products of amplitudes are taken. Particle type T refers not to flavor content but merely to whether the particle is a baryon, antibaryon, baryonium, or a meson. The phase θ_T can be consistently set equal to zero giving $\Gamma_R = 1$.

The individual zero-entropy amplitudes consist of the multiplicative phase (2.1), a multiplicative scalar amplitude, a factor giving the spin dependence in terms of a standard set of four-component or two-component spinors[3,6,8] and multiplicative flavor-conserving δ functions.

If the particles or clusters in Fig. 1 all have a baryon number ± 1 then the boundary lines in Fig. 3 are alternately quarks and diquarks. In the case that two of the objects in Fig. 1 correspond to identical baryons (or antibaryons)—say, e.g., A and C—there must be two zero-entropy terms as shown in Fig. 4. The order of variables for the two terms in Fig. 4 are even permutations of one another as required. The τ phases for the two terms have a relative minus sign so that the sum vanishes when $A = C$ giving the usual Pauli statistics. The relative minus sign occurs because of the alternating quark and diquark boundary lines mentioned above.

III. DISCONTINUITIES OF ZERO-ENTROPY AMPLITUDES

In this section we summarize the phases that come into the discontinuity equations for zero-entropy amplitudes. The discontinuity of a zero-entropy amplitude such as Fig. 1 involves a bilinear product of two zero-entropy amplitudes whose order of variables are determined by the rule shown in Fig. 5 where the dotted line indicates the variable in which the mass-shell discontinuity is taken. Our conventions for the discontinuity equation are such that EDC is the in state in Fig. 5 and FAB is the out state. Figure 5(b) illustrates the case where the dot for the amplitude lies to the right of the discontinuity line but with our in-out convention the same as before. In the graph on the right-hand side of the equalities in Figs. 5(a) and 5(b), there are phases associated with the intermediate-state

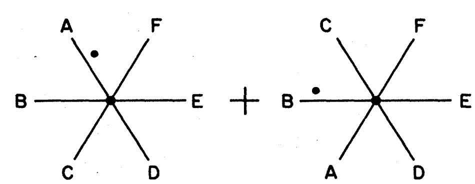

FIG. 4. Pauli statistics for zero-entropy amplitudes.

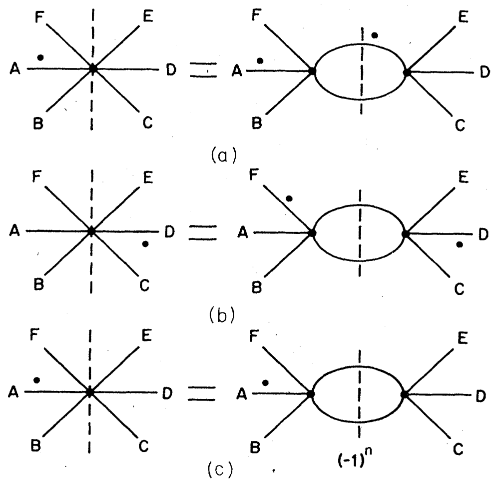

FIG. 5. Discontinuity of a six-point amplitude.

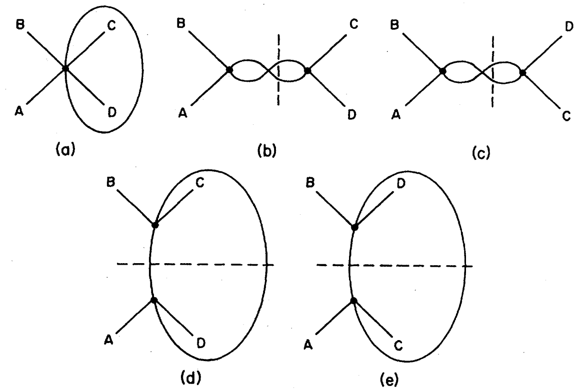

FIG. 6. Cylinder amplitude and discontinuities.

particles from (2.1) as well as the τ phase associated with the second dot all of which are in addition to the phases associated with the graph on the left-hand side. It is convenient to collect these "extra" phases in the bilinear discontinuity expression into a single phase and a simple rule exists for determining it. The net relative phase between the left- and right-hand sides of the discontinuity equation is just $(-1)^n$ where n is the number of closed single-quark loops in the connected sum of intermediate states.[8,3,6] This result follows from a straightforward application of (2.1). For example, we can express Fig. 5(a) as in Fig. 5(c), where no additional phases are to be associated with the right-hand side of the equality beyond those due to a single dot and the external particles, all of which are now common to both sides of the equality.

From a topological point of view the bilinear product of amplitudes in the discontinuity formula has the same topology as the amplitude on the left as long as the intermediate particles are connected in a planar manner. In other words, the bilinear graph can be contracted to give a simple zero-entropy disk with no boundaries or handles. The purpose of the discontinuity relations is to provide the basis for a bootstrap calculation of the zero-entropy amplitudes.[1] Although the phases of the discontinuities will vary as $(-1)^n$, these are absorbed into the scalar function f when dispersion relations are used to calculate the amplitudes from the discontinuity.

IV. PHASES OF NONZERO-ENTROPY AMPLITUDES

In this section we shall discuss the phases to be associated with nonzero-entropy topologies, which, since we do not consider chiral or color switches, can be classified according to the number of boundaries and handles of the two-dimensional surface representing the amplitude. These higher topologies are first generated when unitarity-type bilinear discontinuity products of zero-entropy amplitudes are taken but the intermediate particle lines are sewn together in a nonplanar manner. Since the relative phases of the zero-entropy terms in the topological expansion are determined by (2.1), the phases of the nonzero-entropy terms can also be determined. An important distinction must be made here between graphs representing various topological amplitudes and other graphs representing the discontinuities of the particular topological amplitude.

We begin by considering the cylinder topology which consists of two boundaries. We give an example in Fig. 6 where we have indicated first the topological amplitude which can always be contracted to a single vertex and then four of its discontinuities. The individual momentum or particle lines can be thickened to create the appropriate two-dimensional surface. In Fig. 6 we have not yet included dots to indicate a τ factor to be associated with the diagrams. Also we note that the particles on a boundary can be given a cyclic permutation without changing the topology. Thus interchanging, e.g., particles C and D in the cylinder amplitude in Fig. 6(a) does not result in a new contribution. However, in going from Fig. 6(b) to 6(c) (which correspond to discontinuities), C and D have to be interchanged by sliding one of them around the boundary. This in general results in two different intermediate states both of which must be included in calculating the discontinuity. Thus Figs. 6(b) and 6(c) must be added in calculating the common discontinuity.

This leads us to the first question regarding the phases of nonzero-entropy contributions. In the case of Fig. 6 we must add the discontinuities in Figs. 6(b) and 6(c) and thus must determine the rule for establishing the relative phase between these two terms. The rule is a simple one and is illustrated in Fig. 7. The interpretation of the rule which gives the relative phases between Figs. 7(a) and 7(b) is that with the dot placed before the same particle or cluster in each diagram the discontinuities are to be calculated by a straightforward product of the two zero-

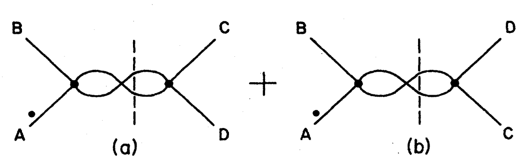

FIG. 7. Two contributions to the same discontinuity.

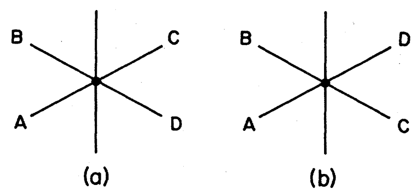

FIG. 8. Zero-entropy diagrams associated with Fig. 7.

entropy amplitudes. All phases associated with the intermediate particles are now dropped and only those phases coming from the external particles and the τ factor associated with the dot placement remain. Although the rule given in Fig. 7 is simple it is not entirely trivial because it holds quite generally regardless of whether the particles are baryons, mesons, etc. In particular, particles in the intermediate states of Fig. 7(a), could be mesons with those in Fig. 7(b) being a baryon-antibaryon pair. Also any phases coming from closed quark or diquark loops have been absorbed into the zero-entropy parts of the amplitude.

We now describe the general technique which has been used to determine the phases in Fig. 7 and which will be subsequently used to find the relative phases between amplitudes corresponding to different topologies. First in Figs. 7(a) and 7(b) we disconnect one of the intermediate particle lines, which leaves a zero-entropy graph in which the two vertices can be contracted as shown in Fig. 8. The additional unlabeled particles in the zero-entropy amplitudes of Figs. 8(a) and 8(b) are not in general the same type in both diagrams. In Fig. 9 we give an example of quark line structure for the two diagrams in Fig. 8, where u, x, y, and z denote quarks or diquarks and flavor.

It is not possible to directly determine the relative phases of the two graphs in Fig. 9 because the amplitudes represent different processes. A simple device, however, enables us to construct amplitudes for the same process and then to relate the phases in Fig. 7. We simply insert a segment of the x line in the right boundary of both 9(a) and 9(b) giving Fig. 10, where we have added the two amplitudes with appropriate dot placement. It is important to note that the location of the dot is correct regardless of which boundary lines are quarks or diquarks. The subscripts o and i refer to "out" and "in." We now deduce the relative phases of the contributions in Figs. 7(a) and 7(b) by connecting the correspondingly labeled in and out particles. In both diagrams a zero-entropy quark or diquark loop is formed as well as a connection which gives the cylinder topology. Our labeling of in and out particles is crucial here—the first particle in the clockwise order

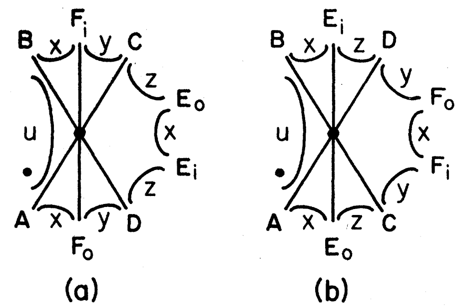

FIG. 10. Diagrams used to determine cylinder phases.

for a zero-entropy loop being an out particle. This conforms to the convention for zero-entropy loops in Figs. 5(b) and 5(c) and will be seen to give the correct factor of -1 for each closed quark loop. Such quark loop phases, however, can be absorbed into the zero-entropy vertices of the diagram. The relative phase between Figs. 7(a) and 7(b) is determined by the phases associated with the external particle F_i in Fig. 10(a) and E_i in Fig. 10(b) as given from (2.1). The phases associated with F_i and E_i are the same regardless of what x, y, and z are, thus leading to the result illustrated in Fig. 7—that the relative phase between the discontinuities is plus one.

Another approach to determining the relative phases of the discontinuities in Fig. 7 is shown in Fig. 11 where 11(a) can be contracted to give 7(a) and 11(b) can be contracted to give 7(b). The two zero-entropy amplitudes on the left of the dotted line are for the same process so their relative phases are known—the same is true for the two amplitudes on the right. Thus the relative phases of the two discontinuities can be determined directly and the result agrees with the earlier argument.

There are two further discontinuities across the AB variable of the amplitude in Fig. 6(a) which result from interchanging A and B in Fig. 7. The relative phases of all these discontinuities is plus one—the dot remaining in the same position relative to A in each discontinuity. The cylinder amplitude of Fig. 6(a) also possesses discontinuities across the variables AD and AC. However, in these variables there do not exist several contributions to the discontinuity analogous to those present in the AB variable (see Fig. 6). It should be noted that in this section we have only given phase rules for discontinuities which are products of zero-entropy amplitudes and in which no closed quark loops explicitly appear. (Closed quark loops will be discussed later.) Within this restriction, the results

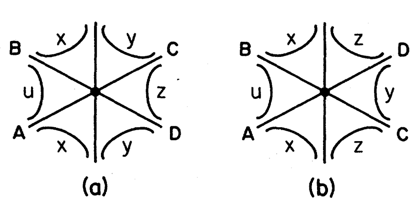

FIG. 9. Thickened version of Fig. 8.

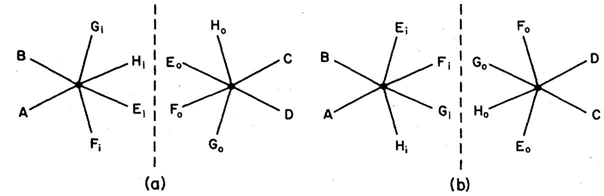

FIG. 11. Alternate diagrams used to determine cylinder phases.

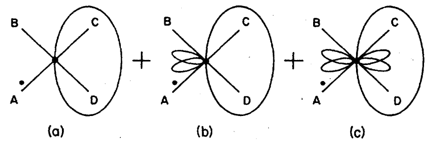

FIG. 12. Diagrams with handles.

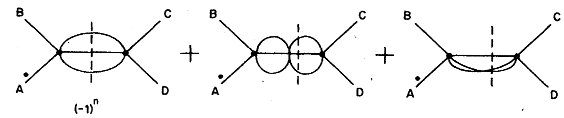

FIG. 14. Phases of discontinuities of single-boundary amplitudes.

we have derived here for the cylinder discontinuities are quite general. Different discontinuities across the same energy variable associated with permuting particles on the same boundary all contribute with the same relative sign regardless of the total number of boundaries. We can thus restrict our efforts to finding the relative phases between amplitudes corresponding to *different* topologies.

V. PHASES ASSOCIATED WITH HANDLES

We first consider the problem of starting with a given topology having a certain number of boundaries and no handles and then ask about the phases of amplitudes in which there are handles added. Starting with the two-boundary cylinder problem just discussed in Sec. IV we have terms shown in Fig. 12, where Fig. 12(b) has one handle and Fig. 12(c) has two handles. In Figs. 12(b) and 12(c) the handles can be drawn anywhere—only the total number of handles is relevant in determining the topology once the boundary configurations are given.

The relative phases for the terms in Fig. 12 are derived in a manner similar to that in Fig. 10. The result is that for any discontinuity of the amplitudes in Fig. 12, the bilinear products are to be taken directly with all phases associated with intermediate particles dropped and only the phases associated with the external particles remaining. As in Sec. IV we consider only discontinuities which are products of zero-entropy amplitudes with no closed quark loops. As an example, we consider the AB discontinuity in Fig. 12. Two contributions to the discontinuity of Fig. 12(a) have already been given in Fig. 7. We give in Fig. 13 several contributions to the AB discontinuity due to the amplitude of Fig. 12(b). Although we have illustrated the result here for the case of two boundaries, the result is general: all discontinuities involving a subset of nonzero-entropy topological amplitudes with the same number of boundaries and the same cyclic order for particles assigned to each of the boundaries differing only in the number of handles having relative phases of $+1$. Again any phases associated with intermediate particles are already accounted for by this rule and are not therefore introduced explicitly.

Consider now the subset of topologies with a single boundary which includes the zero-entropy terms, the torus, etc. Here the same rule applies as above except that one must remember that the discontinuities across the zero-entropy term must include a $(-1)^n$ where n is the number of closed quark loops. The other terms with one or more handles contribute discontinuities with a relative phase of $+1$ with respect to zero entropy. This result is illustrated in Fig. 14.

VI. PHASES ASSOCIATED WITH REARRANGING PARTICLES ON THE BOUNDARIES—THE PAULI PRINCIPLE

The issue we deal with now is how to relate the phases of topological amplitudes which differ only by having a different cyclic order of the particles on the boundaries. We illustrate this situation for the cylinder but the results are again general. The point is that even though the particles on the two boundaries of the cylinder have different cyclic orders as shown in Fig. 15 it is always possible to slide the particles around the boundaries so that the particles in the closed loop are the same as indicated by x in Fig. 15. The relative phases of the amplitudes in Fig. 15 are just those associated with the corresponding planar zero-entropy amplitudes which result from breaking open the x loop. The rule is the one discussed in Sec. II; namely, dots have to be placed on the two amplitudes in such a way as to give even permutations of the baryon variables. A simple specific example is shown in Fig. 16 where we assume that particles D and F are identical baryons. The minus one indicates the relative phase between these and their common discontinuities. When $D=F$ the combination vanishes in accord with the Pauli principle. This illustrates how the Pauli principle is satisfied for each topology separately.

This rule for determining the relative phases for terms with different cyclic orders of particles on the boundaries can be immediately generalized to amplitudes with more than two boundaries. It is always possible to arrange for the particles forming the loops to be the same in each diagram (e.g., particle x in Fig. 15)—then the loops can be

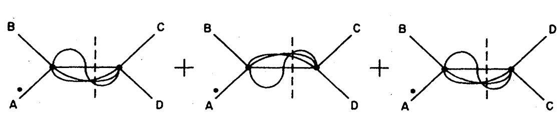

FIG. 13. Discontinuities of amplitudes with handles.

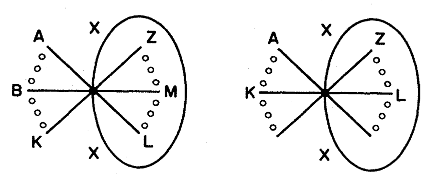

FIG. 15. Different cyclic orders of particles.

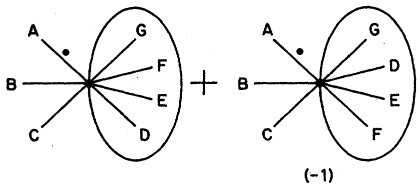

FIG. 16. The case of identical baryons.

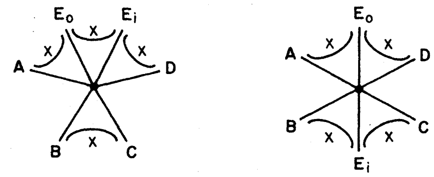

FIG. 18. Diagrams for determining relative phases of zero-entropy and cylinder amplitudes.

severed and relative phases determined from the rules for zero-entropy amplitudes. Even if there are contributions in which some of the particles move from one boundary to another, the same principles given here can be applied, because it is always possible to make the loops in each diagram consist of the same particles as in the other diagrams, thus enabling us to reduce the problem as before to one of comparing zero-entropy amplitudes.

VII. RELATIVE PHASES BETWEEN AMPLITUDES WITH DIFFERENT NUMBERS OF BOUNDARIES

Having found the relative phases associated with different numbers of handles for topological amplitudes with the same boundary structure in Sec. V, and the relative phases when the particles on the boundaries are rearranged in Sec. VI, we now turn to the problem of finding relative phases for amplitudes with different numbers of boundaries.

We discuss first the simplest example of this problem which involves determining the relative phases of discontinuities associated with the planar zero-entropy and the cylinder amplitude, the first having only one boundary and the second, two boundaries. An example of two such amplitudes for the same process is shown in Fig. 17.

It is important to note that not all cylinder amplitudes possess a corresponding zero-entropy amplitude for the same process. In order for this to be the case at least one quark or diquark line on the boundary containing the D and C particles must be the same as one on the other boundary containing the A and B particles. Using the technique introduced in Sec. IV it is possible to deduce the relative phases for discontinuities for the two amplitudes in Fig. 17. In this case the method involves considering the two zero-entropy amplitudes in Fig. 18 whose relative phases can be determined from a knowledge of zero-entropy amplitude phases (see Sec. II).

We have indicated in Fig. 18 the common quark or diquark line denoted by x. Connecting E_0 to E_i we arrive at the two topological terms in Fig. 17 and the result that the relative sign between the discontinuities of the two terms is -1 if particle E is a meson and is $+1$ if particle E is a baryonium. This result is indicated in Fig. 19, where $N(Q)_{AB}$ is the number of quarks (as opposed to antiquarks) in the AB channel of the zero-entropy amplitude. Expressed explicitly in terms of discontinuities Fig. 19 takes the form shown in Fig. 20, where we have indicated that there are other AB discontinuities for the cylinders as discussed earlier. All these cylinder discontinuities in AB must come with the same phase $(-1)^{N(Q)_{AB}}$ as a consequence of previous results (see Fig. 7) that all such discontinuities have the same relative sign. The dot must be located above the same particle in all diagrams. The result regarding the relative phase of the zero-entropy and cylinder term can also be derived directly from the form of the discontinuity in Fig. 20. The right-hand vertex in both terms can be written as a coherent sum of zero-entropy amplitudes shown in Fig. 21, where G and H are either both mesons or both baryonia. When the sum in Fig. 21 is multiplied by the common left-hand vertex amplitude we obtain again the phase rule shown in Fig. 20.

It is also possible in Fig. 19 to consider the BC discontinuity. The relative phases are the same as before and in fact, Fig. 19 states a general result regardless of the channel in which the common discontinuity is taken. The explicit BC discontinuity is shown in Fig. 22.

The discussion just given showing how to relate the phases for two-boundary (cylinder) and one-boundary (zero-entropy) amplitudes can be generalized in a straightforward way to find the relative phase between any two amplitudes for the same process differing by one in the total number of boundaries. We assumed that there are zero handles since the problem of adding handles has already been discussed. The general rule can be adequately illus-

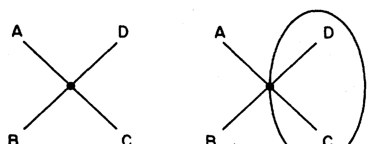

FIG. 17. Planar and cylinder amplitudes.

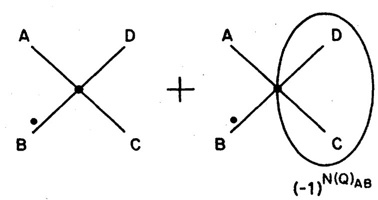

FIG. 19. Relative phase between zero-entropy and cylinder amplitudes.

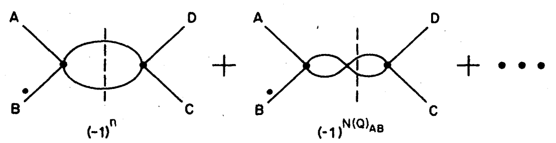

FIG. 20. Relative phases for zero-entropy and cylinder amplitudes.

trated by the following example of relating the phases of a three-boundary and a two-boundary amplitude. Figure 23 gives an example of a three-boundary process. In order for a two-boundary amplitude to exist corresponding to the process in Fig. 23(a) there must exist at least one quark and diquark line which is present on more than one of the boundaries in Fig. 23(a). An example of such a common line is indicated by x in Fig. 23. By cyclically permuting the particles on the two boundaries with the common x line we arrive at the configuration of Fig. 23(b). Thus, a two-boundary amplitude of the form shown in Fig. 24 exists. The relative phase between the graphs in Fig. 23 and Fig. 24 for any common discontinuity may be deduced using the same technique as that employed earlier in this section. The result is also similar: the relative phase is (-1) if $x\bar{x}$ is a meson and $(+1)$ if $x\bar{x}$ is a baryonium. This result is illustrated in Fig. 25.

VIII. PHASES ASSOCIATED WITH PRODUCTS OF NONZERO-ENTROPY AMPLITUDE

Up to now in discussing the relative phases of various discontinuities for a given topological amplitude, we considered only cases where (i) such discontinuities could be written as products of zero-entropy amplitudes (e.g., see Fig. 13), and (ii) no closed quark or diquark loops appeared explicitly in the discontinuities diagrams except in the case of zero entropy.

In this section we give the generalization of our rules to include these cases so far ignored. These rules are deduced from self-consistency arguments of the kind already discussed in previous sections. We shall proceed by illustrating the general rules for specific examples. For example, the cylinder amplitude, in addition to having the discontinuities shown in Fig. 7, also possess discontinuities of the form shown in Fig. 26.

Figure 26(a) includes one closed quark or diquark loop and Fig. 26(b) includes two such loops. These loops are

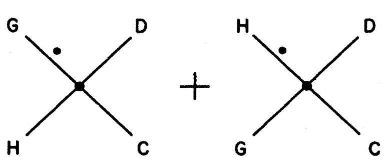

FIG. 21. Coherent sum of right-hand vertices in Fig. 20.

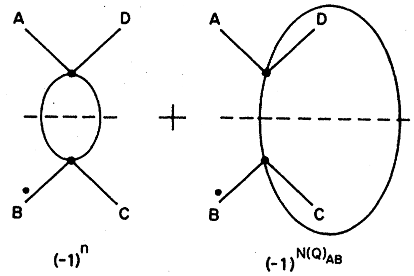

FIG. 22. Relative phases of another discontinuity of zero-entropy and cylinder amplitudes.

indicated by the lines labeled x and y. Since Fig. 26(a) is essentially a zero-entropy loop, it comes in with the usual phase $(-1)^n$ relative to the discontinuities in Fig. 7. To obtain the correct phase of 26(b) relative to the others we need only note the appropriate phases associated with the two amplitudes on the right of the dashed discontinuity line. This is deduced from the result in Fig. 19 and is shown in Fig. 27. The factor $(-1)^{N(Q)_{y\bar{y}}}$ will be $+1$ if y is a diquark line and -1 if y is a quark line. This gives the simple result that Fig. 26(b) contributes with the phase $(-1)^n$ where n is the *total* number of quark loops including those in y. Thus each discontinuity in Fig. 26 has a phase relative to Fig. 7 of $(-1)^n$, n being the number of quark loops in each case.

Another example of a discontinuity involving products of amplitudes which are not both zero entropy is shown in Fig. 28. In this case there are no closed loops. The topology of the diagram in Fig. 28 is that of one boundary and one handle. Since there are no closed quark or diquark loops, it contributes with a phase of plus one relative to the discontinuities shown in Fig. 14. Again this result is based on an understanding of the relative phase between the planar and the cylinder contributions.

Our general result is then the following: discontinuities of a given topological amplitude contribute with relative phases determined by $(-1)^n$ where n is the total number of quark loops. This is true whether the amplitudes in the discontinuity product are zero entropy or not. Relative phases between with different topological amplitudes are still determined by the rules given in earlier sections.

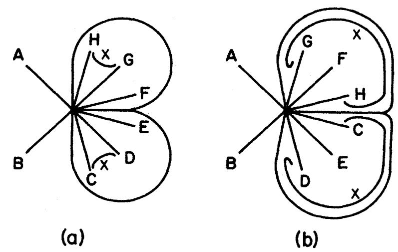

FIG. 23. Three-boundary diagrams.

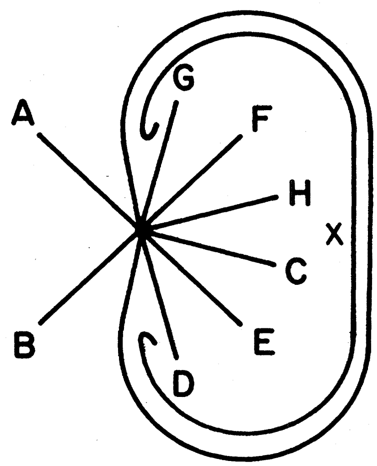

FIG. 24. Two-boundary diagram for process in Fig. 23.

IX. SUMMARY AND CONCLUSIONS

We have derived here the rules for determining the relative phases of the various terms in the topological expansion. For simplicity we have considered only naked topologies (without chiral or color switches) and situations where all quarks have the same patch orientation, although generalizations of our work to include other cases should be straightforward. The present results can be summarized as follows: With each distinct topology we associate a specific topological amplitude which is drawn with momentum lines emanating from a single vertex and with a dot placed before a given external particle. In Fig. 29, several such amplitudes contributing to a four-particle process are shown.

We give rules for (i) the determination of the phases associated with the discontinuities of a given topological amplitude, and (ii) the determination of the relative phases between different topological amplitudes contributing to the particular process.

We first summarize the rules for (i). Discontinuities for a given topological amplitude involve a product of two topological amplitudes both of which may be zero-entropy

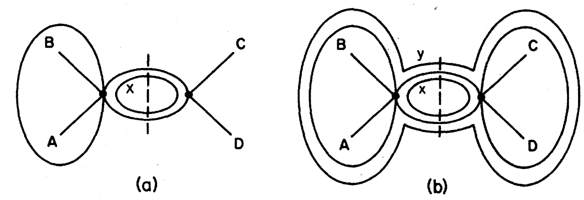

FIG. 26. Nonzero-entropy amplitudes in discontinuity products.

amplitudes or one or both may be nonzero-entropy amplitudes (see, e.g., Figs. 14 and 26). In all of these cases the phase of the discontinuity for the given topological amplitude is $(-1)^n$ where n is the total number of quark loops in the intermediate state. With this rule no phases are to be associated with the intermediate particles. The phases associated with the external particles and the dot location are to be determined from (2.1) and such phases are common to every topological amplitude for the given process.

The rules for (ii) above giving the relative phases between different topological amplitudes must be broken down into several distinct cases which can be summarized as follows.

(a) Topological amplitudes which differ only by having different numbers of handles are to be added with a relative phase of $+1$.

(b) Different topological amplitudes having the same number of boundaries are to be compared by sliding the particles around on the boundaries until the particles comprising the closed momentum loops are of the same type in each amplitude. Then by severing the closed loops the relative phases of such terms are determined by the same rules as those for the corresponding zero-entropy amplitudes (see, e.g., Fig. 4).

(c) To determine the relative phase between two different topological amplitudes for the same process differing by one in the total number of boundaries we begin with the amplitude having the larger number (say, $N+1$) of boundaries. If an N-boundary amplitude for the process exists then as discussed in Sec. VIII, there must exist at least one quark or diquark line which is present on more than one of the boundaries in the $(N+1)$-boundary amplitude. By sliding the particles around on these two

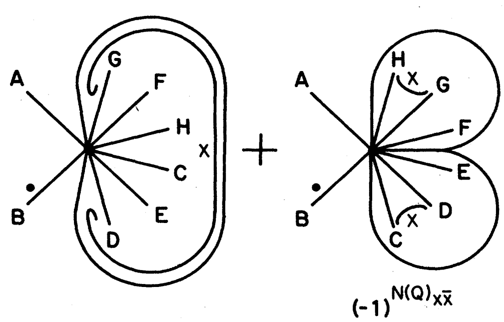

FIG. 25. Relative phases for two- and three-boundary amplitudes.

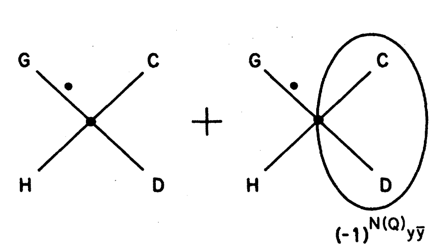

FIG. 27. Application of Fig. 19 to right-hand vertices of Fig. 26.

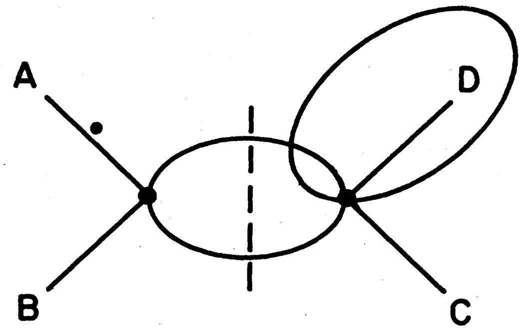

FIG. 28. Discontinuity of amplitude with one-boundary and one handle.

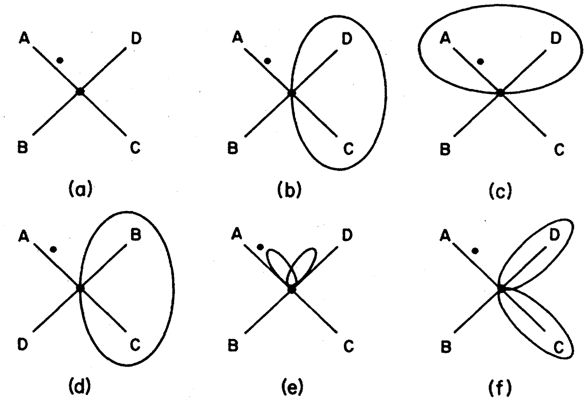

FIG. 29. Different topological amplitudes for a given process.

boundaries this common quark or diquark line can be made to circle the interior of the boundary loop for both boundaries [see Fig. 23(b)]. These two boundary loops can then be combined to give an amplitude with N boundaries (see Fig. 24). The relative phase between these two topological amplitudes is (-1) if the common line above is a quark line and $(+1)$ if the common line is a diquark line.

All other relative phases between different topological amplitudes may be determined by successive and combined applications of the rules given above.

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