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Solution of the light-cone equation for the relativistic bound state

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The ladder approximation to the bound-state equation at equal light-cone time is investigated in the framework of scalar theory. With help of the Fock transformation the equation is reduced to an eigenvalue problem for a compact operator. The eigensolution for the ground state is found.

I. INTRODUCTION

The conventional tool for dealing with the relativistic bound-state problem is the Bethe-Salpeter[1] equation. The one problem which is treated by explicitly covariant methods is the solution of the Wick-Cutkosky model, i.e., the Bethe-Salpeter equation describing the bound state of two spinless bosons of mass m interacting via exchange of a massless scalar field.

Allowing for at most one exchanged boson in the intermediate states one obtains the Bethe-Salpeter equation in the ladder approximation, BSLA (Wick equation[2]). Because of symmetries that are present and the possibility of performing a Wick rotation, Wick[2] and Cutkosky[3] were able to introduce a spectral representation for the bound-state wave function in the momentum space and reduce the eigenvalue problem to one of the Sturm-Liouville type for this spectral function. In this way they were able to obtain a nonperturbative solution of the BSLA for any physically allowable value M of the bound-state mass, i.e., for $0 \leq M \leq 2m$.

An alternative approach to relativistic bound-state problems is the light-cone quantization method[4] which provides a Hamiltonian formalism and Fock-state representation at equal light-cone time $\tau = t + z/c$. The momentum-space bound-state solutions to the system of coupled relativistic equations are functions of the light-cone variables x_i and $\mathbf{k}_\perp$, and thus are immediately suitable for calculations of covariant observables.

This method has been attracting considerable interest over the last few years for its unique and remarkable property, that the perturbative vacuum state is also an eigenstate of the full Hamiltonian. This makes it ideally suited for studying QCD dynamics in large-momentum-transfer reactions,[5] for the Tamm-Dancoff truncation of the Fock space is there a genuine perturbative approach. Therefore, covariant observables, such as structure functions, distribution amplitudes, correlations, anomalous moments, and other hadronic properties could be calculated within a standard perturbative scheme.

Unfortunately, the mathematical complexity of the effective light-cone equation offers a formidable challenge. Even the lowest-order (light-cone ladder approximation or LCLA) equation for the bound state of the Wick-Cutkosky model (i.e., the Weinberg[6] equation) constitutes a serious problem, which remains, to our knowledge, only partially solved. The binding energy was calculated several years ago in the perturbative way for small values of the coupling constant by Feldman, Fulton, and Townsend.[7] An approximate expression for the bound-state wave function was proposed by Karmanov,[8] who investigated the asymptotic properties of the LCLA kernel. Recently, Brodsky, Ji, and Sawicki[9] obtained the asymptotic expression for the large-momentum-transfer behavior of the bound-state wave function by means of the evolution-equation approach. However, the questions of the bound-state mass and the exact form of the wave function have been left open.

The purpose of this paper is to present the nonperturbative solution of the LCLA equation valid for all physically admissible values of the bound-state mass M. This is achieved by means of suitable transformation of variables in the light-cone equation and subsequent application of the Fock transformation,[10] which establishes a one-to-one correspondence between the unit hypersphere S^3 in R^4 and its stereographic projection onto the hyperplane R^3 of those new variables. The reader may realize that the final goal of solving the light-cone equations is achieved by means of a technique similar to that used in nonrelativistic molecular physics.

The paper is organized as follows. In Sec. II the light-cone equation is presented and the relativistic momentum is introduced as the new variable. In Sec. III the Fock transformation is outlined and the resulting eigenvalue problem is formulated. In Sec. IV the numerical solution for the ground state is presented. A short discussion of the result is given in Sec. V.

II. THE LIGHT-CONE BOUND-STATE EQUATION

We shall consider the light-cone description of the relativistic composite system of two scalar particles interacting via exchange of a massless scalar boson (Wick-Cutkosky model[2,3]). The interaction Lagrangian is $L = g{:}\phi^2\phi_0{:}$, where ϕ is a scalar field with mass m and ϕ_0 is a massless field. The bound state can be described by means of the Fock components of the state vector defined on the light-front surface. We denote them by wave functions. The effective equation for the two-body wave function reads[6,9]

$$\psi(x_i,\mathbf{k}_\perp)=\frac{1}{x_1x_2}\frac{1}{M^2-\dfrac{m^2+k_\perp{}^2}{x_1x_2}}g^2\int_0^1[dy]\int[d^2l_\perp]K(x_i,\mathbf{k}_\perp,y_i,\mathbf{l}_\perp\,|\,M^2)\psi(y_i,\mathbf{l}_\perp)\ , \tag{2.1}$$

where $x_i(y_i)$ are the fractions of the total P^+ momentum of the bound state carried by the ith constituent, $x_1+x_2=1$, $[dy]=dy_1dy_2\delta(1-y_1-y_2)$; $\mathbf{k}_\perp$ ($\mathbf{l}_\perp$) are the two-dimensional perpendicular momenta $[d^2l_\perp]=[1/2(2\pi)^3]d^2l_\perp$; and M is the mass of the bound state.

In LCLA the kernel of Eq. (2.1) reads

$$K(x_i,\mathbf{k}_\perp;y_i,\mathbf{l}_\perp\,|\,M^2)=\frac{\theta(y_1-x_1)}{y_1-x_1}\frac{1}{M^2-\dfrac{m^2+k_\perp{}^2}{x_1}-\dfrac{(\mathbf{k}_\perp-\mathbf{l}_\perp)^2}{y_1-x_1}-\dfrac{m^2+l_\perp{}^2}{y_2}}+(1\leftrightarrow 2)\ . \tag{2.2}$$

We note here that the original Wick equation, when projected[9] onto the light cone, also takes the form of Eq. (2.1). The kernel, however, contains not only one-boson-exchange terms as in Eq. (2.2) but also an infinite sum of multiboson-exchange irreducible box diagrams, which provide an additional interaction in the two-body system.

It is useful to introduce the relativistic relative momentum[8] $\mathbf{q}$ as the new variable and work hereafter in polar coordinates. We have

$$\mathbf{k}_\perp=\mathbf{q}_\perp=(q\sin\theta\cos\phi,q\sin\theta\sin\phi)\ ,$$
$$x_{1,2}=\frac{1}{2}\left[1\mp\frac{q\cos\theta}{\epsilon(q)}\right]\ , \tag{2.3}$$

so that

$$q^2=\frac{k_\perp{}^2+m^2x^2}{1-x^2}\ ,$$
$$\cos\theta=-x\left[\frac{k_\perp{}^2+m^2}{k_\perp{}^2+m^2x^2}\right]^{1/2}\ , \tag{2.4}$$

where

$$\epsilon(q)=(m^2+q^2)^{1/2},\quad x=x_1-x_2$$

and

$$dx_1d^2k_\perp=2x_1x_2d^3q/\epsilon(q)\ .$$

In terms of these variables Eqs. (2.1) and (2.2) take the form

$$\left[q^2+m^2-\frac{M^2}{4}\right]\phi(\mathbf{q})=\frac{g^2}{4(2\pi)^3}\int\frac{d^3q'}{\epsilon(q')}V(\mathbf{q},\mathbf{q}',M^2)\phi(\mathbf{q}') \tag{2.5}$$

where

$$V(\mathbf{q},\mathbf{q}',M^2)=[(\mathbf{q}-\mathbf{q}')^2+R(\mathbf{q},\mathbf{q}',M^2)]^{-1}\ ,$$
$$R(\mathbf{q},\mathbf{q}',M^2)=-q\cdot q'\cos\theta\cos\theta'\frac{[\epsilon(q)-\epsilon(q')]^2}{\epsilon(q)\epsilon(q')}+[\epsilon^2(q)+\epsilon^2(q')-\tfrac{1}{2}M^2]\times\left|\frac{q\cos\theta}{\epsilon(q)}-\frac{q'\cos\theta'}{\epsilon(q')}\right|\ , \tag{2.6}$$

and the wave function ϕ is defined by

$$\phi(\mathbf{q})=\frac{m^2+q_\perp{}^2}{m^2+q^2}\psi(\mathbf{q})\equiv(1-x^2)\psi(x_i,\mathbf{k}_\perp)\ , \tag{2.7}$$

where $\psi(\mathbf{q})$ is the wave function of Eq. (2.1) now expressed in terms of the new variables. We note here that the kernel of (2.5) clearly has azimuthal symmetry and is invariant under the parity transformation $\mathbf{q}\to-\mathbf{q}$, $\mathbf{q}'\to-\mathbf{q}'$. We introduce the following parametrization of the bound-state mass M:

$$B=2m-M\ ,$$
$$\bar{B}=B-B^2/4m\ , \tag{2.8}$$
$$\eta^2=(M/2m)^2=1-\bar{B}/m\ .$$

Equation (2.5) now reads

$$\left[\frac{q^2}{m}+\bar{B}\right]\phi(\mathbf{q})=\frac{g^2}{4(2\pi)^3m^2}\int d^3q'\frac{1}{(1+q'^2/m^2)^{1/2}}\times V(\mathbf{q},\mathbf{q}',M^2)\phi(\mathbf{q}')\ . \tag{2.9}$$

The structure of Eq. (2.9) is analogous to that of the Lippmann-Schwinger equation for nonrelativistic positronium with reduced mass $\mu=m/2$,

$$\left[\frac{q^2}{2\mu}+B\right]\phi(\mathbf{q})=\frac{\alpha}{2\pi^2}\int d^3q'\frac{1}{(\mathbf{q}-\mathbf{q}')^2}\phi(\mathbf{q}') \tag{2.10}$$

if we set $\alpha=g^2/16\pi m^2$. This equation has bound-state solutions $\phi^{(0)}_{nlm}(q)$ (Ref. 11) with eigenvalues $B_n/m=\alpha^2/4n^2$. Note that defining $\lambda=\alpha/\pi$ we conform with the original Wick notation.[2]

In the nonrelativistic region ($q/m\ll 1$ and $\bar{B}/m\ll 1$) the kernel of Eq. (2.9) coincides with that of Eq. (2.10). It had been demonstrated by Feldman, Fulton, and Townsend[7] that the Coulomb wave function $\phi^{(0)}_{nlm}$ can serve as a lowest-order solution to Eq. (2.9) for small values of coupling constant α.

The Lippmann-Schwinger equation (2.10) was first solved by Fock,[10] who transformed it in the O(4)-invariant integral equation for the four-dimensional spherical harmonics. The same transformation will be used in the actual case of Eq. (2.9) as presented below.

III. THE FOCK TRANSFORMATION AND THE HYPERSPHERICAL BASIS

The Fock transformation[10] is defined by the one-to-one mapping of the hyperplane R^3 of the variables $\mathbf{q}$ onto the unit sphere S^3 in R^4,

$$R^3 \ni \mathbf{q} \underset{P^{-1}_{q_0}}{\rightarrow} \xi \equiv (\xi_0, \boldsymbol{\xi}) \in S^3 , \tag{3.1}$$

where P_{q_0} is the stereographic projection

$$S^3 \underset{P_{q_0}}{\rightarrow} R^3 ,$$

and the parameter q_0 is related to the binding energy via

$$(q_0/m)^2 = B/m . \tag{3.2}$$

For $\xi \equiv P_{q_0}^{-1}(\mathbf{q})$ one defines

$$\xi_0 = \cos\alpha = \frac{{q_0}^2 - q^2}{{q_0}^2 + q^2} ,$$
$$\boldsymbol{\xi} = \sin\alpha(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.3}$$
$$= \frac{2q_0}{{q_0}^2 + q^2}\mathbf{q} .$$

Here α, θ, and ϕ are spherical coordinates on the sphere S^3, and $q = |\mathbf{q}|$. Conversely,

$$\mathbf{q} = \frac{q_0}{1+\xi_0}\boldsymbol{\xi} \tag{3.4}$$

and

$${q_0}^2 + q^2 = \frac{2{q_0}^2}{1+\xi_0} . \tag{3.5}$$

The invariant measure $d\Omega$ on the sphere S^3 is related to the Euclidean measure d^3q in R^3 as follows:

$$d\Omega = \sin^2\alpha \sin\theta \, d\alpha \, d\theta \, d\phi = \left[\frac{2q_0}{{q_0}^2 + q^2} \right]^3 d^3q , \tag{3.6}$$

with $0 \le \alpha \le \pi$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

The distance $|\mathbf{q} - \mathbf{q}'|$ may be transformed accordingly and one would obtain

$$|\mathbf{q} - \mathbf{q}'|^2 = \frac{({q_0}^2 + q^2)({q_0}^2 + q'^2)}{4{q_0}^2} |\xi - \xi'|^2 . \tag{3.7}$$

With the definition

$$\psi(\Omega) = \frac{1}{4{q_0}^{5/2}}({q_0}^2 + q^2)^2 \phi(\mathbf{q}) , \tag{3.8}$$

the Lippmann-Schwinger equation (2.10) takes the form

$$\psi(\Omega) = \frac{\lambda}{2\pi}\frac{1}{2}\left[\frac{m}{q_0}\right] \int d\Omega' \frac{1}{|\xi - \xi'|^2}\psi(\Omega') , \tag{3.9}$$

where Ω, Ω' are the polar coordinates of the points ξ, ξ' on the unit sphere S^3, respectively. The square of the distance between points on the unit sphere could be written as

$$\frac{1}{|\xi - \xi'|^2} = \sum_\mu \frac{2\pi^2}{n} Y_\mu(\xi) Y_\mu^*(\xi') , \tag{3.10}$$

where the hyperspherical harmonics $Y_\mu(\xi)$ form the orthonormal basis on the sphere S^3

$$\int d\Omega \, Y_\nu(\Omega) Y_\mu(\Omega) = \delta_{\nu\mu} , \tag{3.11}$$

and $\mu = (n, l, m)$, where the integers n, l, m are such that

$$0 \le |m| \le l < n . \tag{3.12}$$

The explicit expression for Y_μ reads[12]

$$Y_{nlm}(\Omega) = (-i)^l \left[\frac{2n(n-l-1)!}{\pi(n+l)!} \right]^{1/2} \sin^l\alpha \times \left[\frac{d}{d\cos\alpha} \right]^l C^1_{n-1}(\cos\alpha) Y_{lm}(\theta,\phi) , \tag{3.13}$$

where the $C^1_{n-1}(\cos\alpha)$ is the Gegenbauer polynomial and Y_{lm} are the spherical harmonics on S^2 in the convention of Messiah,[13] i.e.,

$$Y_{lm}(\theta,\phi) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!} \right]^{1/2} P_{lm}(\cos\theta) e^{im\phi} , \tag{3.14}$$

$$P_{lm} \equiv \frac{(-1)^l}{2^l l!}(\sin\theta)^m \left[\frac{d}{d\cos\theta} \right]^{l+m} (\sin\theta)^{2l} , \; -l \le m \le l . \tag{3.15}$$

It follows from Eqs. (3.10) and (3.11) that the Lippmann-Schwinger equation (3.9) is diagonal in the basis Y_μ with the eigenvalues $q_0/m = (\pi\lambda)/2n$. Thus, the eigenfunctions of Eq. (2.10) are

$$\phi^{(0)}_{nlm} = \frac{4{q_0}^{5/2}}{({q_0}^2 + q^2)^2} Y_{nlm}(\Omega) .$$

The proportionality constant in (3.8) is chosen to ensure the following relation between scalar products on S^3 and R^3:

$$\int d\Omega \, \psi_1^*(\Omega)\psi_2(\Omega) = \int d^3p \frac{{p_0}^2 + p^2}{2{p_0}^2} \phi_1^*(\mathbf{p})\phi_2(\mathbf{p})$$
$$= \int d^3p \, \phi_1^*(\mathbf{p})\phi_2(\mathbf{p}) , \tag{3.16}$$

so that Eq. (3.11) implies orthonormality of $\phi^{(0)}_{nlm}$. [The last step in Eq. (3.16) follows by using the virial theorem.] In the actual case of the light-cone equation (2.9) it is useful to work with a symmetric kernel. To this end we define the wave function

$$\psi(\Omega) = C f^{1/2}(q)({q_0}^2 + q^2)^2 \phi(\mathbf{q}) , \tag{3.17}$$

where now

$$(q_0/m)^2 = \bar{B}/m , \tag{3.2'}$$

C is a normalization constant, and

$$f(q) = (1 + q^2/m^2)^{-1/2} . \tag{3.18}$$

Note that $f^{1/2}$ factors appearing in Eqs. (3.17) and (3.20) below and originating in the phase space of the relativistic kinematics are the "minimal relativity" factors of Ref. 14.

Upon Fock transformation, Eq. (2.9) yields

$$\psi(\Omega) = \frac{\lambda}{2\pi}\frac{1}{2}\left[\frac{m}{q_0}\right] \int d\Omega' \frac{1}{|\xi - \xi'|^2} r(\xi, \xi', q_0/m)\psi(\Omega') , \tag{3.19}$$

where we defined

$$r(\xi,\xi',q_0/m)=f^{1/2}(q)\frac{|\xi-\xi'|^2}{|\xi-\xi'|^2+\Delta(\xi,\xi',q_0/m)}f^{1/2}(q') . \tag{3.20}$$

Explicitly, in terms of polar coordinates on S^3, we have

$$|\xi-\xi'|^2=2[1-\cos\alpha\cos\alpha'-\sin\alpha\sin\alpha'\cos\theta\cos\theta'-\sin\alpha\sin\alpha'\sin\theta\sin\theta'\cos(\phi-\phi')] , \tag{3.21}$$

and

$$\Delta(\xi,\xi',q_0/m)=-\left[\frac{f(q)}{f(q')}+\frac{f(q')}{f(q)}-2\right]\sin\alpha\sin\alpha'\cos\theta\cos\theta'+2\frac{q_0}{m}\left[\frac{1}{1+tg^2\alpha/2}+\frac{1}{1+tg^2\alpha'/2}\right]$$
$$\times|f(q)(1+tg^2\alpha/2)\sin\alpha\cos\theta-f(q')(1+tg^2\alpha'/2)\sin\alpha'\cos\theta'| , \tag{3.22}$$

with

$$f(q)=[1+(q_0/m)^2tg^2\alpha/2]^{-1/2} . \tag{3.18'}$$

Evidently, the kernel of Eq. (3.19), although symmetric in variables ξ,ξ' is not diagonal in the basis Y_μ. However, due to underlying azimuthal symmetry the kernel is still diagonal in magnetic quantum number m and one has the decomposition

$$r(\xi,\xi',q_0/m)=\sum_m\sum_{nl}\sum_{n'l'}C_{nlm,n'l'm}(q_0/m)Y_{nlm}(\Omega)Y_{n'l'm}(\Omega') , \tag{3.23}$$

where

$$C_{nlm,n'l'm'}(q_0/m)=\int d\Omega\int d\Omega' Y^*_{nlm}(\Omega)r(\xi,\xi',q_0/m)Y_{n'l'm'}(\Omega') , \tag{3.24}$$

and, for given m, the sum in (3.23) runs over all (n,l) and (n',l') satisfying condition (3.12). Moreover, the invariance of the problem under the parity transformation ensures that the right-hand side of Eq. (3.24) vanishes unless l and l' are both either even or odd. It follows that the general solution of Eq. (3.19) could be written in the form

$$\psi_m(\Omega)=\sum_{nl}a_{nlm}Y_{nlm}(\Omega) , \tag{3.25}$$

where l's are either even or odd and $n>l\geq|m|$. We combine Eqs. (3.19), (3.10), (3.23), and (3.25). The resulting products of hyperspherical harmonics are next reexpressed as follows:

$$Y_{nlm}(\Omega)Y_{n'l'm'}(\Omega)=\sum_{n''l''m''}[(2l+1)(2l'+1)nn'n''/2\pi^2]^{1/2}(lm,l'm'|l''m'')\times\begin{Bmatrix}\frac{1}{2}(n-1),\frac{1}{2}(n-1),l\\ \frac{1}{2}(n'-1),\frac{1}{2}(n'-1),l'\\ \frac{1}{2}(n''-1),\frac{1}{2}(n''-1),l''\end{Bmatrix}Y_{n''l''m''}(\Omega) , \tag{3.26}$$

where the curly brackets are the $9J$ symbols. With help of the orthogonality property (3.11), one arrives at the system of equations

$$a_{nlm}=\frac{\lambda}{4\pi}\left[\frac{m}{q_0}\right]\sum_{n'l'}d_{nlm,n'l'm}(q_0/m)a_{n'l'm} , \tag{3.27}$$

where

$$d_{nlm,n'l'm'}=\sum_{n''l''m'}\sum_{n_1l_1m_1}\sum_{n_2l_2m_2}[(2l''+1)(2l_1+1)n''n_1n/2\pi^2]^{1/2}(l''m'',l_1m_1|lm)$$
$$\times[(2l''+1)(2l_2+1)n''n_2n'/2\pi^2]^{1/2}(l''m''l_2m_2|l'm')$$
$$\times\begin{Bmatrix}\frac{1}{2}(n'-1),\frac{1}{2}(n''-1),l''\\ \frac{1}{2}(n_1-1),\frac{1}{2}(n_1-1),l_1\\ \frac{1}{2}(n-1),\frac{1}{2}(n-1),l\end{Bmatrix}\begin{Bmatrix}\frac{1}{2}(n''-1),\frac{1}{2}(n''-1),l''\\ \frac{1}{2}(n_2-1),\frac{1}{2}(n_2-1),l_1\\ \frac{1}{2}(n!-1),\frac{1}{2}(n'-1),l'\end{Bmatrix}C_{n_1l_1m_1,n_2l_2m_2}(q_0/m) . \tag{3.28}$$

IV. NUMERICAL RESULTS

Equation (3.19) constitutes the eigenvalue problem

$$\psi=\frac{\lambda}{4\pi}\left[\frac{m}{q_0}\right]D\psi\ , \tag{4.1}$$

for the compact operator D acting on the Hilbert space $H=L_C{}^2(S^3)$ of the square-integrable functions on the sphere S^3. Because of the underlying azimuthal symmetry of the problem, the Hilbert space is the direct sum of subspaces H_m indexed by the magnetic quantum number m. This is quite similar to a situation one encounters when studying the bound-state energy levels of molecules.[12,15] In the hyperspherical basis Y_μ the operator D, when acting on the particular Hilbert subspace H_m, could be represented by the infinite blocklike matrix $D^{(m)}$ with elements $D^{(m)}_{nl,n'l'}=d_{nlm,n'l'm}$ [cf. Eq. (3.28)]. The blocks of several rows (columns) are numbered by n (n'). Within a particular block the rows (columns) are indexed by l (l'). The Hermitian matrix $D^{(m)}$, constructed in this way, is, in fact, real and symmetric due to the invariance of the problem with respect to the parity transformation, as discussed after Eq. (2.7). This ensures a significant gain in computing time and memory. In order to find the ground state of the system, we set the magnetic quantum number $m=0$ and searched for the largest eigenvalue q_0/m for a given value of the coupling constant λ. To make the problem finite, we truncate the harmonic basis Y_μ, $\mu=(nlm)$, at some maximal value N of the principal quantum number n and approximate the matrix $D^{(m=0)}$ by the one calculated in the truncated basis. There are $k=N(N+1)(N+2)$ basis functions contributing to the calculation of the matrix elements $D^{(m=0)}_{nl,n'l'}$, whereas the rank of the matrix is $f=N(N+1)/2$; for example, $k=4,20,56,120$ and $f=3,10,21,36$ for $N=2,4,6,8$, respectively. The ground-state solution of Eq. (3.19) is then approximated by the finite expansion

$$\psi_0(\Omega)=\sum_{n=1}^{N}\sum_{l\ \mathrm{even}} a_{nl0}Y_{nl0}(\Omega)\ . \tag{4.2}$$

The truncation of the harmonic basis Y_μ is the crucial factor limiting the accuracy of the eigenvalues. However, since D is a compact operator, a sequence of matrices of increasing rank f provides a converging approximation in norm on S^3.

The calculations were performed on the IBM3081 machine at the Computer Center of the Virginia Polytechnic Institute and State University. The eigenvalues have been calculated with help of the subroutine RS from the EISPACK-Matrix Eigensystem Routines (Document Number MT08). The Clebsch-Gordan coefficients have been expressed in terms of the $3J$ coefficients of Wigner.[16] The $3J$ and $9J$ coefficients have been calculated with the help of subroutines COEF3J and COEF9J (Centre d'Etudes de Limeil pour Informationes Complementaires document 362/71). It turned out that a stability of the largest eigenvalue better than 0.1% is obtained already for the truncation of the harmonic basis at $N=8$, corresponding to the rank $f=36$ of the matrix $D^{(m=0)}$.

Another difficulty is related to the evaluation of coefficients $C_{nlm,n'l'm'}$ in the expansion (3.23) of the integral kernel. In many molecular problems such coefficients could be calculated analytically (see, for example, the analysis[15] of the problem of a single electron in the presence of many fixed nuclei). In the actual case, the evaluation of integrals over azimuthal angles ϕ and ϕ' could be carried analytically and the remaining integration is performed numerically. This is the most time-consuming part of the computations, since the number of integrals grows as k^2 when the rank f is increased.

In Fig. 1 we present the relation between the mass of the ground state and the coupling constant λ as obtained upon solving the eigenvalue problem (curve LC). For comparison we draw also the curves representing the solution of the Wick equation[2] as obtained by Cutkosky[3] (curve C) and the nonrelativistic Lippmann-Schwinger equation (curve LS). For a strongly bound system one clearly sees a difference between results of relativistic and nonrelativistic dynamics.

In Fig. 2 we represent the details of a weakly bound system. Here our light-cone results are again compared with those based on Wick and Lippmann-Schwinger equations, and, in addition, with the result from perturbation theory.[7] Within the perturbation theory the eigenvalues of the Wick equation and light-cone equation agree to order $\alpha^3\ln\alpha$ and for the ground state one has

$$\bar{B}/m=\frac{\alpha^2}{4}\left[1+\alpha\left[\frac{4}{\pi}\ln\alpha+C\right]\right]\ , \tag{4.3}$$

where $\alpha=\lambda\pi$. The unknown α^3 term is written in such a way as to provide a direct comparison with the $\alpha^3\ln\alpha$ term. The expression (4.3) may seem to be justified as long as the perturbative $\alpha^3\ln\alpha$ corrections do not exceed,

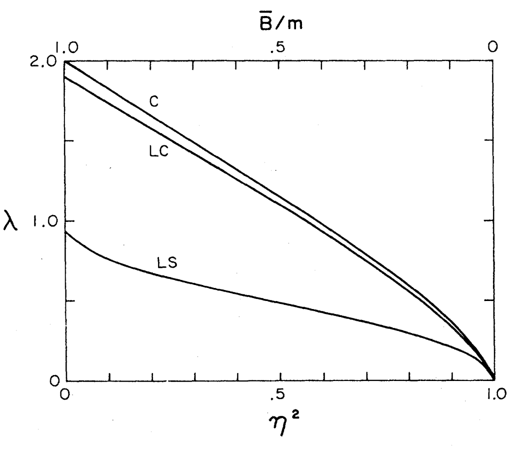

FIG. 1. Coupling constant λ versus η^2 for the ground-state solution of the scalar model in three different ladder approximations. The curve labeled LC represents our solution of the light-cone equation. The curve C corresponds to Cutkosky's solution (Ref. 3) of the Wick (Ref. 2) equation and the curve LS corresponds to the solution of the Lippmann-Schwinger equation (2.10).

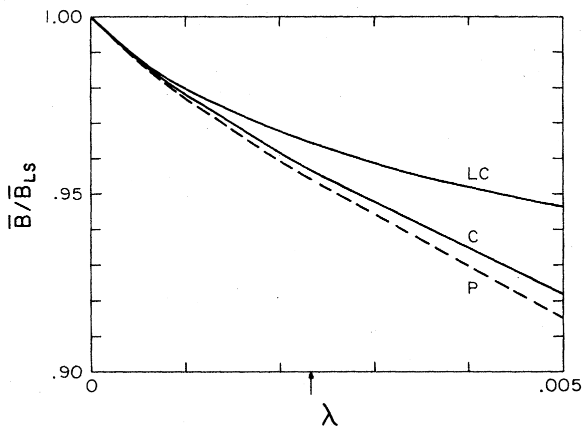

FIG. 2. Binding energy $\bar{B}$ normalized by the nonrelativistic binding energy $\bar{B}_{LS}$ versus coupling constant λ for the ground state of the weakly bound system. The letters LC and C as in Fig. 1. The curve P represents the perturbative approximation to the curves LC and C as obtained in Ref. 7. The arrow indicates the scale of positronium ($\alpha \simeq \frac{1}{137}$).

say, 10% of the difference between unperturbed (nonrelativistic) energies of the first two states of the system. To ensure this one needs $|\alpha \ln\alpha| \leq 0.06$, i.e., $\alpha \leq 0.014$ or $\lambda \leq 0.005$. Therefore a sensible scale of where one expects the perturbation theory to be valid is provided by the scale of positronium, $\alpha \simeq \frac{1}{137} = 0.0073$ ($\lambda \simeq 0.0023$). We find our numerical results consistent with the conclusion of Ref. 7. Indeed, for positronium we have $|(4/\pi)\ln\alpha| = 6.26$, whereas the value of the coefficient C is found to be $C \simeq 0.35$ for the case of the Wick equation and $C \simeq 1.8$ for the case of the light-cone equation. This corresponds to a 1.3% correction due to the $C\alpha^3$ term for the case of the light-cone dynamics and only 0.25% for the Wick dynamics.

Finally, in Fig. 3, we display our results for the configuration mixing in the ground-state wave function (4.2). The probabilities $P_{nl} = |a_{nl0}|^2$ are plotted versus η^2. Surprisingly, the admixture of higher harmonics does not exceed 3% even for the extreme case of vanishing bound-state mass M, i.e., for $\eta = 0$.

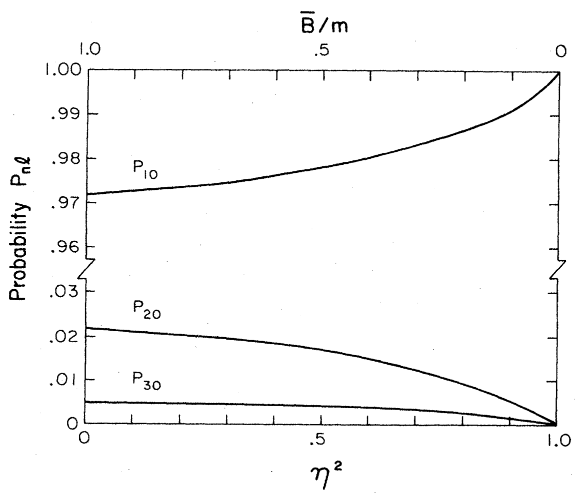

FIG. 3. Configuration mixing in the ground-state solution of the light-cone equation. The probabilities $P_{nl} = |a_{nl0}|^2$ [cf. Eq. (4.2)] are drawn versus η^2.

V. CONCLUSION

In this paper we investigated the ladder approximation to the relativistic bound-state equation in the framework of the Wick-Cutkosky model quantized at equal light-cone time. Introducing suitable relativistic variables and subsequently performing the Fock transformation, we reduced the problem to an eigenvalue problem of the Sturm-Liouville type for a compact operator defined on the Hilbert space $L_C{}^2(S^3)$. The eigensolution for the largest eigenvalue (i.e., ground-state solution) has been numerically found and compared with the result obtained within other ladder-approximation schemes to this model, i.e., covariant ladder approximation to the Wick equation and the nonrelativistic Lippmann-Schwinger equation. There is an evident decrease in the binding energy upon going from nonrelativistic Lippmann-Schwinger to relativistic (light-cone and Wick) ladders. For a weakly bound system, such as positronium, both relativistic equations yield comparable results, in agreement with perturbation theory.

It is interesting to display what the result (4.2) yields for the original light-cone wave function defined by Eqs. (2.1) and (2.2). Using (4.2), (3.17), and (2.7) one obtains after some manipulations

$$\psi(x_i,\mathbf{k}_\perp) = C\left[\frac{m^2+k_\perp{}^2}{x_1x_2}\right]^{1/4}\frac{1}{x_1x_2}\frac{1}{\left[M^2-\dfrac{m^2+k_\perp{}^2}{x_1x_2}\right]^2} \times\left[1+\frac{a_{200}}{a_{100}}\frac{M^2-8m^2+\dfrac{m^2+k_\perp{}^2}{x_1x_2}}{M^2-\dfrac{m^2+k_\perp{}^2}{x_1x_2}}+\cdots\right]. \tag{5.1}$$

It is easy to see that in the nonrelativistic limit ($k_\perp/m \ll 1$, $|x| = |x_1 - x_2| \ll 1$, $B/m \ll 1$) one recovers the ground-state wave function of positronium. On the other hand, for the large momentum transfer limit, one obtains the asymptotic behavior

$$\psi(x_i,\mathbf{k}_\perp) \underset{k_\perp\to\infty}{\sim} \frac{1}{x_1x_2}\left[\frac{x_1x_2}{k_\perp{}^2}\right]^p [1+O(m^2/k_\perp{}^2)] \tag{5.2}$$

with $p = \frac{7}{4}$. This result clearly differs from an approximate form suggested by Karmanov[8] for it is cusp free and analytic. The systematic corrections to Eq. (5.2) originating from higher terms in (5.1) could be alternatively expressed in terms of anomalous dimensions (see Ref. 9).

We note here that the ground-state solution of the Wick equation, when projected onto a light cone takes the form[9]

$$\psi^{\mathrm{LC}}_{\mathrm{Wick}}(x_i,\mathbf{k}_\perp)=\frac{1}{x_1x_2}\frac{1}{\left[M^2-\dfrac{m^2+k_\perp{}^2}{x_1x_2}\right]^2}\frac{g(x_2-x_1,M)^2}{x_1x_2}\,, \tag{5.3}$$

where $g(z,m^2)$ is Wick's spectral function. The asymptotic $k_\perp\to\infty$ behavior again takes the form (5.2) with $p=2$ rather than $p=\frac{7}{4}$. The different powers originate in the minimal relativity factor appearing in the light-cone equation [see Eqs. (3.17) and (3.19)]. However, the effect of the minimal relativity factors on the resulting eigenvalue of the light-cone equation is negligible, most of the effect is due to a nonlocal Δ term in the integral kernel, cf. Eq. (3.20).

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[1]E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951). For an extensive list of references on the Bethe-Salpeter equation see N. Nakanishi, Prog. Theor. Phys. (Kyoto) Suppl. **43**, 1 (1969).

[2]G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[3]R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

[4]S. Chang and S. Ma, Phys. Rev. **180**, 1506 (1969); S.-J. Chang, R. G. Root, and T.-M. Yan, Phys. Rev. D **7**, 1133 (1973); S.-J. Chang and T.-M. Yan, *ibid.* **7**, 1145 (1973); T.-M. Yan, *ibid.* **7**, 1760 (1973); **7**, 1780 (1973).

[5]G. P. Lepage and S. J. Brodsky, Phys. Rev. D **11**, 2157 (1980).

[6]S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[7]G. Feldman, T. Fulton, and J. Townsend, Phys. Rev. D **7**, 1814 (1973).

[8]V. A. Karmanov, Nucl. Phys. **B166**, 378 (1979); **A362**, 331 (1981).

[9]S. J. Brodsky, C.-R. Ji, and M. Sawicki, Phys. Rev. D **32**, 1530 (1985).

[10]V. Fock, Z. Phys. **98**, 145 (1935).

[11]B. Podolsky and L. Pauling, Phys. Rev. **34**, 109 (1929).

[12]T. Shibuya and C. W. Wulfman, Proc. R. Soc. London **A286**, 376 (1965); B. R. Judd, *Angular Momentum Theory of Diatomic Molecules* (Academic, New York, 1975).

[13]A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1966).

[14]G. E. Brown, A. J. Jackson, and T. T. S. Kuo, Nucl. Phys. **A133**, 481 (1969); M. H. Partovi and E. L. Lomon, Phys. Rev. D **2**, 1999 (1970).

[15]C. Duchon, M. Cl. Dumont-Lepage, and J. P. Gazeau, J. Phys. A **15**, 1227 (1982).

[16]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University, Princeton, 1974).