Quantum field theory on discrete space-time. II

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A quantum field theory of bosons and fermions is formulated on discrete Lorentz space-time of four dimensions. The minimum intervals of space and time are assumed to have different values in this paper. As a result the difficulties encountered in the previous paper (complex energy, incompleteness of solutions, and inequivalence between phase representation and momentum representation) are removed. The problem in formulating a field theory of fermions is solved by introducing a new operator and considering a theorem of translation invariance. Any matrix element given by a Feynman diagram is calculated in this theory to give a finite value regardless of the kinds of particles concerned (massive and/or massless bosons and/or fermions).

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I) we have mentioned the philosophy of our discrete space-time and formulated a quantum field theory of bosons on this space-time of two dimensions. In that case we had the solutions of complex energy in the Klein-Gordon equation of discrete space-time version. Since these solutions seemed unfavorable for physical understanding, we have simply discarded them. However, the solutions of real energy alone cannot satisfy the completeness relation. Consequently, we did not have the equivalence between phase representation and momentum representation. This means that we should not discard the solutions of complex energy. On the other hand, it became clear from detailed investigation that the complex energy brings other difficulties into the theory. Therefore, the situation cannot be improved without changing the assumptions of I.

In this paper we give up one of the assumptions of I: the space unit is equal to the time unit, i.e., $\tau=\lambda$ $(c=1)$. The new assumption instead is $\tau=a\lambda$, where a is a certain constant given later.[2] We will then see that the completeness is recovered and the equivalence between phase representation and momentum representation holds again, though the symmetry of space and time is somewhat sacrificed. In Sec. II we describe bosons on discrete space-time of full four dimensions, and show that the points mentioned above are realized in this case. In contrast with the two-dimensional case we see that the propagator of the field does converge always even for the massless case.

When we formulate the theory of fermions on discrete space-time, we find that the difference operator so far used is unsuitable for this purpose, because the use of this operator leads necessarily to a non-Hermitian action and thus to inconsistent field equations. Therefore we must define a new difference operator. The operator should reveal right-left symmetry and should refer to the middle of neighboring space-time points. This would seem to contradict our original philosophy that there is no meaning in considering a smaller length than the unit of space-time. However, when we consider the translation invariance theorem in a wide sense, we see that it is not the case. In fact if we apply the new operator to the case of bosons, we obtain exactly the same results as those obtained before with the old operator (Sec. III).

In Sec. IV we solve the Dirac equation on discrete space-time of four dimensions. The field is quantized canonically and the propagator is obtained. All is straightforward and no problem is found in this case. In the last section (Sec. V) we list the problems remaining, especially the relation to the lattice gauge theory.

II. SCALAR FIELD

As we mentioned in Sec. I we assume that the time unit is not equal to the space unit, i.e., $\tau=a\lambda$ $(a<1)$. If we adopt the unit $\lambda=1$, then the time unit is $\tau=a$. Thus any space-time point is represented by $x=an_0\hat{0}+n_1\hat{1}+n_2\hat{2}+n_3\hat{3}$, where $\hat{\mu}$ $(\mu=0,1,2,3)$ is a unit vector directed along the μ axis and n_μ's are integers.

Now let $\phi(x)$ be a scalar field at x and define a difference operator Δ_μ as follows:

$$\Delta_0\phi(x)\equiv\frac{1}{a}[\phi(x+a\hat{0})-\phi(x)]\ , \tag{2.1}$$

$$\Delta_i\phi(x)\equiv\phi(x+\hat{1})-\phi(x),\quad i=1,2,3\ . \tag{2.2}$$

Throughout this paper we will occasionally use the following abbreviations:

$$t=x_0,\quad \mathbf{r}=(x_1,x_2,x_3),\quad x_\mu x^\mu=t^2-\mathbf{r}^2\ ,$$
$$\dot{\Delta}=\Delta_0,\quad \mathbf{\Delta}=(\Delta_1,\Delta_2,\Delta_3),\quad \Delta_\mu\Delta^\mu=\dot{\Delta}^2-\mathbf{\Delta}^2,\ \text{etc.}$$

If the action sum

$$S\equiv\sum_{x'}\tfrac{1}{2}\{[\Delta_\mu\phi(x')][\Delta^\mu\phi(x')]-m^2\phi^2(x')\} \tag{2.3}$$

is stationary for arbitrary variation at x,

$$\phi(x')\to\phi(x')+\epsilon\delta^4_{x,x'}\ , \tag{2.4}$$

then the field equation is

$$\dot{\Delta}^2\phi(x-a\hat{0})-\sum_{i=1}^{3}\Delta_i{}^2\phi(x-\hat{\imath})+m^2\phi(x)=0\ . \tag{2.5}$$

It is also rewritten as

$$\frac{1}{a^2}[\phi(x+a\hat{0})+\phi(x-a\hat{0})]-\sum_{i=1}^{3}[\phi(x+\hat{i})+\phi(x-\hat{i})]+\left[m^2+6-\frac{2}{a^2}\right]\phi(x)=0\ . \qquad (2.6)$$

The equation is easily solved to give

$$\phi(x)=\int_R d^3\theta[A(\boldsymbol{\theta})e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+A^*(\boldsymbol{\theta})e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}]\ , \qquad (2.7)$$

where $A(\boldsymbol{\theta})$ is an arbitrary function and ω is given by

$$\sin^2\frac{\omega}{2}=a^2\left[\sin^2\frac{\theta_1}{2}+\sin^2\frac{\theta_2}{2}+\sin^2\frac{\theta_3}{2}+\frac{m^2}{4}\right]. \qquad (2.8)$$

The domain of integration R is $-\pi\leqq\theta_i\leqq\pi$ $(i=1,2,3)$. If $a\leqq(3+m^2/4)^{-1/2}$, then ω is real for any $\boldsymbol{\theta}$ in R. For convenience sake, we assume $0\leqq\omega\leqq\pi$. In I a was equal to one and hence ω became a complex number for a certain $\boldsymbol{\theta}$ in R. To avoid it the region which gives complex ω was removed from R. Thus, the solution became incomplete and the equivalence between the x representation and θ representation has been lost in I. On the contrary, the domain of integration in Eq. (2.7) is not restricted at all. This assures the completeness of the solution (2.7) and the equivalence of the two representations as will be shown later on.

Using the Lagrangian

$$\mathscr{L}=\sum_{\mathbf{r}}\{[\Delta_\mu\phi(x)][\Delta^\mu\phi(x)]-m^2\phi^2(x)\}\ , \qquad (2.9)$$

we have the momentum conjugate to $\phi(x)$,

$$\pi(x)=\frac{\partial\mathscr{L}}{\partial\dot{\Delta}\phi(x)}=\frac{1}{2a}[\phi(x+a\hat{0})-\phi(x-a\hat{0})]\ , \qquad (2.10)$$

and thus the Hamiltonian,

$$\mathscr{H}=\sum_{\mathbf{r}}\pi(x)\dot{\Delta}\phi(x)-\mathscr{L}=\frac{1}{2a^2}\sum_{\mathbf{r}}[\phi^2(x)-\phi(x+a\hat{0})\phi(x-a\hat{0})]\ , \qquad (2.11)$$

where we used the field equation (2.6). Similarly, we verify

$$\dot{\Delta}\mathscr{H}=0\ . \qquad (2.12)$$

Following the procedure of canonical quantization, we have the commutation relations

$$[\phi(t,\mathbf{r}),\pi(t,\mathbf{r}')]=i\delta_{\mathbf{r},\mathbf{r}'}{}^3\ , \qquad (2.13)$$

$$[\phi(t,\mathbf{r}),\phi(t,\mathbf{r}')]=[\pi(t,\mathbf{r}),\pi(t,\mathbf{r}')]=0\ . \qquad (2.14)$$

Substituting Eq. (2.10) for $\pi(t,\mathbf{r}')$ in Eq. (2.13), we find

$$[\phi(t,\mathbf{r}),\phi(t+a,\mathbf{r}')]=ia\delta_{\mathbf{r},\mathbf{r}'}{}^3\ . \qquad (2.15)$$

When we rewrite the solution (2.7) in the following normalized form,

$$\phi(x)=\left[\frac{a}{16\pi^3}\right]^{1/2}\int_R\frac{d^3\theta}{(\sin\omega)^{1/2}}[a(\boldsymbol{\theta})e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+a^\dagger(\boldsymbol{\theta})e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}]\ , \qquad (2.16)$$

we then obtain the commutation relations

$$[a(\boldsymbol{\theta}),a^\dagger(\boldsymbol{\theta}')]=\delta^3(\boldsymbol{\theta}-\boldsymbol{\theta}')\ , \qquad (2.17)$$

$$[a(\boldsymbol{\theta}),a(\boldsymbol{\theta}')]=[a^\dagger(\boldsymbol{\theta}),a^\dagger(\boldsymbol{\theta}')]=0\ . \qquad (2.18)$$

In contrast with I the commutation relations (2.14) and (2.15) are obtained from Eqs. (2.17) and (2.18), because the solution (2.16) or (2.7) is complete as mentioned before. Thus, we see that the two representations are equivalent.

Corresponding to the Heisenberg equation we have

$$[\phi(x),\mathscr{H}]=\frac{i}{2a}[\phi(x+a\hat{0})-\phi(x-a\hat{0})]\ . \qquad (2.19)$$

The right-hand side of Eq. (2.19) is not equal to $i\dot{\Delta}\phi(x)$. The Hamiltonian (2.11) is similarly rewritten as

$$\mathscr{H}=\frac{1}{a}\int_R d^3\theta\sin\omega\, a^\dagger(\boldsymbol{\theta})a(\boldsymbol{\theta}) \qquad (2.20)$$

besides zero-point energy.

Using the vacuum $|0\rangle$ defined by

$$a(\boldsymbol{\theta})|0\rangle=0\ , \qquad (2.21)$$

we find that the propagator

$$D_F(x-x';m^2)\equiv\langle 0|T\phi(x)\phi(x')|0\rangle \qquad (2.22)$$

is calculated to give

$$D_F(x;m^2)=\frac{a}{2(2\pi)^3}\int_R\frac{d^3\theta}{\sin\omega}[\theta(t)e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+\theta(-t)e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}]$$
$$=\frac{ia}{(2\pi)^4}\int_R d^3\theta\int_{-\pi}^{\pi}d\phi\frac{\exp[-i(\phi t/a-\boldsymbol{\theta}\cdot\mathbf{r})]}{4\sin^2(\phi/2)-4a^2\sin^2(\boldsymbol{\theta}/2)-a^2m^2+i\epsilon}\ , \qquad (2.23)$$

where we used the abbreviation

$$\sin^2(\boldsymbol{\theta}/2)=\sin^2(\theta_1/2)+\sin^2(\theta_2/2)+\sin^2(\theta_3/2)\ .$$

To investigate the divergence we look for the value at $x=0$,

$$D_F(0;m^2)=\frac{a}{2(2\pi)^3}\int_R \frac{d^3\theta}{\sin\omega}$$
$$=\frac{1}{4(2\pi)^3}\int_R d^3\theta[\sin^2(\theta/2)+m^2/4]^{-1/2}\{1-a^2[\sin^2(\theta/2)+m^2/4]\}^{-1/2}\ , \tag{2.24}$$

which is infinite regardless of $m=0$ or $m\neq 0$, because $a^2\leqq(3+m^2/4)^{-1}$. This is a new result found in the case of three-dimensional space. In I the space dimension was one and $D_F(0;m^2)$ diverged at $m=0$. As a special case, if $a^2=(3+m^2/4)^{-1}$, then

$$D_F(0;m^2)=\frac{1}{4(2\pi)^3 a}\int_R d^3\theta\{[\sin^2(\theta/2)+m^2/4]\cos^2(\theta/2)\}^{-1/2}\ , \tag{2.25}$$

where $\cos^2(\theta/2)$ is a similar abbreviation to $\sin^2(\theta/2)$.

III. SYMMETRIC DIFFERENCE

In this section we will show that the difference operator Δ_μ used in Sec. II is not suitable for describing fermions on discrete space-time. Thus we must introduce a new operator suitable for this purpose. To distinguish the two operators, we call the old one a right difference operator and express it as Δ_μ^R instead of Δ_μ, i.e.,

$$\Delta_0^R\psi(x)=\frac{1}{a}[\psi(x+a\hat{0})-\psi(x)]\ , \tag{3.1}$$

$$\Delta_i^R\psi(x)=\psi(x+\hat{\imath})-\psi(x)\ . \tag{3.2}$$

Let $\psi(x)$ be a spinor field and assume the action

$$S_\psi\equiv-\sum_{x'}\bar\psi(x')(-i\gamma^\mu\Delta_\mu^R+\kappa)\psi(x')\ , \tag{3.3}$$

where $\bar\psi(x)=\psi^\dagger(x)\gamma_0$. The representation of γ matrices are the same as that in Schweber's textbook,[3] i.e.,

$$\gamma_0=\gamma_0^\dagger=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix},\ \ \gamma_i=-\gamma_i^\dagger=\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix},\quad i=1,2,3\ . \tag{3.4}$$

Considering that $\psi(x)$ and $\bar\psi(x)$ are independent functions, we take a variation of $\bar\psi(x')$ at x:

$$\bar\psi_{\alpha'}(x')\to\bar\psi_{\alpha'}(x')+\epsilon\delta_{x,x'}{}^4\delta_{\alpha,\alpha'}\ . \tag{3.5}$$

Then we have the following equation of $\psi(x)$ under the condition that the action (3.3) be stationary for the variation:

$$(-i\gamma^\mu\Delta_\mu^R+\kappa)\psi(x)=0\ . \tag{3.6}$$

The action (3.3) is, on the other hand, rewritten in the form

$$S_\psi=-\sum_{x'}[i\Delta_\mu^L\bar\psi(x')\gamma^\mu+\kappa\bar\psi(x')]\psi(x')\ , \tag{3.7}$$

where Δ_μ^L is called a left difference operator and is defined by

$$\Delta_0^L\bar\psi(x)\equiv\frac{1}{a}[\bar\psi(x)-\bar\psi(x-a\hat{0})]\ , \tag{3.8}$$

$$\Delta_i^L\bar\psi(x)\equiv\bar\psi(x)-\bar\psi(x-\hat{\imath})\ . \tag{3.9}$$

Thus a variation of $\psi(x)$ in S_ψ yields the equation of $\bar\psi(x)$:

$$i\Delta_\mu^L\bar\psi(x)\gamma^\mu+\kappa\bar\psi(x)=0\ . \tag{3.10}$$

Equations (3.6) and (3.10) are, however, inconsistent. In fact, the Hermitian adjoint of Eq. (3.6) is

$$i\Delta_\mu^R\bar\psi(x)\gamma^\mu+\kappa\bar\psi(x)=0\ . \tag{3.11}$$

The reason for this inconsistency is simply that we started with the non-Hermitian action (3.3).

We consider now the following symmetric difference operator:

$$\Delta_\mu^S\equiv\tfrac{1}{2}(\Delta_\mu^R+\Delta_\mu^L)\ , \tag{3.12}$$

$$\Delta_0^S\psi(x)=\frac{1}{2a}[\psi(x+a\hat{0})-\psi(x-a\hat{0})]\ , \tag{3.13}$$

$$\Delta_i^S\psi(x)=\tfrac{1}{2}[\psi(x+\hat{\imath})-\psi(x-\hat{\imath})]\ . \tag{3.14}$$

The action written in terms of this operator,

$$S_\psi\equiv-\sum_x\bar\psi(x)(-i\gamma^\mu\Delta_\mu^S+\kappa)\psi(x)\ , \tag{3.15}$$

is Hermitian this time. The equations given by this action,

$$(-i\gamma^\mu\Delta_\mu^S+\kappa)\psi(x)=0\ , \tag{3.16}$$

$$i\Delta_\mu^S\bar\psi(x)\gamma^\mu+\kappa\bar\psi(x)=0\ , \tag{3.17}$$

are consistent with each other. Operating with $(i\gamma^\mu\Delta_\mu^S+\kappa)$ on Eq. (3.16) from the left, we have

$$(\Delta_\mu^S\Delta^{S\mu}+\kappa^2)\psi(x)=0 \tag{3.18}$$

or, equivalently,

$$\frac{1}{4a^2}[\psi(x+2a\hat{0})+\psi(x-2a\hat{0})]-\frac{1}{4}\sum_{i=1}^{3}[\psi(x+2\hat{\imath})+\psi(x-2\hat{\imath})]+\left\{-\frac{1}{2a^2}+\frac{3}{2}+\kappa^2\right\}\psi(x)=0\ . \tag{3.19}$$

As is seen from Eq. (3.19), we have a relation between ψ's at next-nearest-neighbor space-time points. That is, the values of ψ's at two neighboring points are independent of each other. More exactly the equation gives no relation between $\psi(x)$, $\psi(x+a\hat{0})$, $\psi(x+\hat{1})$, $\psi(x+a\hat{0}+\hat{1})$, $\psi(x+\hat{1}+\hat{j})$ $(i\neq j)$, $\psi(x+a\hat{0}+\hat{1}+\hat{j})$, $\psi(x+\hat{1}+\hat{2}+\hat{3})$, and $\psi(x+a\hat{0}+\hat{1}+\hat{2}+\hat{3})$. This means that there exist $2^4=16$ independent solutions. Each solution corresponds to a set of space-time points out of 16 sets which cover all the space-time points. Almost all of these solutions (except for a certain combination) are spurious solutions (see the Appendix). It would be, however, very difficult to give these spurious solutions physical meaning.[4] Therefore, we must discard them by imposing some restriction on the solutions by hand. It destroys the beauty of the theory very much.

In this way we arrive at a final operator. The operator is denoted by Δ_μ for brevity, though the same notation is used in Sec. II. We define

$$\Delta_0\psi(x)=\frac{1}{a}[\psi(x+\tfrac{1}{2}a\hat{0})-\psi(x-\tfrac{1}{2}a\hat{0})] , \tag{3.20}$$

$$\Delta_i\psi(x)=\psi(x+\tfrac{1}{2}\hat{\imath})-\psi(x-\tfrac{1}{2}\hat{\imath}) . \tag{3.21}$$

These equations seems to contradict the original philosophy of I that the minimum distances in space and time are 1 and a, respectively, because Eq. (3.20) or (3.21) refers to a half-distance $\frac{1}{2}$ or $a/2$. However, it must be noticed here that the time (or space) difference in the right-hand side of Eq. (3.20) or (3.21) is still a (or 1), and the number or density of space-time points is the same as before. In fact, we consider the action

$$S_\psi=-\sum_x\bar{\psi}(x)(-i\gamma^\mu\Delta_\mu+\kappa)\psi(x) , \tag{3.22}$$

where the summation of $x=an_0\hat{0}+n_1\hat{1}+n_2\hat{2}+n_3\hat{3}$ runs over all integers for n_μ and not half-integers.

We introduce here a simple theorem, which we call "translation invariance."

Let $F(x)$ be an arbitrary smooth function of space-time point x and c be a real four-vector, then

$$\sum_x F(x)=\sum_x F(x+c) . \tag{3.23}$$

The smooth function here means that $F(x)$ has the Fourier transform

$$F(x)=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}d^4\theta f(\theta)e^{i(\theta_0 t/a-\boldsymbol{\theta}\cdot\mathbf{r})} . \tag{3.24}$$

The proof is as follows: From Eq. (3.24) we have

$$F(x+c)=\int\cdots\int d^4\theta f(\theta)e^{i(\theta_0 t/a-\boldsymbol{\theta}\cdot\mathbf{r})}e^{i(\theta_0 c_0/a-\boldsymbol{\theta}\cdot\mathbf{c})}$$

Therefore,

$$\begin{aligned}\sum_x F(x+c)&=(2\pi)^4\int\cdots\int d^4\theta f(\theta)\delta(\theta_0/a)\delta^3(\boldsymbol{\theta})\\&\quad\times e^{i(\theta_0 c_0/a-\boldsymbol{\theta}\cdot\mathbf{c})}\\&=(2\pi)^4\int\cdots\int d^4\theta f(\theta)\delta(\theta_0/a)\delta^3(\boldsymbol{\theta})\\&=\sum_x F(x) .\end{aligned}$$

In particular, if $c=an\hat{0}$ or $c=n\hat{\imath}$ (n an integer), Eq. (3.23) is nothing but a change in numbering of space-time points. Thus, Eq. (3.23) is obvious. When $c=\pm(a/2)\hat{0}$ or $c=\pm(\frac{1}{2})\hat{\imath}$, Eq. (3.23) gives

$$\sum_x\bar{\psi}_\alpha(x)\psi_\beta(x\pm\tfrac{1}{2}a\hat{0})=\sum_x\bar{\psi}_\alpha(x\mp\tfrac{1}{2}a\hat{0})\psi_\beta(x) , \tag{3.25}$$

$$\sum_x\bar{\psi}_\alpha(x)\psi_\beta(x\pm\tfrac{1}{2}\hat{\imath})=\sum_x\bar{\psi}_\alpha(x\mp\tfrac{1}{2}\hat{\imath})\psi_\beta(x) . \tag{3.26}$$

Making use of Eqs. (3.25) and (3.26), we see that the action (3.22) is rewritten in the form

$$S_\psi=-\sum_x[i\Delta_\mu\bar{\psi}(x)\gamma^\mu+\kappa\bar{\psi}(x)]\psi(x) . \tag{3.27}$$

This is the Hermitian adjoint of Eq. (3.22), and hence S_ψ is Hermitian. From the action (3.22) or (3.27) we have the field equations

$$(-i\gamma^\mu\Delta_\mu+\kappa)\psi(x)=0 , \tag{3.28}$$

$$i\Delta_\mu\bar{\psi}(x)\gamma^\mu+\kappa\bar{\psi}(x)=0 , \tag{3.29}$$

which are, of course, consistent.

Before we solve these equations, we will apply the new operator to a scalar field and see whether or not we will have the same result as what we had in Sec. II. Assume the action

$$S_\phi=\sum_x\tfrac{1}{2}\{[\Delta_\mu\phi(x)][\Delta^\mu\phi(x)]-m^2\phi^2(x)\} . \tag{3.30}$$

The seeming form of Eq. (3.30) is exactly the same as Eq. (2.3), but the difference operators in the two equations are different. From Eq. (3.30) we obtain the field equation

$$(\Delta_\mu\Delta^\mu+m^2)\phi(x)=0 . \tag{3.31}$$

When we write it down explicitly,

$$\frac{1}{a^2}[\phi(x+a\hat{0})+\phi(x-a\hat{0})]-\sum_{i=1}^{3}[\phi(x+\hat{\imath})+\phi(x-\hat{\imath})]+\left[m^2+6-\frac{2}{a^2}\right]\phi(x)=0 , \tag{3.32}$$

we find it the same as Eq. (2.6). Therefore, Eqs. (2.5) and (3.31) are completely the same equation. From an aesthetical point of view Eq. (3.31) is much better than Eq. (2.5). The explicit form of Eq. (2.3) is

$$S=\sum_x\tfrac{1}{2}\left[\frac{1}{a^2}[\phi(x+a\hat{0})-\phi(x)]^2-\sum_{i=1}^{3}[\phi(x+\hat{\imath})-\phi(x)]^2-m^2\phi^2(x)\right] . \tag{3.33}$$

Applying the translation invariance (3.23) to Eq. (3.33), we have

$$S=\sum_x\tfrac{1}{2}\left[\frac{1}{a^2}[\phi(x+\tfrac{1}{2}a\hat{0})-\phi(x-\tfrac{1}{2}a\hat{0})]^2-\sum_{i=1}^{3}[\phi(x+\tfrac{1}{2}\hat{\imath})-\phi(x-\tfrac{1}{2}\hat{\imath})]^2-m^2\phi^2(x)\right] , \tag{3.34}$$

which is exactly the same as S_ϕ of Eq. (3.30). As we have seen above, we may say that the symmetric difference defined by Eqs. (3.20) and (3.21) is not unnatural.

IV. SPINOR FIELD

We have seen in the foregoing section that the action (3.22) with the symmetric difference (3.20) and (3.21) leads to the consistent field equations (3.28) and (3.29). In this section we solve Eq. (3.28), then quantize the spinor field, and obtain its propagator.

When we operate with $(i\gamma^\mu\Delta_\mu+\kappa)$ on Eq. (3.28) from the left, we have

$$(\Delta_\mu\Delta^\mu+\kappa^2)\psi(x)=0 \; . \tag{4.1}$$

The solution of this equation is already known in Sec. II. Thus, we write the solution of Eq. (3.28) in the form

$$\psi(x)=\begin{cases} u(\boldsymbol{\theta})e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})} \; , \\ v(\boldsymbol{\theta})e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})} \; , \end{cases} \tag{4.2}$$

where

$$\sin^2(\omega/2)=a^2[\sin^2(\boldsymbol{\theta}/2)+\kappa^2/4] \; ,$$

$$0\leqq\omega\leqq\pi, \quad a^2\leqq(3+\kappa^2/4)^{-1} \; .$$

Then $u(\boldsymbol{\theta})$ and $v(\boldsymbol{\theta})$ satisfy

$$\left[-\frac{2}{a}\gamma_0\sin\frac{\omega}{2}+\sum_{i=1}^{3}2\gamma_i\sin\frac{\theta_i}{2}+\kappa\right]u(\boldsymbol{\theta})=0 \; , \tag{4.3}$$

$$\left[\frac{2}{a}\gamma_0\sin\frac{\omega}{2}-\sum_{i=1}^{3}2\gamma_i\sin\frac{\theta_i}{2}+\kappa\right]v(\boldsymbol{\theta})=0 \; , \tag{4.4}$$

respectively. These equations are the same as those in the usual continuous space-time, if the energy-momentum p_μ is replaced by

$$\left[\frac{2}{a}\sin\frac{\omega}{2},2\sin\frac{\theta_1}{2},2\sin\frac{\theta_2}{2},2\sin\frac{\theta_3}{2}\right] .$$

Therefore, we can obtain the solutions easily,[3] but we do not write them here. Instead we show the relations which $u(\boldsymbol{\theta})$ and $v(\boldsymbol{\theta})$ satisfy

$$\bar{u}_r(\boldsymbol{\theta})u_s(\boldsymbol{\theta})=-\bar{v}_r(\boldsymbol{\theta})v_s(\boldsymbol{\theta})=\delta_{r,s} \; , \tag{4.5}$$

$$\bar{v}_r(\boldsymbol{\theta})u_s(\boldsymbol{\theta})=\bar{u}_r(\boldsymbol{\theta})v_s(\boldsymbol{\theta})=0 \; , \tag{4.6}$$

$$\sum_{s=1}^{2}u_s(\boldsymbol{\theta})\bar{u}_s(\boldsymbol{\theta})=\frac{1}{2\kappa}\left[\frac{2}{a}\gamma_0\sin\frac{\omega}{2}-\sum_{i=1}^{3}2\gamma_i\sin\frac{\theta_i}{2}+\kappa\right] , \tag{4.7}$$

$$-\sum_{s=1}^{2}v_s(\boldsymbol{\theta})\bar{v}_s(\boldsymbol{\theta})=\frac{1}{2\kappa}\left[-\frac{2}{a}\gamma_0\sin\frac{\omega}{2}+\sum_{i=1}^{3}2\gamma_i\sin\frac{\theta_i}{2}+\kappa\right] . \tag{4.8}$$

Assume the Lagrangian

$$\mathscr{L}_\psi=-\sum_{\mathbf{r}}\bar{\psi}(x)(-i\gamma_\mu\Delta^\mu+\kappa)\psi(x) \; . \tag{4.9}$$

We then have the momentum conjugate to $\psi_\alpha(x)$:

$$\pi_{\psi_\alpha}(x)=\partial\mathscr{L}_\psi/\partial\dot{\Delta}\psi_\alpha(x)=-i\psi_\alpha^\dagger(x) \; . \tag{4.10}$$

Hence the Hamiltonian is

$$\mathscr{H}_\psi=\sum_{r}\bar{\psi}(x)(-i\boldsymbol{\gamma}\cdot\boldsymbol{\Delta}+\kappa)\psi(x) \; , \tag{4.11}$$

and

$$\dot{\Delta}\mathscr{H}_\psi=0 \tag{4.12}$$

is easily verified.

The quantization is straightforward. The equal-time commutation relations are

$$\{\psi_\alpha(t,\mathbf{r}),\psi_\beta^\dagger(t,\mathbf{r}')\}=\delta_{\alpha,\beta}\delta_{\mathbf{r},\mathbf{r}'}{}^3 \; , \tag{4.13}$$

$$\{\psi_\alpha(t,\mathbf{r}),\psi_\beta(t,\mathbf{r}')\}=\{\psi_\alpha^\dagger(t,\mathbf{r}),\psi_\beta^\dagger(t,\mathbf{r}')\}=0 \; . \tag{4.14}$$

Thus the Heisenberg equation holds for $\psi(x)$ and $\psi^\dagger(x)$:

$$[\psi(x),\mathscr{H}_\psi]=i\dot{\Delta}\psi(x) \; , \tag{4.15}$$

$$[\psi^\dagger(x),\mathscr{H}_\psi]=i\dot{\Delta}\psi^\dagger(x) \; . \tag{4.16}$$

However, it does not hold for a nonlinear quantity, e.g.,

$$[\psi_\alpha(x)\psi_\beta(y),\mathscr{H}_\psi]=i[\dot{\Delta}\psi_\alpha(x)]\psi_\beta(y)+i\psi_\alpha(x)\dot{\Delta}\psi_\beta(y) \neq i\dot{\Delta}[\psi_\alpha(x)\psi_\beta(y)] \; . \tag{4.17}$$

This clearly comes from the distinction between derivatives and differences.

When we write down the solutions of Eqs. (3.28) and (3.29) in the normalized forms,

$$\psi(x)=\left[\frac{a\kappa}{16\pi^3}\right]^{1/2}\int_R d^3\theta\left[\sin\frac{\omega}{2}\right]^{-1/2}\sum_s[a_s(\boldsymbol{\theta})u_s(\boldsymbol{\theta})e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+b_s^\dagger(\boldsymbol{\theta})v_s(\boldsymbol{\theta})e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}] \; , \tag{4.18}$$

$$\bar{\psi}(x)=\left[\frac{a}{16\pi^3}\right]^{1/2}\int_R d^3\theta\left[\sin\frac{\omega}{2}\right]^{-1/2}\sum_s[b_s(\boldsymbol{\theta})\bar{v}_s(\boldsymbol{\theta})e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+a_s^\dagger(\boldsymbol{\theta})\bar{u}_s(\boldsymbol{\theta})e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}] \; , \tag{4.19}$$

then from the commutation relations (4.13) and (4.14) we obtain

$$\{a_r(\boldsymbol{\theta}),a_s^\dagger(\boldsymbol{\theta}')\}=\{b_r(\boldsymbol{\theta}),b_s^\dagger(\boldsymbol{\theta}')\}=\delta_{r,s}\delta^3(\boldsymbol{\theta}-\boldsymbol{\theta}') , \tag{4.20}$$

$$\{a_r(\boldsymbol{\theta}),a_s(\boldsymbol{\theta}')\}=\{b_r(\boldsymbol{\theta}),b_s(\boldsymbol{\theta}')\}=\{a_r(\boldsymbol{\theta}),b_s(\boldsymbol{\theta}')\}=\{a_r(\boldsymbol{\theta}),b_s^\dagger(\boldsymbol{\theta}')\}=0 . \tag{4.21}$$

The Hamiltonian (4.11) is

$$\mathscr{H}_\psi=\frac{2}{a}\int_R d^3\theta \sin\frac{\omega}{2}\sum_s [a_s^\dagger(\boldsymbol{\theta})a_s(\boldsymbol{\theta})+b_s^\dagger(\boldsymbol{\theta})b_s(\boldsymbol{\theta})] . \tag{4.22}$$

The vacuum is defined by

$$a_s(\boldsymbol{\theta})\,|\,0\rangle=b_s(\boldsymbol{\theta})\,|\,0\rangle=0 . \tag{4.23}$$

Then the propagator of a spinor field defined by

$$S_F(x-x';\kappa)\equiv\langle 0\,|\,T\psi(x)\bar{\psi}(x')\,|\,0\rangle \tag{4.24}$$

is written in the form

$$S_F(x;\kappa)=(i\gamma^\mu\Delta_\mu+\kappa)\widetilde{D}_F(x;\kappa^2) , \tag{4.25}$$

where $\widetilde{D}_F(x;\kappa^2)$ is given by

$$\widetilde{D}_F(x;\kappa^2)\equiv\frac{a}{2(2\pi)^3}\int_R \frac{d^3\theta}{\sin(\omega/2)}[\theta(t)e^{-i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}+\theta(-t)e^{i(\omega t/a-\boldsymbol{\theta}\cdot\mathbf{r})}] . \tag{4.26}$$

To examine the divergence problem we investigate the propagator (4.25) at $x=0$ and find it finite. The domain of integration is finite and the denominator of the integrand has no zero point except for the case $\kappa=0$, though the measure at this zero point is still finite in the three-dimensional case.

V. DISCUSSION

We have formulated a free scalar field and a free spinor field on discrete space-time of four dimensions. Thus, we can calculate any transition matrix elements perturbationally, in principle, if the relevant interaction Lagrangian is given, and we know that they give finite values. Therefore, our original motivation is sufficiently satisfied. However, there are still many interesting problems, if we want to apply the theory to the recent subjects. In what follows we give a few examples.

(i) Determination of interactions: All the interactions so far proposed in the theories of continuous space-time can be adopted as the interactions of our theory of discrete space-time without large modification. However, it is interesting to determine the interaction through the gauge theory, which is widely accepted at the present day as a very promising theory. Therefore, we consider it now the most urgent and important problem to settle the gauge theory in our framework. In doing it the lattice gauge theory may provide us with useful information.

(ii) Relation to the lattice gauge theory: When the gauge theory is formulated on discrete space-time, an interesting question is whether or not both theories, the lattice gauge theory and our theory, come to the same conclusion. While we assumed a Lorentz space-time, in the lattice gauge theory they assumed a Euclidean space-time. This seems to us very crucial, though Wilson says "the use of Euclidean space instead of a Lorentz space is not a serious restriction."[5] We do not think, however, that the difference of quantization in two theories, the Feynman path integral in the lattice gauge theory and the canonical quantization in our theory, produces a serious distinction.

(iii) Unitarity of scattering matrix: As we mentioned above, we can calculate scattering matrix elements for any Feynman diagrams and always obtain finite values, in principle. However, we cannot prove the convergence of a perturbation series for a certain physical process, though each term of the series is finite. Hence the unitarity of the total matrix remains unproved. In the case of continuous space-time the scattering matrix can be expressed in a compact form, i.e.,

$$T\exp\left[-i\int \mathscr{H}_{\rm int}dt\right] .$$

This enables us to prove the unitarity. In our case we have at present no idea how to express it in a compact form. Besides this we still have many problems in such a formal theory.

APPENDIX

We give here a definition of the "spurious solution" for difference equations and show two examples of difference equations, one of which has no spurious solution and the other which has a spurious solution.

Generally, a difference equation is reduced to a differential equation, if the distance of difference is brought to zero. Correspondingly, the usual solutions of the difference equation tend to the solutions of the differential equation. However, it may occur that a solution of a certain difference equation does not tend to any solution of the corresponding differential equation. Such a solution of the difference equation is called a spurious solution.

1. A difference equation with no spurious solution

We consider the equation

$$\frac{y[(n+1)a]-y[na]}{a}=-ky[na] , \tag{A1}$$

where n is an integer and k is a constant. If the distance of difference a tends to zero, Eq. (A1) goes to the differential equation

$$\frac{dy(x)}{dx}=-ky(x) . \tag{A2}$$

Equation (A1) is rewritten in the form

$$y[(n+1)a]-(1-ka)y[na]=0 , \tag{A3}$$

and the solution is

$$y[na]=c(1-ka)^n . \tag{A4}$$

The arbitrary constant c is determined by a boundary condition, e.g., $c=y[0]$. If a tends to zero with $na=x$ fixed, we have

$$\lim_{a\to 0} y[na]=\lim_{a\to 0} c(1-ka)^{kx/ka}=ce^{-kx} , \tag{A5}$$

which is the solution of the corresponding differential equation (A2).

2. A difference equation with a spurious solution

Next we consider the equation

$$\frac{y[(n+1)a]-y[(n-1)a]}{2a}=-ky[na] . \tag{A6}$$

The corresponding differential equation of Eq. (A6) is also Eq. (A2). Equation (A6) is rewritten as

$$y[(n+1)a]+2kay[na]-y[(n-1)a]=0 , \tag{A7}$$

and has two independent solutions:

$$\begin{aligned} y[na]=&c_1[(1+k^2a^2)^{1/2}+ka]^{-n}\\ &+c_2(-1)^n[(1+k^2a^2)^{1/2}+ka]^n , \end{aligned} \tag{A8}$$

where c_1 and c_2 are arbitrary constants. If a tends to zero, the first term of Eq. (A8) goes to

$$\lim_{a\to 0} c_1[(1+k^2a^2)^{1/2}+ka]^{-n}=c_1e^{-kx} , \tag{A9}$$

which is the solution of Eq. (A2), while the second term goes to

$$\begin{aligned} \lim_{a\to 0} c_2(-1)^n[(1+k^2a^2)^{1/2}+ka]^n&=c_2e^{kx}\lim_{a\to 0}(-1)^{x/a}\\ &=c_2e^{kx}\lim_{n\to\infty}(-1)^n , \end{aligned} \tag{A10}$$

which is indefinite and is not a solution of Eq. (A2). Therefore, the second term of Eq. (A8) is the spurious solution.

In a special case $k=0$, Eq. (A6) is

$$y[(n+1)a]=y[(n-1)a] . \tag{A11}$$

This shows that y's at neighboring points, i.e., $y[na]$ and $y[(n+1)a]$, are independent. The solution is

$$\begin{aligned} y[na]&=c_1+c_2(-1)^n\\ &=\tfrac{1}{2}c_1'[1+(-1)^n]+\tfrac{1}{2}c_2'[1+(-1)^n] . \end{aligned} \tag{A12}$$

For n=even or odd y is equal to c_1' or c_2', respectively. In order to discard the spurious solution we must impose a condition $c_2=0$ or $c_1'=c_2'$ by hand.

From our point of view the difference equation is fundamental and the differential equation is just an approximation. Thus if we want to use the spurious solutions for some purpose and to give some physical meaning, then we need not to impose any condition.

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[1]H. Yamamoto, Phys. Rev. D **30**, 1727 (1984); this paper will be referred to as I.

[2]The author is very grateful to Dr. S. Koretune at Fukui Medical College for the suggestion to assume $\tau\neq\lambda$.

[3]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

[4]The spurious solution is a subject of discussion also in the lattice gauge theory, though the term "spurious solution" is not used there. See L. Susskind, Phys. Rev. D **16**, 3031 (1977). In recent works this problem is known by the name of "fermion doubling." See, for example, J. M. Rabin, Nucl. Phys. **B201**, 315 (1982); M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rep. **95**, 201 (1983).

[5]K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).