Gauge-invariant statistical mechanics and average action principle for the Klein-Gordon particle in geometric quantum mechanics

E. Santamato
Dipartimento di Fisica Nucleare, Struttura della Materia e Fisica Applicata di Napoli, Pad. 20-Mostra d'Oltremare, 80125 Napoli, Italy
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A gauge-invariant average action principle is presented that permits the formulation of a completely classical theory which is proved to be equivalent to Klein-Gordon quantum mechanics. In this approach, here called geometric quantum mechanics, the particle motion as well as the space-time geometry are determined simultaneously from the same average action principle. Quantum effects are proved to be related to space-time affine connections rather than to space-time metric tensor components. In this way geometric quantum mechanics is made compatible with axiomatic approaches to both space-time structure and probability calculus.

I. INTRODUCTION

Different axiomatic approaches to space-time structure, based on primitive concepts like light rays and freely falling particles, end up with assigning to space-time a Weyl instead of the more restricted Riemann geometry of general relativity.[1] Recently, it has been proposed that the existing gap between Weyl and Riemann geometry could be closed if quantum mechanics is enclosed in the total scheme.[2]

On the other hand, it is commonly accepted that quantum mechanics should be, in essence, a statistical theory. But it is already well known that any natural axiomatic approach to probability theory ends up with Laplace rules of combining probabilities and not with Feynman quantum rules. Also this gap should be closed in a logically consistent theory.

In two recent papers (hereafter referred to as I and II, respectively) it has been shown that traditional quantum mechanics is equivalent, in some sense, to classical statistical mechanics in Weyl spaces.[3,4] The main results of these works can be summarized as follows.

Either of these points of view are equivalent.

(a) The space-time is a Riemannian manifold. The statistical behavior of a spinless particle is described by the Klein-Gordon wave equation and probabilities combine according to Feynman quantum rules.

(b) The space-time is a generic affinely connected manifold, whose actual geometric structure is determined by the matter content. The statistical behavior of a spinless particle is described by classical statistical mechanics and probabilities combine according to Laplace rules.

In nonrelativistic applications the words "space-time," "Riemannian," and "Klein-Gordon" are to be replaced with "space," "Euclidean," and "Schrödinger," respectively.

It is evident that point of view (b) is the one consistent with the axiomatic approach to both space-time structure and probability theory. Traditional quantum mechanics, based on wave equations and *ad hoc* probability calculus [point of view (a)], appears to be merely a convenient mathematical construct to overcome the complications arising from a nontrivial space-time geometric structure.

In this paper the theory presented in I and II is reformulated starting from first principles and extended to include gauge invariance, i.e., invariance with respect to an arbitrary choice of the space-time calibration.[5] Moreover, the restrictive hypothesis of assuming a Weyl geometry from the beginning is released, both the particle motion and the space-time geometric structure being derived from a single average action principle. For this reason it seems appropriate to call the present theory *geometric quantum mechanics.*

The space-time is supposed to be a *generic* four-dimensional differential manifold with torsion-free connections $\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}$ and a metric tensor $g_{\mu\nu}$ having the signature $(+,-,-,-)$. Units are used where $\hbar=c=1$.

The metric tensor components $g_{\mu\nu}$ are regarded as arbitrarily prescribed fields, so that the theory describes the motion of a spinless particle in an external gravitational field.

In Riemann geometry the connections are *assumed* to coincide with the Christoffel symbols $\{^{\lambda}_{\mu\nu}\}$. As shown in I and II, the departure of the actual connections from the Christoffel symbols accounts for quantum effects. Therefore, to include quantum phenomena, the connections are to be determined from the space-time matter content.

The overall physical picture is quite analogous to the situation prevailing in general relativity: Geometry is not prescribed; rather, it is determined by the physical reality.

A *result* of geometric quantum mechanics is that the actual space-time connections are *integrable* semimetric Weyl connections [see Eqs. (11) and (12) below].

As is well known, in a Weyl space any physical quantity is characterized by its covariance properties and by its Weyl type.[2,5] By definition, the Weyl-type metric tensor and connections are $w(g_{\mu\nu})=1$ and $w(\Gamma^{\lambda}_{\mu\nu})=0$, respectively.

The paper is organized as follows. In Sec. II the fundamental average action principle is introduced and its lack of invariance under electromagnetic gauge transformations is noted. In Sec. III the particle random motion is derived from this principle. In Sec. IV the space-time affine connections are obtained from the same average action principle and, finally, in Sec. V the connection with traditional relativistic quantum mechanics is established. Moreover, in order not to interrupt the thread of the article, three Appendices are devoted to the technical aspects of the theory.

II. THE AVERAGE ACTION PRINCIPLE

Following the physical picture outlined in I and II, the particle is supposed to undergo a motion in space-time with deterministic trajectories and random initial conditions taken on an arbitrary spacelike three-dimensional hypersurface. In this way, according to the usual statistical interpretation of quantum mechanics, the theory describes a relativistic "Gibbs ensemble" of particles.[6]

Both the particle motion and the actual space-time connections can be obtained from the average stationary action principle

$$\delta\left[E\left[\int_{\tau_1}^{\tau_2} L(x(\tau),\dot{x}(\tau))d\tau\right]\right]=0 , \tag{1}$$

where E denotes the expectation value, L is the particle Lagrangian, and τ is an arbitrary parameter along the particle trajectory. The action integral appearing in Eq. (1) must be parameter invariant, coordinate invariant, and gauge invariant. All these requirements are met if L is positively homogeneous of the first degree in $\dot{x}^\mu=dx^\mu/d\tau$ and transforms as a scalar of Weyl type $w(L)=0$. The underlying probability measure must be gauge invariant, too.

A suitable Lagrangian for the particle is

$$L(x,dx)=(m^2-R/6)^{1/2}ds+A_\mu dx^\mu , \tag{2}$$

where $ds=(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}d\tau$ is the arc length, R is the space-time scalar curvature, and m a parameterlike scalar field of Weyl type $w(m)=-\frac{1}{2}$, which is to be interpreted as the particle rest mass. The factor $\frac{1}{6}$ in front of the space-time curvature is essentially arbitrary and its value has been chosen for further convenience. The vector field A_μ may be interpreted as the four-potential, due to an externally applied electromagnetic field, and the curvature-dependent factor in front of ds in Eq. (2) as the "effective" particle mass.

As pointed out in Ref. 2, the field m is merely the gauge transformation of the field $m=\text{const}$; therefore, no field equation is needed for it. Since the quotient of two such parameters m_1 and m_2 remains constant, a local mass measurement is possible in Weyl space-time.

Lagrangian (2) is gauge invariant, provided the *covariant* components A_μ of the four-potential have Weyl type $w(A_\mu)=0$.

The field A_μ may be split uniquely in its gradient and divergence-free parts, viz. $(\partial_\mu=\partial/\partial x^\mu)$,

$$A_\mu=\overline{A}_\mu-\partial_\mu S . \tag{3}$$

The divergence-free part $\overline{A}_\mu$ may be interpreted as the electromagnetic four-potential in the Lorentz gauge. The Weyl types of S and $\overline{A}_\mu$ are both zero.

For electrically neutral particles or in the absence of external electromagnetic fields $\overline{A}_\mu=0$, by definition, and the total four-potential A_μ reduces to a gradient.

In classical electrodynamics, the gradient part of the four-potential does not affect the particle motion, since classical mechanics is based on a stationary action principle with *fixed end points;* therefore two Lagrangians differing by a total differential are to be regarded as physically equivalent.

This is no longer true in a theory based on the action principle (1), where an *average* is involved on the particle random motion. In fact, the end-point terms in the variation of the action integral do not average to zero, in general, and, therefore, Lagrangians differing by a total differential are no longer dynamically equivalent. As a consequence, the four-potentials A_μ and $\overline{A}_\mu$ are not equivalent, either.

All these considerations may be summarized by saying that the average action principle (1) is *not* invariant under electromagnetic gauge transformations $A_\mu\rightarrow A_\mu+\partial_\mu S$.

But it is already well known that quantum mechanics provides a theory that is not invariant under electromagnetic gauge transformations, since the four-potential (and not the electromagnetic fields) appears explicitly in the wave equation.[7]

Then, principle (1) is at least compatible with general symmetries of quantum mechanics. We will show in the following that quantum mechanics can be *derived* indeed from principle (1), once the appropriate variations are taken, and that it is just the gradient part of the electromagnetic four-potential which leads ultimately to the existence of the quantum particle wave function.

III. THE PARTICLE RANDOM MOTION

The set of all space-time trajectories accessible to the particle (i.e., the particle *path space*) may be obtained from principle (1) by performing the variation with respect to the particle trajectory, with fixed metric tensor, connections, and an underlying probability measure.

This kind of variational problem may be handled in the framework of the Hamilton-Jacobi theory in the calculus of variations. As shown in Appendix A, the solution is given by the so-called Carathéodory *complete figure*[8] associated with the Lagrangian

$$\overline{L}(x,dx)=(m^2-R/6)^{1/2}ds+\overline{A}_\mu dx^\mu \tag{4}$$

written out in terms of the divergence-free part of the four-potential alone.[9]

The resulting complete figure is a geometric entity formed by a one-parameter family of hypersurfaces $S(x)=\text{const}$, where $S(x)$ obeys the Hamilton-Jacobi equation $(g^{\mu\nu}g_{\nu\rho}=\delta^\mu_\rho)$

$$g^{\mu\nu}(\partial_\mu S-\overline{A}_\mu)(\partial_\nu S-\overline{A}_\nu)=m^2-R/6 , \tag{5}$$

and by a congruence of curves intersecting this family, given by

$$dx^\mu/ds=g^{\mu\nu}(\partial_\nu S-\overline{A}_\nu)/[g^{\rho\sigma}(\partial_\rho S-\overline{A}_\rho)(\partial_\sigma S-\overline{A}_\sigma)]^{1/2} . \tag{6}$$

The congruence (6) yields the actual particle path space. The underlying probability measure on the path space may be defined on an arbitrary three-dimensional hypersurface intersecting all of the members of the congruence (6) without tangencies.[6]

The measure is completely identified by its probability current density j^μ (see Appendix B).

Moreover, the measure being independent of the arbitrary choice of the above-mentioned hypersurface, j^μ must be conservative,[6] viz.,

$$\partial_\mu j^\mu = 0 \ . \tag{7}$$

Since the trajectories are deterministically defined by Eq. (6), j^μ must be parallel to the particle four-velocity (6), and hence we can write

$$j^\mu = \rho\sqrt{-g}\, g^{\mu\nu}(\partial_\nu S - \bar{A}_\nu) \ , \tag{8}$$

with some $\rho > 0$.

Gauge invariance of the underlying measure as well as of the Carathéodory complete figure requires that j^μ transforms as a vector *density* of Weyl type $w(j^\mu)=0$ and S as a scalar of Weyl type $w(S)=0$. From definition (8) we see that ρ transforms as a scalar of Weyl type $w(\rho) = -1$. The quantity ρ may be called the scalar probability density of the particle random motion.

IV. THE SPACE-TIME GEOMETRY

The actual space-time affine connections are obtained again from principle (1) by performing the variation with respect to the fields $\Gamma^\lambda_{\mu\nu}$ for a fixed metric tensor, particle trajectory, and probability measure. To solve this variational problem it is expedient to transform the average action principle (1) in the form of a four-volume integral, viz.,

$$\delta\left[\int_\Omega d^4x[(m^2 - R/6)(g_{\mu\nu} j^\mu j^\nu)]^{1/2} + A_\mu j^\mu\right] = 0 \ , \tag{9}$$

where Ω is the space-time region occupied by the congruence (6) and j^μ is given by Eq. (8). The equivalence between variational principles (1) and (9) is proved in Appendix B. The main advantage of putting variational principle (1) in the form (9) is that in this way familiar variational techniques of field theories can be exploited.

Since the connection fields $\Gamma^\lambda_{\mu\nu}$ are contained only in the curvature term R, the variational problem (9) can be further reduced to

$$\delta\left[\int_\Omega \rho R\sqrt{-g}\; d^4x\right] = 0 \ . \tag{10}$$

In obtaining Eq. (10) the Hamilton-Jacobi equation (5) has been used. The geometric principle (10) states that the average space-time curvature must be stationary under a variation of the fields $\Gamma^\lambda_{\mu\nu}$ (principle of stationary average curvature).

The extremal connections $\Gamma^\lambda_{\mu\nu}$ arising from principle (10) are found in Appendix C, using standard techniques of field theories, as

$$\Gamma^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + \tfrac{1}{2}(\phi_\mu \delta^\lambda_\nu + \phi_\nu \delta^\lambda_\mu - g_{\mu\nu} g^{\lambda\rho}\phi_\rho) \ , \tag{11}$$

with

$$\phi_\mu = \partial_\mu(\ln\rho) \ . \tag{12}$$

As previously anticipated, Eqs. (11) and (12) show that the resulting connections are integrable Weyl connections with a gauge field ϕ_μ. The same relationship between geometry and space-time matter content was exploited also in I and II.

All the relevant equations of the present theory may be written out in a manifestly double-covariant (i.e., coordinate- and gauge-invariant) form.

The Hamilton-Jacobi equation (5) and the continuity equation (7) may be condensed in a single complex equation for $S(x)$, viz.,

$$\exp(+iS) g^{\mu\nu}(iD_\mu - \bar{A}_\mu)(iD_\nu - \bar{A}_\nu)\exp(-iS) - (m^2 - R/6) = 0 \tag{13}$$

and the fundamental relation (12), yielding the gauge field ϕ_μ, may be written as

$$D_\mu \rho = 0 \ . \tag{14}$$

In Eqs. (13) and (14) D_μ denotes the double-covariant Weyl derivative with respect to the coordinate x^μ (Ref. 2). We see that in geometric quantum mechanics the probability density (and not the rest mass m, as claimed in Ref. 2) behaves as a constant with respect to double-covariant derivation.

When written out explicitly, Eqs. (13) and (14) form a set of two coupled partial differential equations for the quantities $\rho(x)$ and $S(x)$. To any solution $\{\rho, S\}$ of these equations it corresponds to a particular random motion for the particle.[10]

V. THE KLEIN-GORDON EQUATION

The connection with traditional quantum mechanics is made by observing that Eqs. (13) and (14) may be cast in the familiar Klein-Gordon form, viz.,

$$[(i/\sqrt{-g})\partial_\mu\sqrt{-g} - \bar{A}_\mu] g^{\mu\nu}(i\partial_\nu - \bar{A}_\nu)\psi - (m^2 - \dot{R}/6)\psi = 0 \ , \tag{15}$$

where $\psi = \sqrt{\rho}\exp(-iS)$ and $\dot{R}$ is the Riemannian scalar curvature built out from the metric tensor $g_{\mu\nu}$ only. The curvature $\dot{R}$ is related to the Weyl curvature R by the formula[5]

$$R = \dot{R} - 3[\tfrac{1}{2} g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\phi_\nu)] \ . \tag{16}$$

According to point of view (a) considered at the beginning of this work, in the Klein-Gordon equation (15) any explicit reference to the underlying space-time Weyl structure has disappeared. In this sense, we may say that the Weyl structure is *hidden* in the Klein-Gordon theory. It should be stressed, however, that in geometric quantum mechanics no physical meaning is given to the complex quantity ψ and to the Klein-Gordon equation (15). Rather, the dynamical and statistical behavior of the particle, regarded as the classical particle, is determined by Eqs. (13) and (14), which, although fully equivalent to the Klein-Gordon equation (15), are expressed in terms of quantities having a more direct physical interpretation.[10]

As a final physical remark, we observe that the presence of the Riemannian curvature term in Eq. (15) is needed to have gauge invariance. When gravitational fields are negligible, however, the curvature $\dot{R}$ vanishes and Eq. (15) reduces to the Klein-Gordon of special relativity, written out in curvilinear coordinates.

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APPENDIX A

In this appendix it is shown that the Carathéodory complete figure formed by the congruence (6) solves the variational problem (1) of the text.

In order to understand the meaning of the average involved in Eq. (1) of the text we need the notion of the Gibbs ensemble in relativistic mechanics. This notion is given in Ref. 6. Roughly, a relativistic Gibbs ensemble of particles may be assimilated to an *incoherent* globule of matter moving in space-time. More exactly, a relativistic Gibbs ensemble is given by (i) a congruence of timelike curves in space-time (the path space of the particle) and (ii) a probability measure defined on this congruence.

We may construct a relativistic Gibbs ensemble as follows.

Let K, a three-parameter congruence of time-like curves in space-time, be given by

$$x^\mu = x^\mu(\tau, u^k) \ , \tag{A1}$$

where u^k $(k=1,2,3)$ are the parameters and τ is an arbitrary parameter along each curve of the congruence. For the sake of simplicity, let us suppose that the congruence covers a region Ω of space-time simply; i.e., one and only one curve of K passes through each point of Ω. Then, we may regard Eq. (A1) as a change of coordinates from the x^μ to the new coordinates y^μ $(y^0=\tau, y^k=u^k)$, whose Jacobian is nonzero in Ω.

Let us consider the action integral

$$\int_{\tau_1}^{\tau_2} L(x(\tau,u^k),\dot{x}(\tau,u^k))d\tau \ , \tag{A2}$$

with Lagrangian L homogeneous of the first degree in the derivatives $\dot{x}^\mu = \partial x^\mu/\partial\tau$.

This integral depends, of course, on the parameters u^k. Since we have one-to-one correspondence between the values of u^k and members of the congruence K, we may introduce the notion of probability that the particle follows a sample path having parameters u^k in some three-dimensional region B as

$$\text{prob}(B) = \int_{B\subset R^3} \mu(u^k)du^1du^2du^3 \ , \tag{A3}$$

where $\mu(u^k)$ is some probability density defined on R^3.

Therefore, the average action integral appearing in Eq. (1) of the text may be written in a more transparent form as

$$\begin{aligned} I &= E\left[\int_{\tau_1}^{\tau_2} L\,d\tau\right] \\ &= \int_{R^3}\int_{\tau_1}^{\tau_2} \mu(u^k)L(x^\mu(\tau,u^k),\dot{x}^\mu(\tau,u^k))d\tau\,du^1du^2du^3 \ . \end{aligned} \tag{A4}$$

The last term on the right-hand side is a four-dimensional volume integral extended to the zone between the hyperplanes $y^0=\tau_1$ and $y^0=\tau_2$ in the y coordinates.

In the x coordinates these hyperplanes are mapped on two three-dimensional hypersurfaces in space-time, namely, the hypersurfaces $\tau(x^\mu)=\tau_1$ and $\tau(x^\mu)=\tau_2[\tau(x^\mu)$ is obtained by solving Eqs. (A1) with respect to $\tau]$. These hypersurfaces, however, being merely the result of the parametrization of the congruence K, are to be regarded as essentially arbitrary.

The integrand in Eq. (A4) depends on the four unknown functions x^μ (y^ν), on their first derivative $\partial x^\mu/\partial y^0$, and on the coordinates y^ν themselves. Therefore, the variational problem $\delta I=0$ is reduced to a standard problem of variational calculus for multiple integrals, well known in field theories. The solution of this problem yields the functions $x^\mu(\tau,u^k)$, i.e., the actual congruence that renders the average action stationary.

The Lagrangian density in Eq. (A4) is represented by

$$\Lambda = \mu(u^k)L(x^\mu(\tau,u^k),x^\mu_{,\tau}(\tau,u^k)) \ ,$$

in which $x^\mu_{,\tau}=\dot{x}^\mu$ and τ, u^k are the independent variables.

According to the standard prescription, the Euler-Lagrange expressions are given by

$$E(\Lambda) = \frac{\partial}{\partial u^k}\left[\frac{\partial\Lambda}{\partial x^\mu_{,k}}\right] + \frac{\partial}{\partial\tau}\left[\frac{\partial\Lambda}{\partial x^\mu_{,\tau}}\right] - \frac{\partial\Lambda}{\partial x^\mu} \ ,$$

where $x^\mu_{,k}=\partial x^\mu/\partial u^k$. In this case, however, $\partial\Lambda/\partial x^\mu_{,k}=0$, and consequently the fixed equations $E(\Lambda)=0$ reduce to

$$\frac{\partial}{\partial\tau}\left[\mu\frac{\partial L}{\partial x^\mu_{,\tau}}\right] - \mu\frac{\partial L}{\partial x^\mu} = 0 \ ;$$

that is, since μ does not depend explicitly on τ,

$$\frac{\partial}{\partial\tau}\left[\frac{\partial L}{\partial\dot{x}^\mu}\right] - \frac{\partial L}{\partial x^\mu} = 0 \ , \tag{A5}$$

which coincides with the Euler-Lagrange equations associated with the action integral (A2). This means that the actual congruence must be a *congruence of extremals*, or, equivalently, that the particle obeys equations of motion (A5) *with probability one*.

Even if the congruence is extremal, however, we are left with nonvanishing surface terms in the variation of I, viz.,

$$\begin{aligned} \delta I = \int_{R^3} &\mu(u^k)du^1du^2du^3 \\ &\times\left[\frac{\partial L}{\partial\dot{x}^\mu}(\tau_2,u^k)\delta x^\mu(\tau_2,u^k) \right.\\ &\left. -\frac{\partial L}{\partial\dot{x}^\mu}(\tau_1,u^k)\delta x^\mu(\tau_1,u^k)\right] = 0 \ . \end{aligned} \tag{A6}$$

The terms containing the Euler-Lagrange equations of motion have been set to zero, since the congruence is supposed to be extremal.

In Eq. (A6) the quantities δx^μ at $\tau=\tau_1$ and $\tau=\tau_2$ are the displacements between the points P and $P+\delta P$, where the curve x^μ and the varied curve $x^\mu+\delta x^\mu$ intersect the hypersurfaces $\tau=\tau_1$ and $\tau=\tau_2$; then $\delta x^\mu(\tau_1,u^k)$ and $\delta x^\mu(\tau_2,u^k)$ are tangential to these hypersurfaces, respectively.

But, as previously noted, the hypersurfaces $\tau(x^\mu)=\text{const}$ are essentially arbitrary, so that the displacements $\delta x^\mu(\tau_1,u^k)$ and $\delta x^\mu(\tau_2,u^k)$ must be regarded as arbitrary as well. Therefore, $\delta I=0$ implies

$$\partial L/\partial \dot{x}^\mu(\tau,u^k)=0 . \qquad \text{(A7)}$$

Now, let us specialize Lagrangian L with Lagrangian (2) of the text. Comparing this Lagrangian with $\bar{L}$, as defined in Eq. (4) of the text, we find

$$\partial L/\partial \dot{x}^\mu=\partial \bar{L}/\partial \dot{x}^\mu-\partial_\mu S . \qquad \text{(A8)}$$

Then, condition (A7) yields

$$\partial \bar{L}/\partial \dot{x}^\mu=\partial_\mu S \qquad \text{(A9)}$$

along each member of the congruence K.

Moreover, we observe that Lagrangians L and $\bar{L}$, differing by the total differential dS, lead to the same Euler-Lagrange equations and therefore we may safely replace L by $\bar{L}$ in Eqs. (A5).

In conclusion, the congruence that renders the average action (A1) stationary must be (i) a congruence of curves that are extremal with respect to Lagrangian $\bar{L}$ and (ii) a congruence satisfying the integrability conditions (A9).

Now, it is a well known result of the Hamilton-Jacobi theory in the calculus of variations that such a congruence is given by Eq. (6) of the text, provided $S(x^\mu)$ obeys the Hamilton-Jacobi equation associated with the Lagrangian $\bar{L}(x^\mu,\dot{x}^\mu)$, namely, Eq. (5) of the text.[8]

APPENDIX B

In this appendix the current density j^μ is introduced and the equivalence between the average action (1) and the four-volume integral (9) of the text is proved. This provides a useful connection between ensemble averages and four-volume integrals appearing in field theories.

The four-dimensional integral (A4) is expressed in terms of the y coordinates (τ,u^k). Obviously, it can be expressed in terms of the x coordinates too.

To this purpose it is convenient to introduce the *current density* j^μ associated with the relativistic Gibbs ensemble.

The surface element normal to the hypersurface $\tau(x^\mu)=\text{const}$ is given by

$$d\sigma_\mu=\pi_\mu du^1du^2du^3 , \qquad \text{(B1)}$$

where π_μ are the Jacobian determinants

$$\pi_0=\frac{\partial(x^1,x^2,x^3)}{\partial(u^1,u^2,u^3)} , \quad \pi_1=\frac{\partial(x^0,x^2,x^3)}{\partial(u^1,u^2,u^3)} , \ldots , \qquad \text{(B2)}$$

etc. Then, the current density j^μ is defined according to

$$\mu=j^\mu\pi_\mu , \qquad \text{(B3)}$$

so that Eq. (A3) becomes

$$\text{prob}(B)=\int_{B\subset R^3}\mu\, du^1du^2du^3=\int_{B\subset R^3}j^\mu d\,\sigma_\mu . \qquad \text{(B4)}$$

Notice that Eq. (B3) does not define j^μ completely, its direction being still undetermined. Then, we are free to choose the current direction parallel to the congruence K at each point; i.e., $j^\mu=\lambda\dot{x}^\mu$ for some λ.

The independence of the underlying measure on the chosen hypersurface $\tau=\text{const}$ is expressed analytically by the fact that $\mu=\mu(u^1,u^2,u^3)$ does not depend on τ explicitly.

This implies $\partial_\mu j^\mu=0$, since by Gauss's theorem we have

$$\int_{\tau(x^\mu)=\tau_2}j^\mu d\sigma_\mu-\int_{\tau(x^\mu)=\tau_1}j^\mu d\sigma_\mu=\int_\Omega \partial_\mu j^\mu d^4x=0 , \qquad \text{(B5)}$$

where Ω is the strip enclosed between the essentially arbitrary hypersurfaces $\tau(x^\mu)=\tau_1$ and $\tau(x^\mu)=\tau_2$.

The same result may be found by taking the derivative of Eq. (B3) with respect to τ and using well-known properties of Jacobian determinants.

Passing to the x coordinates, the integral (A4) becomes

$$I=\int_\Omega \mu L J^{-1}d^4x , \qquad \text{(B6)}$$

where

$$J=\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(\tau,u^1,u^2,u^3)}$$

is the appropriate Jacobian. We observe that, by definition, $J=(\partial x^\mu/\partial\tau)\pi_\mu=\dot{x}^\mu\pi_\mu$ so that we have

$$I=\int_\Omega \mu[L(x^\mu,\dot{x}^\mu)/(\dot{x}^\mu\pi_\mu)]d^4x . \qquad \text{(B7)}$$

The Lagrangian L being homogeneous of the first degree in the $\dot{x}^\mu$, the term in square brackets in Eq. (B7) is homogeneous of degree zero in the $\dot{x}^\mu$. Then, we may replace $\dot{x}^\mu$ with the current $j^\mu=\lambda\dot{x}^\mu$ without affecting the integral, obtaining

$$I=\int_\Omega L(x^\mu,j^\mu)d^4x , \qquad \text{(B8)}$$

where definition (B3) has been used.

We have just proved that the average action I may be converted in a four-volume integral of the Lagrangian $L(x^\mu,j^\mu)$, obtained by replacing formally the variables $\dot{x}^\mu$ with the current components j^μ. When this formal substitution is made in Lagrangian (2) of the text, Eq. (9) is obtained. This substitution does not alter the functional dependence of the average action integral I on the connection fields $\Gamma^\lambda_{\mu\nu}$, so that variational problems (1) and (9) of the text are equivalent as long as the variation is performed with respect to these fields.

APPENDIX C

In this appendix Eqs. (11) and (12) of the text are derived. Similar results have been already obtained by various authors in a completely different context, namely, the Brans-Dicke theory of gravitation.[11]

Let us start from Eq. (9) of the text, which is to be varied with respect to the connection fields $\Gamma^{\lambda}_{\mu\nu}$, with fixed j^{μ} and $g_{\mu\nu}$.

The connections being only contained in the curvature R, we find

$$\int_{\Omega} d^4x [g_{\mu\nu} j^{\mu} j^{\nu}/(m^2 - R/6)]^{1/2} \delta R = 0 . \tag{C1}$$

Once the variation is performed, we may safely insert into Eq. (C1) the actual current j^{μ}, given by Eq. (8) of the text with $S(x^{\mu})$ obeying the Hamilton-Jacobi equation (5) of the text. After this substitution is made, Eq. (C1) reduces to Eq. (10) of the text, viz.,

$$\int_{\Omega} d^4x \, \rho\sqrt{-g}\, \delta R = \delta \left[\int_{\Omega} d^4x \, \rho\sqrt{-g}\, R \right] = 0 . \tag{C2}$$

In order to evaluate δR, we pose

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \{^{\lambda}_{\mu\nu}\} , \tag{C3}$$

where $\{^{\lambda}_{\mu\nu}\}$ are the Christoffel symbols, built out of the metric tensor $g_{\mu\nu}$.

Being the difference between two connections defined on the same manifold, the fields $T_{\mu\nu}{}^{\lambda}$ transform as the components of a tensor. For the sake of mathematical simplicity the connections $\Gamma^{\lambda}_{\mu\nu}$ are assumed to be symmetric, i.e., $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$, so that we have also $T_{\mu\nu}{}^{\lambda} = T_{\nu\mu}{}^{\lambda}$.

The curvature R with respect to the connections $\Gamma^{\lambda}_{\mu\nu}$ is related to the Riemannian curvature $\dot{R}$ with respect to the Christoffel $\{^{\lambda}_{\mu\nu}\}$ by a well-known formula of differential geometry, viz.,

$$R = \dot{R} + \nabla_{\lambda}\omega^{\lambda} - g^{\mu\nu} T_{\sigma\nu}{}^{\lambda} T_{\mu\lambda}{}^{\sigma} + g^{\mu\nu} T_{\mu\nu}{}^{\lambda} T_{\lambda\sigma}{}^{\sigma} , \tag{C4}$$

where

$$\omega^{\lambda} = g^{\mu\nu} T_{\mu\nu}{}^{\lambda} - g^{\mu\lambda} T_{\mu\nu}{}^{\nu} . \tag{C5}$$

Taking the variation of Eq. (C4) we get

$$\delta R = \nabla_{\lambda}(\delta\omega^{\lambda}) - g^{\mu\nu}(T_{\sigma\nu}{}^{\lambda}\delta T_{\mu\lambda}{}^{\sigma} + T_{\mu\lambda}{}^{\sigma}\delta T_{\sigma\nu}{}^{\lambda} - T_{\mu\nu}{}^{\lambda}\delta T_{\lambda\sigma}{}^{\sigma} - T_{\lambda\sigma}{}^{\sigma}\delta T_{\mu\nu}{}^{\lambda}) . \tag{C6}$$

Notice that, during the variation, the metric tensor components $g_{\mu\nu}$ are held constant, so that $\delta\Gamma^{\lambda}_{\mu\nu} = \delta T_{\mu\nu}{}^{\lambda}$ and the Riemannian covariant derivative ∇_{μ} commutes with the δ operation. We assume the variations $\delta\Gamma^{\lambda}_{\mu\nu}$ to be symmetric too, viz., $\delta\Gamma^{\lambda}_{\mu\nu} = \delta\Gamma^{\lambda}_{\nu\mu}$.

In Eq. (C6) the contravariant vector $\delta\omega^{\lambda}$ is found from Eq. (C5) as

$$\delta\omega^{\lambda} = g^{\mu\nu}\delta T_{\mu\nu}{}^{\lambda} - g^{\mu\nu}\delta T_{\mu\nu}{}^{\nu} . \tag{C7}$$

Equation (C6) can be rearranged as follows,

$$\delta R = \nabla_{\lambda}(\delta\omega^{\lambda}) + (g^{\mu\nu} T_{\lambda\sigma}{}^{\sigma} - g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} - g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu})\delta T_{\mu\nu}{}^{\lambda} + g^{\mu\nu} T_{\mu\nu}{}^{\lambda}\delta T_{\lambda\sigma}{}^{\sigma} . \tag{C8}$$

Since $\delta T_{\mu\nu}{}^{\lambda} = \delta T_{\nu\mu}{}^{\lambda}$, the last term in Eq. (C8) can be symmetrized, obtaining

$$g^{\mu\nu} T_{\mu\nu}{}^{\lambda}\delta T_{\lambda\sigma}{}^{\sigma} = \tfrac{1}{2}(g^{\sigma\rho} T_{\sigma\rho}{}^{\mu}\delta^{\nu}_{\lambda} + g^{\sigma\rho} T_{\sigma\rho}{}^{\nu}\delta^{\mu}_{\lambda})\delta T_{\mu\nu}{}^{\lambda} . \tag{C9}$$

Inserting Eqs. (C8) and (C9) into Eq. (C2) and performing a partial integration we find

$$\int_{\Omega} d^4x \sqrt{-g}\, \rho \{ -(\nabla_{\lambda}\rho/\rho)\delta\omega^{\lambda} + \delta T_{\mu\nu}{}^{\lambda}[g^{\mu\nu} T_{\lambda\sigma}{}^{\sigma} - g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} - g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu} + \tfrac{1}{2}(g^{\sigma\rho} T_{\sigma\rho}{}^{\mu}\delta^{\nu}_{\lambda} + g^{\sigma\rho} T_{\sigma\rho}{}^{\nu}\delta^{\mu}_{\lambda})]\} = 0 , \tag{C10}$$

where surface terms have been set to zero.

Putting $\phi_{\lambda} = (\nabla_{\lambda}\rho/\rho) = \partial_{\lambda}(\ln\rho)$ [see Eq. (12) of the text] and inserting Eq. (C7) into Eq. (C10) we get

$$\int_{\Omega} d^4x \sqrt{-g}\, \rho \delta T_{\mu\nu}{}^{\lambda}[-g^{\mu\nu}\phi_{\lambda} + g^{\mu\nu} T_{\lambda\sigma}{}^{\nu} - g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} - g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu} + \tfrac{1}{2}(\phi^{\nu}\delta^{\mu}_{\lambda} + \phi^{\mu}\delta^{\nu}_{\lambda} + (g^{\sigma\rho} T_{\sigma\rho}{}^{\mu}\delta^{\nu}_{\lambda} + g^{\sigma\rho} T_{\sigma\rho}{}^{\nu}\delta^{\mu}_{\lambda})] = 0 , \tag{C11}$$

where $\phi^{\mu} = g^{\mu\nu}\phi_{\nu}$, as usual. Now, since $\delta T_{\mu\nu}{}^{\lambda} = \delta\Gamma^{\lambda}_{\mu\nu}$ are arbitrary fields, we are led to the "field equations"

$$g^{\mu\nu}\phi_{\lambda} = g^{\mu\nu} T_{\lambda\sigma}{}^{\sigma} - g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} - g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu} + \tfrac{1}{2}\delta^{\mu}_{\lambda}(\phi^{\nu} + g^{\sigma\rho} T_{\sigma\rho}{}^{\nu}) + \tfrac{1}{2}\delta^{\nu}_{\lambda}(\phi^{\mu} + g^{\sigma\rho} T_{\sigma\rho}{}^{\mu}) . \tag{C12}$$

Contracting with respect to the indices μ and λ we get

$$\phi^{\nu} + g^{\sigma\rho} T_{\sigma\rho}{}^{\nu} = 0 . \tag{C13}$$

Inserting this relation back into Eq. (C12), the terms in parentheses vanish and the field equations may be put in the simpler form

$$g^{\mu\nu}\phi_{\lambda} = g^{\mu\nu} T_{\lambda\sigma}{}^{\sigma} - g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} - g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu} . \tag{C14}$$

These equations can be further simplified as follows. Multiplying Eq. (C14) by $g_{\mu\nu}$ and contracting, we obtain

$$T_{\lambda\sigma}{}^{\sigma} = 2\phi_{\lambda} , \tag{C15}$$

which inserted back into Eq. (C14) yields

$$g^{\mu\nu}\phi_{\lambda} = g^{\mu\sigma} T_{\lambda\sigma}{}^{\nu} + g^{\nu\sigma} T_{\lambda\sigma}{}^{\mu} , \tag{C16}$$

or, equivalently, lowering all indices,

$$g_{\mu\nu}\phi_{\lambda} = T_{\lambda\mu\nu} + T_{\lambda\nu\mu} . \tag{C17}$$

Equations (C17) can be solved with respect to $T_{\lambda\mu\nu}$ explicitly by subtracting Eq. (C17) from the sum of the two equations obtained from Eq. (C17) itself after a cyclic permutation of the indices λ,μ,ν and exploiting the symmetry of $T_{\lambda\mu\nu}$ with respect to λ and μ.

The result is

$$T_{\mu\nu\lambda} = \tfrac{1}{2}(g_{\mu\lambda}\phi_{\nu} + g_{\nu\lambda}\phi_{\mu} - g_{\mu\nu}\phi_{\lambda}) . \tag{C18}$$

Raising the index λ and inserting Eq. (C18) into Eq. (C3) we find Eq. (11) of the text, with the gauge field ϕ_{μ} given by Eq. (12) of the text.

[1]See, for instance, J. Ehlers, F. B. E. Pirani, and B. Schild, in *General Relativity,* edited by L. O'Raifeartaigh (Oxford University Press, Oxford, 1972).

[2]J. Audretsch, Phys. Rev. D **27**, 2872 (1983).

[3]E. Santamato, Phys. Rev. D **29**, 216 (1984), referred to as I.

[4]E. Santamato, J. Math. Phys. **25**, 2477 (1984), referred to as II.

[5]H. Weyl, *Space, Time, Matter* (Dover, New York, 1952), Chap. II.

[6]The notion of the relativistic Gibbs ensemble is given in R. Hakim, J. Math. Phys. **8**, 1315 (1967). See also J. L. Synge, *Relativity: the General Theory* (North-Holland, Amsterdam, 1960), Chap. IV.

[7]Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[8]H. Rund, *The Hamilton-Jacobi Theory in the Calculus of Variations* (Krieger, Huntington, New York, 1973), Chap. 3.

[9]Lagrangians (2) and (4) differ by the total differential dS, then they lead to the same Euler-Lagrange equations of motion.

[10]For a more detailed discussion about this point as well as for the antiparticle interpretation of the theory see II. It is worth noting that Feynman's interpretation of antiparticles [Phys. Rev. **76**, 49 (1949)] emerges naturally from the present approach.

[11]U. Lindstrom, Nuovo Cimento B **32**, 298 (1976); see also N. Van der Berg, J. Math. Phys. **22**, 2245 (1981). The field equations (11) are found in their integrated form directly, since the curvature R contains the first derivatives of $\Gamma^{\lambda}_{\mu\nu}$ only linearly.