Supersymmetry and the relationship between the Coulomb and oscillator problems in arbitrary dimensions

V. Alan Kostelecký* and Michael Martin Nieto†
Theoretical Division, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545

D. Rodney Truax‡
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545

(Received 26 December 1984; revised manuscript received 11 July 1985)

We construct the relationship between the radial equations of the d-dimensional hydrogen atom and the D-dimensional simple harmonic oscillator. The supersymmetric partners of each of these systems are obtained, and a series of maps between the various systems is delineated. We present an explicit physical example of our maps. Finally, we generalize to D dimensions the recent work of Balantekin on the supersymmetric oscillator with spin-orbit coupling, and we demonstrate that this supersymmetry is different from the one we study.

I. INTRODUCTION

The advent of supersymmetry has had a significant impact on theoretical physics in a number of distinct disciplines.[1] One subfield that has recently been receiving much attention is supersymmetric quantum mechanics,[2–5] in which the Hamiltonians of distinct systems are related by a supersymmetry algebra. In the simplest case, the eigenspectra of two systems are found to be identical except for the supersymmetric ground state, which is associated with only one spectrum.

In this work, we are concerned with clarifying the relationship between two distinct supersymmetric systems: the supersymmetric simple-harmonic-oscillator *radial* equation and the supersymmetric hydrogen-atom *radial* equation.[5] We emphasize that we consider the radial equations, rather than the full set of multidimensional equations discussed in Sec. V. There exists[6,7] a map between the radial equations of the three-dimensional Coulomb potential and the harmonic oscillator in arbitrary dimensions in the absence of supersymmetry, but no such map has been identified for the supersymmetric extension. Our first goal is the construction of such a map.

We begin in Sec. II with a presentation of the map between the radial equations of the d-dimensional Coulomb problem and the D-dimensional simple harmonic oscillator. To our knowledge, this generalization of the works of Refs. 6 and 7 has not been presented before in the literature.

Section III contains a discussion of the supersymmetric Coulomb and oscillator systems. The supersymmetric Coulomb radial equation for three dimensions has been discussed previously,[5] but we need the extension to arbitrary dimensions to construct the map between the two systems. Also, we analyze the supersymmetric radial equation of the D-dimensional harmonic oscillator.

The map between the two systems is presented explicitly in Sec. IV. There, we also describe the maps between the standard Coulomb problem and the supersymmetric oscillator, and between the supersymmetric Coulomb problem and the standard oscillator. In Sec. V we discuss a physical example that illustrates the application of these maps.

Recently, the supersymmetric oscillator in three dimensions with generalized spin-orbit couplings and the associated spectrum-generating algebras have been analyzed.[3,4] In Sec. IV we extend these results to D dimensions. As in three dimensions, this supersymmetry manifests itself in the spin-orbit spectra. In contrast, the supersymmetry in our radial equation relates the principal-quantum-number spectra to each other. The two supersymmetries are distinct.

In Sec. VII we close with a few comments.

II. GENERALIZED CONNECTION BETWEEN COULOMB AND OSCILLATOR PROBLEMS

The radial equation of the d-dimensional Coulomb problem may be written as[8]

$$\left[\frac{d^2}{dx^2}+\frac{(d-1)}{x}\frac{d}{dx}-\frac{l(l+d-2)}{x^2}+\frac{k}{x}-\frac{1}{4}\right]u(x,d,n,l)=0\,, \quad (2.1)$$

where $x=r/kr_0$, $r_0=\hbar^2/2\mu e^2$, $k=n+\frac{1}{2}(d-3)$, and $n\geq l+1$. The energy eigenvalues ϵ_n are given by

$$\epsilon_n=-\epsilon_0/[n+\tfrac{1}{2}(d-3)]^2\,, \quad (2.2)$$

where $\epsilon_0=me^4/2\hbar^2$.

The solution to Eq. (2.1) is[8]

$$u(x,d,n,l)=c(d,n,l)e^{-x/2}x^{l}L_{n-l-1}^{(2l+d-2)}(x)\ , \tag{2.3}$$

with normalization constant

$$c(d,n,l)=r_0^{-d/2}[n+\tfrac{1}{2}(d-3)]^{-(d+1)/2} \times[\Gamma(n-l)]^{1/2}[2\Gamma(n+l+d-2)]^{-1/2}\ . \tag{2.4}$$

Note that the Laguerre polynomials $L_n^{(\alpha)}$ are those defined in handbooks on mathematical functions[9] and are not the more limited $L_{n+\alpha}^{\alpha}$ often used in discussions of the hydrogen eigenfunctions.

The radial equation of the D-dimensional simple harmonic oscillator may be written[6] as

$$\left[\frac{d^2}{dX^2}+\frac{(D-1)}{X}\frac{d}{dX}-\frac{L(L+D-2)}{X^2} -X^2+K\right]U(X,D,N,L)=0\ , \tag{2.5}$$

where $X=R/R_0$, $R_0=(m\omega/\hbar)^{1/2}$, $K=2N+D$, and $N\geq L$. The energy eigenvalues E_n are given by

$$E_N=\tfrac{1}{2}\hbar\omega(2N+D)\ . \tag{2.6}$$

The solution to Eq. (2.5) is[6]

$$U(X,D,N,L)=C(D,N,L)e^{-X^2/2}X^{L}L_{N/2-L/2}^{(L+D/2-1)}(X^2), \tag{2.7}$$

with normalization constant

$$C(D,N,L)=R_0^{-D/2}[2\Gamma(\tfrac{1}{2}N-\tfrac{1}{2}L+1)]^{1/2} \times[\Gamma(\tfrac{1}{2}N+\tfrac{1}{2}L+\tfrac{1}{2}D)]^{-1/2}\ . \tag{2.8}$$

(i) Direct map between solutions (2.3) and (2.7). The map taking Eq. (2.1) into Eq. (2.5) is $x=X^2$. Restricting D, N, and L to integers, we find that the solution (2.3) for $u(x,d,n,l)$ can be related to $U(X,D,N,L)$ by

$$u(x,d,n,l)=\Lambda_0 U(X,2d-2,2n-2,2l)\ , \tag{2.9}$$

where

$$\Lambda_0=\{\tfrac{1}{2}R_0^{2d-2}/r_0^{d}[n+\tfrac{1}{2}(d-3)]^{d+1}\}^{1/2}\ . \tag{2.10}$$

The d- and n-dependent constant Λ_0 arises because $u(x,d,n,l)$ and $U(X,D,N,L)$ are normalized to unity in d and D dimensions, respectively.

The identification (2.9) yields the solution

$$\begin{aligned} D&=2d-2\ ,\\ N&=2n-2\ ,\\ L&=2l\ . \end{aligned} \tag{2.11}$$

Note that the three-dimensional Coulomb problem is in one-to-one correspondence with half the states of the four-dimensional oscillator,[10] namely, those states with even values of N and L.

(ii) General map between solutions (2.3) and (2.7). There exists a further degree of freedom in the map $x=X^2$. A generalization of Eq. (2.9) sets[7]

$$u(x,d,n,l)=\Lambda_\lambda X^{-\lambda}U(X,D,N,L)\ , \tag{2.12}$$

for integer λ and constant Λ_λ. Because of the factor $X^{-\lambda}$ in front of U, U must contain $X^{L+\lambda}$. This implies that $L\rightarrow L+\lambda$. Hence, $L_{N/2-L/2}^{(L+D/2-1)}(X^2)$ remains the same if $N\rightarrow N+\lambda$ and $D\rightarrow D-2\lambda$. Even the normalization of U is unchanged, except for the factor $R_0^{-\lambda}$. Therefore, we find that

$$u(x,d,n,l)=\Lambda_\lambda X^{-\lambda}U(X,2d-2-2\lambda,2n-2+\lambda,2l+\lambda)\ , \tag{2.13}$$

$$\Lambda_\lambda=\Lambda_0 R_0^{-\lambda}\ , \tag{2.14}$$

yielding

$$\begin{aligned} D&=2d-2-2\lambda\ ,\\ N&=2n-2+\lambda\ ,\\ L&=2l+\lambda\ . \end{aligned} \tag{2.15}$$

Clearly, λ must be an integer if N, L, n, and l are integers. It is a general feature of this map that the spectrum of the d-dimensional Coulomb problem is related to half the spectrum of the D-dimensional oscillator for any even integer D. However, the quantities in Eq. (2.15) have parameter spaces that are further restricted by the properties chosen for the map.

For instance, suppose we wish to map *all* states of the d-dimensional Coulomb problem into a harmonic oscillator. Since on physical grounds we known that $D\geq 2$, $N\geq 0$, $L\geq 0$, we must impose $d\geq 2+\lambda$, $n\geq\frac{1}{2}(2-\lambda)$, $l\geq-\frac{1}{2}\lambda$. This yields the bound $-2l\leq\lambda\leq d-2$. Further requiring $n\geq 1$, $l\geq 0$ restricts the bound to $0\leq\lambda\leq d-2$. We conclude that all states of the d-dimensional Coulomb problem can be mapped into the appropriate harmonic oscillator, except for $d=1$.[11]

As an example, consider the three-dimensional Coulomb problem. Assuming we wish to map *all* its states into the harmonic oscillator, we must impose $0\leq\lambda\leq 1$. First, take $\lambda=0$. Then, the hydrogenic s orbitals ($n\geq 1, l=0$) are related to the ($N=2n-2\geq 0$, $L=0$) states of the four-dimensional harmonic oscillator. Similarly, the hydrogenic p states ($n\geq 2$, $l=1$) correspond to the ($N=2n-2\geq 2$, $L=0$) states of the same oscillator. In the latter case, note the missing ground state. As a rule, the lowest-lying oscillator states are excluded, one by one, with each higher value of l.

Next, suppose $\lambda=1$. The hydrogenic states are then mapped into the odd-integer states of the two-dimensional harmonic oscillator. The hydrogenic s orbitals ($n\geq 1$, $l=0$) map into the ($N=2n-1\geq 1$, $L=1$) oscillator states, while the hydrogenic p orbitals ($n\geq 2$, $l=1$) map into the ($N=2n-1\geq 3$, $L=1$) oscillator states. Again, the lowest states are successively excluded with increasing l.

III. THE SUPERSYMMETRIC COULOMB AND OSCILLATOR PROBLEMS

In a supersymmetric quantum-mechanical system,[2] there exist operators Q_i, $i=1,\ldots,M$, that commute with the Hamiltonian,

$$[Q_i,H_{ss}]=0\ , \tag{3.1}$$

and that anticommute to generate the Hamiltonian,

$$\{Q_i,Q_j\}=H_{ss}\delta_{ij}\ . \tag{3.2}$$

The smallest such system has $M=2$. The simplest realization involves a two-component wave function ψ and two-by-two matrix operators Q_i and H_{ss}.

We choose to work with matrices Q and $Q^\dagger$, defined by

$$Q=\frac{1}{\sqrt{2}}(Q_1+iQ_2)\ ,$$
$$Q^\dagger=\frac{1}{\sqrt{2}}(Q_1-iQ_2)\ , \tag{3.3}$$

so that

$$\{Q,Q^\dagger\}=H_{ss}\ . \tag{3.4}$$

In particular, we take

$$Q=\left[p-\frac{i}{2}\frac{dU}{dz}\right]\sigma_-\ ,$$
$$Q^\dagger=\left[p+\frac{i}{2}\frac{dU}{dz}\right]\sigma_+\ , \tag{3.5}$$

where

$$\sigma_+=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad \sigma_-=\begin{bmatrix}0&0\\1&0\end{bmatrix}, \tag{3.6}$$

and $U(z)$ is a function of z to be related to the potential of the supersymmetric Hamiltonian. The form (3.5) implies that Q and $Q^\dagger$ act as lowering and raising operators, respectively, on the Pauli spinor ψ.

Anticommutation yields

$$H_{ss}=\begin{bmatrix}H_+&0\\0&H_-\end{bmatrix} \tag{3.7}$$

with

$$H_\pm=-\frac{d^2}{dz^2}+V_\pm\ ,$$
$$V_\pm=\frac{1}{4}\left[\frac{dU}{dz}\right]^2\mp\frac{1}{2}\frac{d^2U}{dz^2}\ . \tag{3.8}$$

One can show[2] that the ground state of H_{ss} is associated only with H_+ and that it has zero eigenvalue. Its eigenvector is given by

$$\psi_0\sim\exp(-\tfrac{1}{2}U)\ . \tag{3.9}$$

The remaining eigenvalues of H_+ are degenerate with those of H_-.

To construct the supersymmetric Hamiltonian for the hydrogen atom in d dimensions, we proceed as in the three-dimensional case.[5] First, we multiply u by the quantity $x^{(d-1)/2}$, obtaining

$$v(x,d,n,l)=-x^{(d-1)/2}u(x,d,n,l)\ . \tag{3.10}$$

Here, $x^{(d-1)/2}$ is the square root of the volume element in d dimensions. Its use removes the first-order derivative term in Eq. (2.1), so that v satisfies an equation that has only a d^2/dx^2 derivative term, as does Eq. (3.8). For a given l, Eq. (2.3) shows that the ground state has $n=l+1$. Therefore, combining Eqs. (2.3), (3.9), and (3.10) yields

$$U\left[x=\frac{y}{k_0}\right]=\frac{y}{k_0}-(2l+d-1)(\ln y-\ln k_0)\ , \tag{3.11}$$

where

$$k_0=l+1+\tfrac{1}{2}(d-3)\ . \tag{3.12}$$

Substituting this into Eq. (3.8) gives

$$V_\pm=-\frac{1}{y}+\frac{[l+\frac{1}{2}(d-1)\mp1][l+\frac{1}{2}(d-1)]}{y^2}$$
$$+\frac{1}{4[l+1+\frac{1}{2}(d-3)]^2}\ . \tag{3.13}$$

Effectively, by changing the y^{-2} term, the supersymmetry shifts the spectrum $n\geqq l+1$ into the spectrum $n'\geqq l+2$ and renormalizes the ground state to zero by adding the constant term in Eq. (3.13). Therefore, the supersymmetric partner has angular momentum quantum number $l'=l+1$. This is the left-hand relationship of Fig. 1. A discussion of the evidence for the phenomenological manifestation among atomic spectra of the $d=3$ version of this supersymmetry may be found in Ref. 5.

Similarly, we may construct the supersymmetric partner of the harmonic-oscillator Hamiltonian. We convert Eq. (2.5) into the form (3.8) with the substitution

$$V(X,D,N,L)=-X^{(D-1)/2}U(X,D,N,L)\ . \tag{3.14}$$

For a given L, Eq. (2.7) shows that $N=L$ is the ground state. Therefore, combining Eqs. (2.7), (3.9), and (3.14) gives

$$U=X^2-(2L+D-1)\ln X\ . \tag{3.15}$$

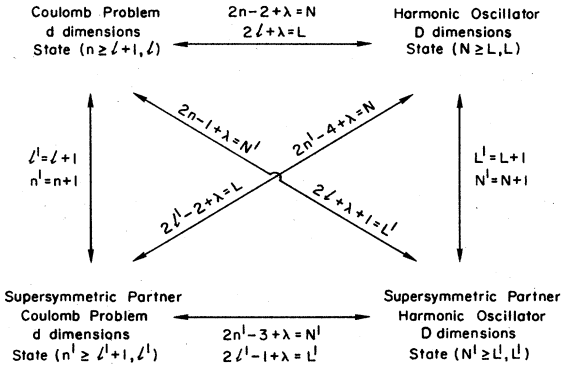

FIG. 1. Relationships between the supersymmetric quantum-mechanical Coulomb and oscillator systems. In all cases, $D=2d-2-2\lambda$.

Using this in (3.8) yields

$$V_{\pm}=X^2+\frac{[L+\frac{1}{2}(D-1)\mp 1][L+\frac{1}{2}(D-1)]}{X^2}$$
$$-(2L+D-1\pm 1)\,. \tag{3.16}$$

Again, the angular momentum term, which is proportional to X^{-2}, shows that the angular momentum quantum number L' of the supersymmetric partner is given by $L'=L+1$. As before, the spectrum $N\geqq L$ is shifted into $N'\geqq L+1$. This is the right-hand relationship of Fig. 1.

Note that in this case the added constant in Eq. (3.16) is different for the two potentials $V_{\pm}$. We illustrate the necessity of this by considering the supersymmetric partners $L=2$ and $L'=3$. Recall that the eigenvalue K in Eq. (2.5) has the value $K=2N+D$. For the ground state, $N=L$ and so $K=2L+D$. However, the ground state of the bosonic spectrum of the supersymmetric harmonic oscillator must be zero. Hence, the eigenvalue K_+ for the bosonic system is $K_+=2N+D-(2L+D)=2(N-L)$ where the constant term $2L+D$ corresponds to the constant term in V_+ and renormalizes the ground-state energy to zero. In our example, $L=2$, so $K_+=2(N-2)$. Since $N=L, L+2,\ldots$, according to Eq. (2.7), $K_+=0, 4, 8,\ldots$. Now, the constant term in V_- is $2L+D-2=2L'+D-4$ and so the eigenvalue K_- for the fermionic spectrum is $K_-=2N'+D-(2L'+D-4)=2(N'-L')+4$. In our example, the fermionic spectrum is given by $K_-=2N'-2$; it has $L'=3$ and so $K_-=4, 8, 12,\ldots$. Without the new constant term for the fermionic spectrum, the eigenvalues K_- would have been $K_-=2N'+D-(2L'+D-2)=2, 6, 10,\ldots$ and the spectra K_+, K_- would have been disjoint. Thus, we see that the correct supersymmetric spectrum is automatically achieved by the appearance of the constant term $(2L+D-1\pm 1)$ in Eq. (3.16).

IV. CONNECTION BETWEEN SUPERSYMMETRIC COULOMB AND OSCILLATOR PROBLEMS

At this stage, we are in a position to obtain relationships between the models constructed in Secs. II and III. We have already provided the map (2.15) taking the d-dimensional Coulomb problem into the D-dimensional simple harmonic oscillator. It is straightforward to extend these maps to the supersymmetric case.

In fact, the solution is obtained in the same fashion as Eqs. (2.15) but with (l,L,n,N) replaced by $(l'-1,L'-1,n'-1,N'-1)$, respectively. This yields

$$\begin{aligned} D&=2d-2-2\lambda\,,\\ N'&=2n'-3+\lambda\,,\\ L'&=2l'-1+\lambda\,, \end{aligned} \tag{4.1}$$

which is the bottom map of Fig. 1.

We can interpret the solution (4.1) in the following way. The relationship between the fermionic Hamiltonians is dimensionally the same as that expressed in the bosonic relationship (2.15). Recall that in Eq. (2.15) only half of the oscillator L levels is involved. In the fermionic relationship (4.1), however, the other half of the L levels occurs.

For example, if we choose $d=3$ and set $\lambda=0$, we can relate the $(n'\geqq l'+1,l')$ states of the supersymmetric Coulomb partner with the $(2n'-3,2l'-1)$ states of the four-dimensional supersymmetric oscillator partner. If we set $\lambda=1$, then the $(n'\geqq l'+1,l')$ states of the supersymmetric Coulomb partner are mapped into the $(2n'-2,2l')$ states of the two-dimensional supersymmetric oscillator partner. Therefore, the supersymmetric Coulomb levels are shifted by one unit with respect to both the hydrogen l' and the supersymmetric oscillator L states.

By the same technique, or by following two consecutive maps along the edges of Fig. 1, we can also determine the relationship between the supersymmetric hydrogen atom and the normal oscillator, and between the hydrogen atom and the supersymmetric oscillator. The map between the supersymmetric hydrogen atom and the normal oscillator is

$$\begin{aligned} D&=2d-2-2\lambda\,,\\ N&=2n'-4+\lambda\,,\\ L&=2l'-2+\lambda\,. \end{aligned} \tag{4.2}$$

In this case, the oscillator solution is shifted by two units in L relative to Eq. (2.15). Similarly, the map between the normal hydrogen atom and the supersymmetric oscillator is

$$\begin{aligned} D&=2d-2-2\lambda\,,\\ N'&=2n-1+\lambda\,,\\ L'&=2l+1+\lambda\,. \end{aligned} \tag{4.3}$$

This also represents a shift in L of two units, relative to (4.1).

The last two maps are the diagonals of Fig. 1 and complete its contents.

V. A PHYSICAL EXAMPLE

Here, we present an example identifying the corners of Fig. 1 with physical systems.

Suppose we consider the radial part of the Schrödinger equation describing the hydrogen atom. Restricting ourselves to the s orbitals, we can view them as forming the upper left-hand corner of Fig. 1, with $d=3$, $l=0$, and $n\geqq 1$.

It may be argued[5] that, physically , the supersymmetric partner of the s-orbital spectrum of hydrogen is the s-orbital spectrum of lithium. The s orbitals of lithium are described in this picture by spherical harmonics with $l=0$ and associated Laguerre polynomials with $l=1$. We refer the reader to Ref. 5 for details. Thus, the s-orbital spectrum of lithium may be placed in the lower left-hand corner of Fig. 1, with $d=3$, $l'=1$, and $n'\geqq 2$.

As may be seen from our diagram, the s-orbital spectra of hydrogen and lithium can be mapped into many different harmonic oscillators. For our example, let us choose $\lambda=1$, so that we are restricted to a two-dimensional oscillator. Then, any physical system with

the same equations of motion as a two-dimensional quantum-mechanical pendulum will provide us with the upper right-hand corner of our figure.

As an explicit example, we take the twofold degenerate E vibrational mode of the H_3^+ molecular ion.[12] This vibration forms a two-dimensional harmonic oscillator in the limit in which anharmonic effects and couplings to other degrees of freedom in the molecular ion can be ignored. Since $\lambda=1$ and since we took $l=0$ for the hydrogen atom, we must consider the $L=1$, $N\geqq 1$ oscillator states, where for simplicity we assume the ground-state configuration for the rotational modes. This spectrum may be placed in the upper right-hand corner of Fig. 1.

To motivate a possible physical system[13] for the lower right-hand corner of Fig. 1, we first consider the vibrational modes of the molecular ion H_3^+ with $L=0$, again assuming a ground-state rotational configuration. Here, it is known[12] that the rotation-vibrational state with $N=0$ is excluded by the Pauli principle. This is *not* the case, however, for the molecular ion D_3^+ with $L=0$, because the nuclear-spin wave function is bosonic rather than fermionic.

Returning to our suggestion for the lower right-hand corner of the figure, note that since the molecular ion D_3^+ has a wave function that is bosonic rather than fermionic, the levels of this system will not be the same as those of H_3^+, even disregarding the mass difference. We therefore suggest that a supersymmetry may be present between some part of the total spectra of H_3^+ and D_3^+.

At this stage, we have provided a physical model for Fig. 1. The reader may now check that our maps correctly interrelate the spectra we have described.

Other physical examples may readily be found, relating spectra with different values of l for the Coulomb problem[5] to spectra with different values of L for the oscillator.

VI. D-DIMENSIONAL OSCILLATOR WITH SPIN-ORBIT COUPLING

Next, we turn to a generalization to D dimensions of the analysis of Balantekin[4] on the supersymmetric three-dimensional oscillator with spin-orbit couplings. First, we construct the basis for a dynamical supersymmetry algebra such that the Hamiltonian of the system can be written in terms of Casimir operators of the superalgebra and its subalgebras. The Hamiltonian we analyze describes a D-dimensional simple harmonic oscillator with constant spin-orbit coupling ζ:

$$H=\tfrac{1}{2}(\mathbf{P}^2+\mathbf{R}^2)+\zeta(2\mathbf{L}\cdot\mathbf{S}+\tfrac{1}{2}D)\ . \tag{6.1}$$

This is the D-dimensional generalization of the model A of Ref. 4.

To discuss the dynamical supersymmetry associated with this model, we need to introduce the generators of a Clifford algebra. If D is even, the generators will be denoted by Γ_p, $p=1,\ldots,D$. If D is odd, they will be denoted by Γ_p, $p=1,\ldots,D-1$, and we append in addition a Dth quantity Γ_D defined by

$$\Gamma_D=-i\prod_{p=1}^{D-1}\Gamma_p\ . \tag{6.2}$$

The Γ_p satisfy the relationship

$$\{\Gamma_p,\Gamma_q\}=2\delta_{pq}\ . \tag{6.3}$$

The quantities Γ_p and, in the case of odd D, the quantity Γ_D may be used to construct a representation of the rotation group SO(D). The generators of SO(D) are given as

$$S_{pq}=-\tfrac{1}{4}i[\Gamma_p,\Gamma_q]\ , \tag{6.4}$$

where for D odd the index p now includes the Dth quantity Γ_D. For example, if $D=4$ the Γ_p may be represented by the Euclidean γ matrices, γ_p^E; the S_{pq} then are the Euclidean σ_{pq}^E matrices.

At this point, we introduce five operators, K_+, K_-, K_0, F_+, F_-, defined by

$$\begin{aligned}
K_0&=\tfrac{1}{2}\sum_{p=1}^{D}(a_p^\dagger a_p+\tfrac{1}{2})\ ,\\
K_+&=\tfrac{1}{2}\sum_{p=1}^{D}a_p^\dagger a_p^\dagger\ ,\\
K_-&=\tfrac{1}{2}\sum_{p=1}^{D}a_p a_p\ ,\\
F_+&=\tfrac{1}{2}\sum_{p=1}^{D}a_p^\dagger\Gamma_p\ ,\\
F_-&=\tfrac{1}{2}\sum_{p=1}^{D}a_p\Gamma_p\ ,
\end{aligned} \tag{6.5}$$

where the quantities $a_p^\dagger, a_p$ are the usual harmonic-oscillator raising and lowering operators satisfying

$$[a_p,a_q^\dagger]=\delta_{pq}\ . \tag{6.6}$$

It can be shown that the operators (6.5) form a basis of the noncompact orthosymplectic superalgebra osp(1 | 2), which has graded commutation relations

$$\begin{aligned}
&[K_\pm,F_\pm]=0,\ \ [K_\pm,F_\mp]=\mp F_\pm\ ,\\
&[K_0,F_\pm]=\pm\tfrac{1}{2}F_\pm\ ,\\
&\{F_\pm,F_\pm\}=K_\pm,\ \ \{F_+,F_-\}=K_0\ ,\\
&[K_+,K_-]=-2K_0,\ \ [K_0,K_\pm]=\pm K_\pm\ .
\end{aligned} \tag{6.7}$$

These commutation relations were established in Ref. 4.

The Casimir operator of this superalgebra is

$$C_2(\text{osp}(1\,|\,2))=A^2+\tfrac{1}{2}A\ , \tag{6.8}$$

where

$$A=[F_+,F_-]\ . \tag{6.9}$$

From Eqs. (6.5) we find

$$\begin{aligned}
A&=-\sum_{p<q}^{D}L_{pq}S_{pq}-\tfrac{1}{4}D\\
&=-\sum_{r=1}^{D(D-1)/2}L_rS_r-\tfrac{1}{4}D\\
&=-\mathbf{L}\cdot\mathbf{S}-\tfrac{1}{4}D\ ,
\end{aligned} \tag{6.10}$$

where

$$L_{pq}=i(a_q^\dagger a_p-a_p^\dagger a_q)\ . \tag{6.11}$$

In Eq. (6.10), we have replaced the double indices pq with a single index r, taking values $r=1,\ldots,\frac{1}{2}D(D-1)$, using the formula

$$r=\tfrac{1}{2}(p-1)(2D-p)+q-p\ . \tag{6.12}$$

Substitution for A in Eq. (6.9) yields

$$C_2(\mathrm{osp}(1\,|\,2))=\tfrac{1}{4}(\mathbf{S}+\mathbf{L})^2+\tfrac{1}{32}D(D-3)\ . \tag{6.13}$$

The Hamiltonian (6.1) can be expressed in terms of K_0, $C_2(\mathrm{osp}(1\,|\,2))$, and the Casimir operator $C_2(\mathrm{sp}(2))$ of the subalgebra sp(2) given by[4]

$$C_2(\mathrm{sp}(2))=A^2+A=\tfrac{1}{4}\mathbf{L}^2+\tfrac{1}{16}D(D-4)\ . \tag{6.14}$$

We find

$$H=2K_0+4\zeta C_2(\mathrm{osp}(1\,|\,2))-4\zeta C_2(\mathrm{sp}(2))\ , \tag{6.15}$$

as given in Ref. 4. The Hamiltonian (6.1) thus exhibits a dynamical supersymmetry with superalgebra chain

$$\mathrm{osp}(1\,|\,2)\rightarrow\mathrm{sp}(2)\rightarrow\mathrm{so}(2)\ . \tag{6.16}$$

Let us turn to a construction of the supersymmetric quantum mechanics of this system[14] in terms of the spectrum-generating algebra. Following Ref. 4, we introduce the eight operators

$$\begin{aligned}&\hat{K}_0=K_0\sigma_0,\quad \hat{K}_\pm=K_\pm\sigma_0\ ,\\ &\hat{V}_\pm=\sqrt{2}F_\pm\sigma_+,\quad \hat{W}=\sqrt{2}F_\pm\sigma_-\ ,\\ &\hat{Y}=-A\sigma_3\ ,\end{aligned} \tag{6.17}$$

where σ_+ and σ_- are given by Eq. (3.6), and where

$$\sigma_0=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\ . \tag{6.18}$$

These operators satisfy graded commutation relations that generate the algebra of osp(2 | 2):

$$\begin{aligned}&[\hat{K}_0,\hat{K}_\pm]=\pm\hat{K}_\pm\ ,\\ &[\hat{K}_+,\hat{K}_-]=-2\hat{K}_0\ ,\\ &[\hat{K}_0,\hat{V}_\pm]=\pm\tfrac{1}{2}\hat{V}_\pm,\quad [\hat{K}_0,\hat{W}_\pm]=\pm\tfrac{1}{2}\hat{W}_\pm\ ,\\ &[\hat{K}_\pm,\hat{V}_\pm]=[\hat{K}_\pm,\hat{W}_\pm]=0\ ,\\ &[\hat{K}_\pm,\hat{V}_\mp]=\mp\hat{V}_\pm,\quad [\hat{K}_\pm,\hat{W}_\mp]=\mp\hat{W}_\pm\ ,\\ &\{\hat{V}_\pm,\hat{V}_\pm\}=\{\hat{W}_\pm,\hat{W}_\pm\}=0\ ,\\ &\{\hat{V}_+,\hat{V}_-\}=\{\hat{W}_+,\hat{W}_-\}=0\ ,\\ &\{\hat{V}_\pm,\hat{W}_\pm\}=\hat{K}_\pm\ ,\\ &\{\hat{V}_+,\hat{W}_-\}=\hat{K}_0-\hat{Y},\quad \{\hat{V}_-,\hat{W}_+\}=\hat{K}_0+\hat{Y}\ ,\\ &[\hat{Y},\hat{K}_0]=[\hat{Y},\hat{K}_\pm]=0\ ,\\ &[\hat{Y},\hat{V}_\pm]=\tfrac{1}{2}\hat{V}_\pm,\quad [\hat{Y},\hat{W}_\pm]=-\tfrac{1}{2}\hat{W}_\pm\ .\end{aligned} \tag{6.19}$$

The quantum-mechanical supersymmetry algebra (3.4) may be identified as a subalgebra of osp(2 | 2). If we choose

$$Q=\sqrt{2}\hat{W}_+,\quad Q^\dagger=\sqrt{2}\hat{V}_-\ , \tag{6.20}$$

then

$$\begin{aligned}\{Q,Q^\dagger\}&=2(\hat{K}_0+\hat{Y})\\ &=2\begin{bmatrix}K_0-A&0\\0&K_0+A\end{bmatrix}.\end{aligned} \tag{6.21}$$

The supersymmetric pair of Hamiltonians is then

$$\begin{aligned}H_+&=\tfrac{1}{2}\sum_{p=1}^{D}(a_p^\dagger a_p+a_pa_p^\dagger)+(2\mathbf{L}\cdot\mathbf{S}+\tfrac{1}{2}D)\\ &=2K_0+4C_2(\mathrm{osp}(1\,|\,2))-4C_2(\mathrm{sp}(2))\ ,\\ H_-&=\tfrac{1}{2}\sum_{p=1}^{D}(a_p^\dagger a_p+a_pa_p^\dagger)-(2\mathbf{L}\cdot\mathbf{S}+\tfrac{1}{2}D)\\ &=2K_0-4C_2(\mathrm{osp}(1\,|\,2))+4C_2(\mathrm{sp}(2))\ .\end{aligned} \tag{6.22}$$

At this stage, we are in a position to compare the supersymmetric quantum mechanics of Eq. (6.22) and that of Eq. (3.16). Although both systems involve a supersymmetric oscillator, the supersymmetries are different. The supersymmetry of Eq. (3.16) acts only on the radial part of the D-dimensional oscillator. The remaining $(D-1)$-dimensional piece is not affected. In contrast, the supersymmetry of Eq. (6.22) does not involve the radial variable alone. Rather, it relates the full D-dimensional oscillator with spin-orbit coupling to its partner. The distinction between the two systems is clear if the spin-orbit coupling term in Eq. (6.22), $(\mathbf{L}\cdot\mathbf{S}+\frac{1}{2}D)$, is taken to zero. The resulting Hamiltonians H_+ and H_- are identical, and the supersymmetry has disappeared.

Furthermore, we emphasize that the realization (6.20) of supersymmetric quantum mechanics is in terms of $2t\times 2t$ matrices, where t is the number of rows or columns of Γ_p. Since the realization (3.5) uses 2×2 matrices, Eqs. (6.22) cannot be written in the form (3.8). Therefore, the two supersymmetries are quite distinct.

VII. CONCLUSION

In this work, we have provided the reader with a picture of the relationships between the supersymmetric quantum-mechanical Coulomb and oscillator problems in arbitrary dimensions. The resulting scheme is an elegant intermeshing of the angular momentum structure of the many Hamiltonians involved. Our discussion of the supersymmetric quantum mechanics associated with the D-dimensional harmonic oscillator with spin-orbit couplings should alert the reader to the important distinctions between the different types of supersymmetry that may appear in a quantum-mechanical problem.

ACKNOWLEDGMENTS

We are greatly indebted to Baha Balantekin for his kindness in providing us with advance copies of his paper[4] as well as a summary of his latest findings. The financial support of the United States Department of Energy and, for D.R.T., the Natural Sciences and Engineering Council of Canada is gratefully acknowledged.

---

*Present address: Department of Physics, Indiana University, Bloomington, IN 47405. Address during calendar year 1986: Theoretical Division, CERN, 1211 Genève 23, Switzerland.

†Present address: Niels Bohr Institutet, Blegdamsvej 17, 2100-København Ø, Denmark.

‡On leave of absence from the Department of Chemistry, University of Calgary, Alberta, Canada T2N 1N4.

[1]For an overview of the recent impact of supersymmetry, see *Supersymmetry in Physics,* edited by V. A. Kostelecký and D. K. Campbell (North-Holland, Amsterdam, 1985).

[2]E. Witten, Nucl. Phys. **B185**, 513 (1981); P. Salomonson and J. W. van Holten, *ibid.* **B196**, 509 (1982); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983); C. M. Bender, F. Cooper, and B. Freedman, Nucl. Phys. **B219**, 61 (1983); M. de Crombrugghe and V. Rittenberg, Ann. Phys. (N.Y.) **151**, 99 (1983); E. Gozzi, Phys. Lett. **129B**, 432 (1983); R. Akhoury and A. Comtet, Nucl. Phys. **B246**, 253 (1984); M. Bernstein and L. S. Brown, Phys. Rev. Lett. **52**, 1933 (1984); T. Curtright and G. Ghandour, Phys. Lett. **136B**, 50 (1984); E. D'Hoker and L. Vinet, *ibid.* **137B**, 72 (1984); S. Fubini and E. Rabinovici, Nucl. Phys. **B245**, 17 (1984); M. M. Nieto, Phys. Lett. **145B**, 208 (1984).

[3]D. Lancaster, Nuovo Cimento **79A**, 28 (1984); C. A. Blockley and G. E. Stedman, University of Canterbury, New Zealand, Report 1984 (unpublished).

[4]A. B. Balantekin, Ann. Phys. (N.Y.) (to be published); private communication.

[5]V. A. Kostelecký and M. N. Nieto, Phys. Rev. Lett. **53**, 2285 (1984); Phys. Rev. A **32**, 1293 (1985); *ibid.* (to be published).

[6]J. D. Louck and W. H. Schaffer, J. Mol. Spectrosc. **4**, 285 (1960); J. D. Louck, *ibid.* **4**, 298 (1960); **4**, 334 (1960).

[7]D. Bergmann and Y. Frishman, J. Math. Phys. **6**, 1855 (1965).

[8]M. M. Nieto, Am. J. Phys. **47**, 1067 (1979).

[9]See, for example, M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U.S. GPO, Washington, D.C., 1964).

[10]E. D'Hoker and L. Vinet, Columbia University Report No. CU-TP-292, 1985 (unpublished). Here, connection is established between the three-dimensional Coulomb problem and the four-dimensional harmonic oscillator.

[11]For a discussion of the special properties of the one-dimensional Coulomb problem, see Ref. 8.

[12]R. N. Porter, Ber. Bunsenges. Phys. Chem. **86**, 407 (1982).

[13]To our knowledge, no *physical* two-dimensional harmonic-oscillator supersymmetry has yet been identified. Indeed, one goal of this work is to motivate the search for supersymmetries in physical systems.

[14]It is interesting to note that the value of ζ appears to be restricted to ± 1 by the quantum-mechanical superalgebra.