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Functional measure in Kaluza-Klein theories

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The principal features of classical Kaluza-Klein theories for scalar, vector, and gravitational fields are reviewed and summarized. It is then argued that existing forms of the Kaluza-Klein *Ansatz* are potentially inconsistent on the quantum level due to functional-measure discrepancies. The canonical functional measures for integer-spin fields, derived elsewhere, are used to demonstrate the partial quantum consistency of toroidally compactified Kaluza-Klein theories of scalar, vector, and gravitational fields in an arbitrary number of dimensions. It is shown that the use of any of the other popular functional measures found in the literature would lead to the inconsistency of Kaluza-Klein compactifications. It is argued that the quantum consistency of field theories based on the canonical functional measure is an automatic consequence of the transformation properties of that measure under field redefinitions, with the full quantum consistency of all Kaluza-Klein theories following as a special case of this general rule. Finally it is suggested that nontrivial measure factors may act to stabilize the Kaluza-Klein Casimir effect.

I. INTRODUCTION

In its modern incarnation, the unification of gravity with the other forces originally proposed by Kaluza and Klein[1] has attracted an enormous amount of interest. The fundamental idea involved is simple and powerfully compelling: spacetime is assumed to possess more than four dimensions, and the higher-dimensional analogue of pure Einsteinian gravitation is found to contain ordinary gravitation together with vector and scalar field theories once the higher-dimensional theory is reduced to four dimensions.

This process of dimensional reduction to a four-dimensional effective-field theory occurs automatically if we assume that the physical ground state of our extra space dimensions is a compact manifold of microscopic size, in contrast with the noncompact Minkowski-space ground state for our four observed spacetime dimensions. Given an appropriate choice for the structure of this compact manifold, the four-dimensional vector fields which appear after compactification can be arranged to transform under the adjoint representation of any desired compact symmetry group.[2] Those nongravitational forces in nature which were actually portions of dimensionally reduced gravitation would have coupling constants which were simply related to the geometrical size and shape of our compact manifold.[3] General considerations (as well as several quantum calculations[4-6]) would lead us to believe that the characteristic size of the compact manifold must not be much larger than the characteristic length scale of gravitation, namely, the Planck length.

In Sec. II I briefly review the principal features of classical Kaluza-Klein theory, as applied to dimensionally reduced theories of scalar, vector, and tensor (i.e., gravitational) fields. For simplicity and ease of discussion, I shall confine most of my attention to the case of a single compact dimension added to n-dimensional Minkowski space (i.e., theories based on a full spacetime ground-state topology of $M^n \times S^1$). This approach generalizes trivially to the case of an arbitrary number of toroidally compactified dimensions [i.e., theories based on a ground-state topology of $M^n \times (S^1)^m$]. Such a review is necessary both in order to establish a framework for the following discussion and because of the errors and confusion found in much of the previous literature.

In Sec. III I shall suggest that although all the various Kaluza-Klein *Ansätzes* described in Sec. II (and widely assumed in the literature) are acceptable as classical theories, they are potentially inconsistent on the quantum level because of discrepancies in the functional measure associated with the path-integral formulation of the theory in its dimensionally reduced form. Although such discrepancies would be removed by several popular regularization techniques (which formally set all nontrivial measure factors equal to unity), it is argued that these regularizations are inappropriate in situations in which the underlying background topology of spacetime is not flat, most notably in Kaluza-Klein theories. A general criterion by which a quantum theory based on a Kaluza-Klein *Ansatz* can be judged consistent is pointed out.

In Sec. IV I show that toroidally compactified Kaluza-Klein theories of scalar and vector fields are self-consistent on the quantum level. Next, I show that toroidally compactified Kaluza-Klein theories of scalar and vector fields coupled to quantized gravitation are also self-consistent, at least with regard to the purely zero modes of the functional-measure factors. Finally, I demonstrate that toroidally compactified Kaluza-Klein theories of gravitation which are based on the canonical functional measure are self-consistent in this same way, but that such theories based on other functional measures for gravitation are not self-consistent. I note that these results partially rely on an interesting relation between massive and massless scalar, vector, and spin-2 fields.

In Sec. V I use formal arguments to demonstrate that the specific quantum consistency results derived in Sec. IV are actually automatic consequences of the structure of the canonical functional measure for integer-spin fields, which guarantees the quantum consistency of all such theories connected by field redefinitions. This fact strongly suggests that the canonical functional measure is actually the correct measure for a quantum field theory, and indicates the quantum consistency of all Kaluza-Klein compactifications, as well as the Higgs mechanism, the background-field method, and other common procedures in modern quantum field theory. I also show that the canonical functional measures for half-integer-spin fields and for auxiliary fields also ensure this automatic quantum consistency for those theories as well.

In Sec. VI I suggest that the nontrivial functional-measure factors in Kaluza-Klein theories of gravitation may serve to stabilize the Casimir effect in the one-loop effective potential, preventing the compact manifold from shrinking to zero size. A direct computation seems to indicate that this stabilization does occur for the $(4+1)$-dimensional case.

Throughout this paper I shall use units in which $\hbar=c=k=1$ and all quantities are measured in GeV. My metric convention will be spacelike, $\eta_{\mu\nu}=\text{diag}(-1,+1,+1,\ldots,-1)$, which is most convenient in Kaluza-Klein theories. I will adopt the usage of the rationalized Newton's constant, $\widetilde{G}=8\pi G$, with the n-dimensional (rationalized) Planck mass being given by $M_{\text{Planck}}=(\widetilde{G})^{1/(n-2)}$. In general, greek letters will range over the noncompact spacetime coordinates, written as x's, small latin letters will range over the compact spacetime coordinates, written as y's, and capital latin letters will range over all spacetime coordinates, written as z's; tildes will denote the higher-dimensional fields.

II. CLASSICAL FEATURES OF KALUZA-KLEIN THEORIES

Consider the classical theory of a massless scalar field in $n+1$ dimensions. The action is

$$\begin{aligned} S&=\int d^nx\,dy[-\tfrac{1}{2}(\partial_M\widetilde{\phi})(\partial^M\widetilde{\phi})] \\ &=\int d^nx\,dy[-\tfrac{1}{2}(\partial_\mu\widetilde{\phi})(\partial^\mu\widetilde{\phi})-\tfrac{1}{2}(\partial_y\widetilde{\phi})(\partial^y\widetilde{\phi})]\;, \end{aligned} \tag{2.1}$$

with $\widetilde{\phi}=\widetilde{\phi}(x,y)$. Now suppose that our $(n+1)$st spacetime dimension is compact, namely, that the physical ground state of our space (about which all of our field configurations represent small perturbations) is not M^{n+1}, the $(n+1)$-dimensional Minkowski space, but instead $M^n\times S^1$, the direct product of n-dimensional Minkowski space with the circle. If the value of our scalar field is to be consistently defined, it must be periodic in the spatially periodic y coordinate, i.e., $\widetilde{\phi}(x^\alpha,y)=\widetilde{\phi}(x^\alpha,y+2\pi L)$, where $2\pi L$ is the circumference of the compact space S^1. [Actually, we could also choose to define our scalar field as being "periodic with a twist" in the compact dimension, e.g., choosing $\widetilde{\phi}(x^\alpha,y)=-\widetilde{\phi}(x^\alpha,y+2\pi L)$. However, such twisted boundary conditions lead to the absence of zero modes in the dimensionally reduced version of the theory, resulting in a lack of low-energy dynamics.] Now if $\widetilde{\phi}(x,y)$ is piecewise regular in the y coordinate, it can be expanded in a Fourier series

$$\widetilde{\phi}(x^\beta,y)=\sum_{k=-\infty}^{\infty}\phi^{(k)}(x^\beta)\exp(iky/L)\;. \tag{2.2}$$

It is important to note that each of the Fourier modes $\phi^{(k)}(x^\alpha)$ in the decomposition of $\widetilde{\phi}(x^\alpha,y)$ is a completely independent field. Under this Fourier decomposition, our scalar action in (2.1) takes the form

$$S=2\pi L\int d^4x\left[-\tfrac{1}{2}(\partial_\mu\phi_{(0)})(\partial^\mu\phi_{(0)})+\sum_{k=1}^{\infty}-(\partial_\mu\phi_{(k)})(\partial^\mu\phi_{(-k)})+\frac{k^2}{L^2}\phi_{(k)}\phi_{(-k)}\right], \tag{2.3}$$

since

$$\int_0^{2\pi L}dy\exp[i(k+k')y/L]=2\pi L\delta_{k-k',0}\;.$$

Aside from the (classically) irrelevant factor of $2\pi L$ multiplying the action (which can be absorbed by a field redefinition), this is identical to the kinetic action for an infinite set of massive four-dimensional charged scalar fields, with masses given by $m^2=k^2/L^2$ and charges proportional to k. If we are confining our attention to energies low compared to the compactification energy scale $1/L$, all of the scalar modes except the zero mode would contribute negligibly to our results and can usually be ignored. This turns out to be a general result: in all Kaluza-Klein theories, whether involving interactions or not, the full kinetic term of the theory in $n+1$ dimensions reduces in n dimensions to a kinetic term and a mass operator, with the masses of the eigenmodes being proportional to the mode number and being inversely proportional to the compactification length scale. Assuming a sufficiently small compactification scale (on the order of M_{Planck}), we are usually justified in neglecting all but the zero-mode portions of the Fourier expansion in our effective-field theory at normal energies.

Now consider the more complicated case of a massless vector field $\widetilde{A}_M$ in $n+1$ dimensions. The full $(n+1)$-dimensional action is

$$\begin{aligned} S&=\int d^nx\,dy(-\tfrac{1}{4}\widetilde{F}_{MN}\widetilde{F}^{MN}) \\ &=\int d^nx\,dy[-\tfrac{1}{2}(\partial_M\widetilde{A}_N)(\partial^M\widetilde{A}^N)+\tfrac{1}{2}(\partial_M\widetilde{A}_N)(\partial^N\widetilde{A}^M)]\;, \end{aligned} \tag{2.4}$$

where $\widetilde{F}_{MN}=\partial_M\widetilde{A}_N-\partial_N\widetilde{A}_M$. Dimensionally reducing by deleting all but the zero-mode (y-independent) portions of our fields and absorbing the extra factor of $2\pi L$ into a classical field redefinition, we obtain (after suppressing the zero-mode marker) our modified action

$$S=\int d^nx[-\tfrac{1}{2}(\partial_\mu\widetilde{A}_N)(\partial^\mu\widetilde{A}_N)+\tfrac{1}{2}(\partial_\mu\widetilde{A}_\nu)(\partial^\nu\widetilde{A}^\mu)]\;. \tag{2.5}$$

Since our $(n+1)$-dimensional vector field transforms under the n-dimensional Lorentz group as the combination of an n-vector and an n-scalar, we may parametrize it as

$$\tilde{A}_M=\phi^a(A_\mu,\phi^b)\ , \tag{2.6}$$

with ϕ being a scalar and a,b being arbitrary constants. Inserting this Kaluza-Klein *Ansatz* into our $(n+1)$-dimensional action (2.5) yields

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{2}(a+b)^2\phi^{2a+2b-2}(\partial_\mu\phi)(\partial^\mu\phi)\\&-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial^\mu A^\nu-\partial^\nu A^\mu)\\&+\frac{a^2}{2}A_\nu\phi^{2a-2}(\partial_\mu\phi)(A^\mu\partial^\nu\phi-A^\nu\partial^\mu\phi)\\&+aA_\nu\phi^{2a-1}(\partial_\mu\phi)(\partial^\nu A^\mu-\partial^\mu A^\nu).\end{aligned} \tag{2.7}$$

Since a,b are arbitrary, they may be chosen for purposes of convenience, and the most convenient choice is clearly $a=0$ and $b=1$, which eliminates scalar-vector mixings and produces a particularly simple form for the dimensionally reduced action, namely,

$$\int d^nx\,dy(-\tfrac{1}{4}\tilde{F}_{MN}\tilde{F}^{MN})\mapsto\int d^nx[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)]\ . \tag{2.8}$$

Thus, the theory of a free, massless $(n+1)$-vector particle reduces to the theory of a free, massless n-vector particle and a free, massless n-scalar particle. This is a very neat and elegant result.

Similar results can be found in the case of the dimensional reduction of a spin-2 field such as gravitation. Let us begin with the Einstein-Hilbert action for gravitation in $n+1$ dimensions,

$$S=-\frac{1}{2\tilde{G}}\int d^nx\,dy(-\tilde{g})^{1/2}\tilde{R}\ . \tag{2.9}$$

The most general possible parametrization of our $(n+1)$-dimensional metric tensor in terms of n-dimensional generally covariant fields of rank two, one, and zero (i.e., fields which transform under the n-dimensional general coordinate transformation group) is

$$\tilde{g}_{MN}=A(\sigma)\begin{bmatrix}g_{\mu\nu}+B(\sigma)A_\mu A_\nu & C(\sigma)A_\nu\\ C(\sigma)A_\mu & D(\sigma)\end{bmatrix}, \tag{2.10}$$

where $A(\sigma)$, $B(\sigma)$, $C(\sigma)$, and $D(\sigma)$ are arbitrary scalar functions. Each component of the $(n+1)$-metric tensor may be expanded in harmonics of the compact dimensions, in this case a Fourier series

$$\tilde{g}_{MN}(x^B)=\sum_{k=-\infty}^{\infty}\tilde{g}^{\,k}_{MN}(x^\beta)\exp(iky/L)\ . \tag{2.11}$$

Now if we substitute this expansion into (2.9), apply the orthonormality condition for the Fourier modes, and delete all but the zero-mode terms from our effective action, we will obtain the dimensionally reduced form of the gravitational action. In general, this will be a very complicated expression, involving a wide range of mixings and interactions between the scalar, vector, and tensor fields. However, certain choices of our four arbitrary scalar functions will simplify this result considerably.

First, if we choose scalar functions which satisfy the condition $D(\sigma)B(\sigma)=[C(\sigma)]^2$, our zero-mode $(n+1)$-dimensional volume measure factorizes conveniently as

$$\begin{aligned}\tilde{g}\equiv\det(\tilde{g}_{MN})&=[A(\sigma)]^{n+1}D(\sigma)\det(g_\mu\nu)\\&\equiv[A(\sigma)]^{n+1}D(\sigma)g\ .\end{aligned} \tag{2.12}$$

Without this factorization, our n-dimensional action would contain an infinite number of field interactions of arbitrarily high order. (This important condition is maintained in most recent papers on Kaluza-Klein theories, but unfortunately is carelessly ignored in one of the most influential.[2]) Next, we can eliminate explicit scalar-graviton mixings by choosing to impose the condition

$$\frac{(n-1)A'(\sigma)}{A(\sigma)}+\frac{D'(\sigma)}{D(\sigma)}=0\ . \tag{2.13}$$

Finally, we can eliminate explicit vector-scalar mixings by choosing to set

$$\frac{B'(\sigma)}{B(\sigma)}=\frac{C'(\sigma)}{C(\sigma)}\ . \tag{2.14}$$

These three conditions combine to yield the unique relationships (modulo two unimportant constant factors which can be absorbed into field redefinitions)

$$[A(\sigma)]^{-(n-1)}=B(\sigma)=C(\sigma)=D(\sigma)\ . \tag{2.15}$$

If we choose to relabel $B(\sigma)=C(\sigma)=D(\sigma)$ as σ, we obtain the uniquely convenient gravitational Kaluza-Klein *Ansatz*

$$\tilde{g}_{MN}=\sigma^{-1/(n-1)}\begin{bmatrix}g_{\mu\nu}+\sigma A_\mu A_\nu & \sigma A_\nu\\ \sigma A_\mu & \sigma\end{bmatrix}, \tag{2.16}$$

which yields the relations

$$\tilde{g}\equiv\det(\tilde{g}_{MN})=\sigma^{-2/(n-1)}\det(g_{\mu\nu}) \tag{2.17}$$

and

$$\tilde{g}^{MN}=\sigma^{1/(n-1)}\begin{bmatrix}g^{\mu\nu} & -A^\nu\\ -A^\mu & \dfrac{1}{\sigma}+A_\mu A^\mu\end{bmatrix}. \tag{2.18}$$

These produce the dimensionally reduced action

$$S=\frac{1}{2\tilde{G}}\int d^nx(-g)^{1/2}\left[-R-\tfrac{1}{4}\sigma F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}\left[\frac{n-2}{n-1}\right]\frac{\partial_\mu\sigma\partial^\mu\sigma}{\sigma^2}\right]. \tag{2.19}$$

This is the actual justification for the $n=4$ gravitational *Ansatz* presented in Ref. 4, and gradually becoming more popular in the literature.

Thus, *on the classical level,* we may choose a form of the Kaluza-Klein *Ansatz* in which a field theory of pure gravitation in $n+1$ dimensions looks like a field theory of gravity, electromagnetism, and a massless scalar field, all in n dimensions.

The above results were based on the dimensional reduction of pure $n+1$ gravitation around an $M^n \times S^1$ physical ground state, but use of the preceding procedures for the reduction of scalar and vector field theories, along with simple iteration and field redefinition allows us to similarly obtain the dimensionally reduced form of gravitation around an $M^n \times (S^1)^m$ ground state. Such a toroidally compactified Kaluza-Klein theory yields n-dimensional gravitation, m free n-dimensional Abelian vector fields, and $m(m+1)/2$ free scalar fields. It should be noted that all of these theories actually do involve implicit scalar-tensor mixings in the field equations. The scalar fields correspond to dilation operations on the field theory.

III. QUANTUM KALUZA-KLEIN THEORIES AND THE FUNCTIONAL MEASURE

Most of the above results for classical Kaluza-Klein theories were based on the special features of classical field theory, not least of which is the ability to make arbitrary field redefinitions of our canonical variables. In a quantum field theory, such field redefinitions must be matched by corresponding changes in the functional measure associated with the path-integral formulation of the theory, and, in general, do not merely change the form of the naive action alone. Extra terms in the measure (or equivalently in the effective action) must be taken into account if the quantum versions to the two theories connected by field redefinitions are to be identical.[7]

This very important feature of quantum field theories—the issue of the functional measure and its behavior under field redefinitions—has been ignored in the vast majority of discussions concerning quantum field theory. This is for two very simple reasons. First, the functional measure for most ordinary field theories is trivial, with the measure factor being equal to unity. Second, and more importantly, any nontrivial measure factors are formally set equal to unity under several very popular regularization schemes such as dimensional regularization or ζ-function regularization. The dominance of these regularizations has virtually eliminated functional-measure factors from the recent thoughts of most theorists.

However, as has been argued elsewhere,[8,9] dimensional regularization may be inappropriate in situations in which the underlying background topology of spacetime is not flat. This is because the dimensionality of such a spacetime may be extended in several different ways, with the regularized values of divergent quantities being dependent on the extension chosen, and hence ambiguously defined. For example,

$$M^4 \times S^1 \mapsto M^\omega \times S^1 \text{ or } M^4 \times S^\omega \text{ or } M^4 \times (S^1)^\omega , \quad (3.1)$$

or any combination of these. Furthermore, dimensional regularization does not respect the chiral or conformal symmetries of our theory. ζ-function regularization may be understood to suffer from these same difficulties because of its underlying similarity to dimensional regularization,[8,9] and is anyway self-consistent to only one loop. For these reasons, we should hesitate to ignore divergent terms which are equated to zero under these regularizations, but which survive under other, more intuitively simple regularizations such as working on a lattice or using a simple cutoff. Maintaining such a cautious approach, terms derived from the functional measure of a quantum field theory should be retained.

Following this line of reasoning, let us consider the Lagrangian path-integral formulation of a quantum field theory based on some field $\tilde{Q}$ in $n+m$ dimensions. Formally, we have

$$Z = \int [d\tilde{Q}] \exp\left[i \int d^n x \, d^m y \, \tilde{\mathscr{L}}[\tilde{Q}] \right] , \quad (3.2)$$

with Z being the generating functional for our theory and with $[d\tilde{Q}]$ being the correct functional measure to be used (we will discuss its form later on). Now suppose that we compactify our theory to one in n dimensions via the Kaluza-Klein approach (i.e., enforce periodic boundary conditions on m of the spatial coordinates in the argument of $\tilde{Q}$, changing our background space to $M^n \times B^m$, with B^m being some m-dimensional compact manifold). Under this compactification scheme, our original field $\tilde{Q}$, which transformed under some representation of the appropriate symmetry group in $n+m$ dimensions will decompose into some combination of independent field Q_k, each of which transforms under the same symmetry group in n dimensions, namely,

$$\tilde{Q} = \tilde{Q}[Q_1, \ldots, Q_l] . \quad (3.3)$$

Depending on the symmetry properties of the compact space B^m, these new field Q_k may also transform under additional "gauge symmetries." All of this corresponds to the factorization of $\tilde{g}_{MN}$ into a combination of $g_{\mu\nu}$, A_μ, and ϕ which we saw in the previous section.

Now if the process of Kaluza-Klein compactification is to be consistent, the quantum theory based on the generating functional in (3.2) should be identical to the quantum theory obtained by inserting the field redefinition *Ansatz* (3.3). This requirement is simply that

$$\begin{aligned} Z &= \int [d\tilde{Q}] \exp\left[i \int d^n x \, d^m y \, \tilde{\mathscr{L}}[\tilde{Q}] \right] \\ &= \int [dQ_k] \exp\left[i \int d^n x \, d^m y \, \mathscr{L}[Q_k] \right] , \end{aligned} \quad (3.4)$$

with $[dQ_k]$ being the functional measure for the quantum fields Q_k. In this expression, $\mathscr{L}[Q_k]$ is simply defined by

$$\mathscr{L}[Q_k] \equiv \tilde{\mathscr{L}}[\tilde{Q}[Q_k]] , \quad (3.5)$$

and, as discussed in the preceding section, our field redefinition *Ansatz* $\tilde{Q}[Q_k]$ is chosen in order to yield a convenient form for $\mathscr{L}$ (or more precisely, for the dimensional reduction of $\mathscr{L}$). (Actually, this statement is not quite right. Contrary to popular wisdom, and the claims of Coleman[7] and 't Hooft,[10] there are actually extra terms appearing in the effective Lagrangian after such a change of variables, as was shown in a paper by Gervais and Jervicki[11] which has received insufficient notice. However, these additional terms enter only at two loops and higher, so I will neglect them in the context of this paper.)

However, this classical consistency requirement that (3.5) be satisfied is not sufficient to assure that (3.4) is satisfied; the functional measures must also be equal. That is, $[d\tilde{Q}]$ must factorize into

$$[d\tilde{Q}]=[dQ_1]\cdots[dQ_l]\ . \tag{3.6}$$

Unless this condition is satisfied, the classically correct dimensional reduction of a Kaluza-Klein theory will be destroyed by extra terms in the effective action corresponding to discrepancies in the functional measure. Such terms would enter at one loop, and since they derive from the functional measure can presumably be interpreted as quantum anomalies of the theory.[12]

Next, the behavior of the functional measure under the process of dimensional reduction itself should be examined. Our new fields Q_k can be expanded in harmonics of the compact manifold, which eigenfunctions of the compact portion of our kinetic operator

$$Q_k(x,y)=\sum_{\bar{n}} Q_k^{\bar{n}}(x)h^{\bar{n}}(y)\ , \tag{3.7}$$

with $\bar{n}$ labeling the particular harmonic $h(y)$. Dimensional reduction is achieved by performing a functional integration over all the nonzero modes, and discarding the additional terms produced in the effective action, which are suppressed by powers of the compactification length scale. The surviving portion of the Lagrangian will contain terms involving only the y-independent zero-mode fields and which are of mass dimension n or lower, n being the dimension of our noncompact manifold.

Now since the harmonics on B^m constitute a complete orthonormal basis set of functions, the expansion in (3.7) is perfectly legitimate for all configurations $Q_k(x,y)$ which are piecewise regular in y. Furthermore, our functional measure ranging over all coordinate points can be rewritten as a functional measure ranging over all noncompact coordinate points and over all eigenmodes $\bar{n}$. That is,

$$\prod_{x,y} dQ_k(x,y)=\prod_{x,\bar{n}} dQ_k^{\bar{n}}(x)\ . \tag{3.8}$$

However, for the dimensional reduction scheme to produce our desired zero-mode theory without being destroyed by anomalies corresponding to functional-measure discrepancies, our functional measure must factorize into

$$[d\tilde{Q}]=\prod_k [dQ_k]=\prod_{k,\bar{n}} [dQ_k^{\bar{n}}]\ . \tag{3.9}$$

It should be noted that the functional measure for one field mode can (and generally does) contain other modes and other fields.

Although it is most reassuring if our functional measure factorizes exactly, any discrepancies in the non-zero-mode field factors should not be viewed as being as serious as discrepancies in the zero-mode field factors. This is because attempts at realistic Kaluza-Klein theories are invariably based on compactification scales close to the Planck length. Therefore, the effects of higher mode fields in the action become significant only at energy scales for which the (completely unknown) higher-mass-dimension terms of full quantum gravity are also becoming significant, and our existing field theory is becoming unreliable. Furthermore, most of the specific functional measures derived in Ref. 9 which will be utilized below are only valid for energy scales low compared to M_{Planck}, and hence low compared to our compactification scale.

IV. THE QUANTUM CONSISTENCY OF KALUZA-KLEIN THEORIES

The abstract results presented above will become much more clear once we investigate the consistency of specific Kaluza-Klein theories. Let us begin with the simplest possible theory, a massless scalar field in $(n+1)$-dimensional space, the quantum version of our first example in Sec. II. We have

$$\tilde{\mathscr{L}}=-\tfrac{1}{2}(\partial_M\tilde{\phi})(\partial^M\tilde{\phi})\ , \tag{4.1}$$

with

$$Z=\int[d\tilde{\phi}]\exp\left[i\int d^nx\,dy\,\tilde{\mathscr{L}}[\tilde{\phi}]\right]\ . \tag{4.2}$$

Now the functional measure for a scalar field (in the absence of quantized gravitation) is trivial, being the flat measure[9]

$$[d\tilde{\phi}]=\prod_{x,y} d\tilde{\phi}(x,y)=\prod_{x,k} d\phi^{(k)}(x)=\prod_k [d\phi^{(k)}]\ . \tag{4.3}$$

Thus, the functional measure for a compactified $(n+1)$-dimensional massless scalar field factorizes exactly into the correct functional measures for each of the massive n-dimensional scalar field modes. The Kaluza-Klein procedure is perfectly consistent on the quantum level, being unbroken by functional-measure discrepancies in this case.

The analysis is only slightly more complicated for the case of the compactification of a massless vector field in $n+1$ dimensions. As shown in Sec. II, we have

$$\tilde{\mathscr{L}}=-\tfrac{1}{2}(\partial_M\tilde{A}_N)(\partial^M\tilde{A}_N)+\tfrac{1}{2}(\partial_M\tilde{A}_N)(\partial^N\tilde{A}^M)\ , \tag{4.4}$$

with the most convenient parametrization of $\tilde{A}_M$ being $\tilde{A}_M=(A_\mu,\phi)$. Now the functional measure for a massless vector field is once again flat,[9] being given by

$$[d\tilde{A}_M]=\prod_{x,y} d\tilde{A}_M=\prod_{x,y} dA_\mu d\phi=\prod_{x,k} dA_\mu^{(k)}d\phi^{(k)} =\prod_k [dA_\mu^{(k)}][d\phi^{(k)}]\ . \tag{4.5}$$

(Actually, the functional measure is more precisely given by $\prod_{x,y} d\tilde{A}_\mu d\bar{\eta}\,d\eta$, with $\bar{\eta}$ and η being the Faddeev-Popov ghost fields. However, for Abelian-gauge fields uncoupled to quantized gravitation, the functional integrations over the ghost fields are trivial and can be absorbed into our overall normalization factor.) Again, the functional measure factorizes perfectly, this time into the product of the measures for each mode of the n-dimensional vector and scalar fields obtained by the Kaluza-Klein procedure. This demonstrates the quantum consistency of the Kaluza-Klein compactification of a massless vector field in $n+1$ dimensions.

Taken together, the preceding two results may be combined and iterated to prove the quantum consistency of the toroidal compactification of a massless vector theory in $n+m$ dimensions.

Now let us turn to the somewhat more complicated case of the compactification of a massless scalar field coupled to quantized gravitation. (For the moment, we are simply interested in checking the quantum consistency of the compactification procedure for the scalar field, postponing the question of the quantum consistency of the gravitational compactification.) Our Lagrangian in $n+1$ dimensions is

$$\tilde{\mathscr{L}}=-\tfrac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{MN}(\partial_M\tilde{\phi})(\partial_N\tilde{\phi}) , \tag{4.6}$$

with the functional measure for the scalar field given by

$$[d\tilde{\phi}]=\prod_{x,y}(\tilde{g}^{00})^{1/2}(\tilde{g})^{1/4}d\tilde{\phi} . \tag{4.7}$$

If we expand out $\tilde{\mathscr{L}}$ using the parametrization derived in Sec. II, we obtain

$$\tilde{\mathscr{L}}=-\tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\tilde{\phi}\partial^\nu\tilde{\phi}+\sqrt{-g}A^\mu\partial_\mu\tilde{\phi}\partial_y\tilde{\phi} -\frac{1}{2\sigma}\partial_y\tilde{\phi}\partial_y\tilde{\phi}-\tfrac{1}{2}\sqrt{-g}A_\mu A^\mu\partial_y\tilde{\phi}\partial_y\tilde{\phi} . \tag{4.8}$$

Now the functional measure for our compactified theory depends only on those terms in the Lagrangian which are quadratic in time derivatives. Therefore, only the first term in (4.8) contributes, and this term can be rewritten in an eigenmode basis as

$$\mathscr{L}=\sum_{k,k'}(g^{00}g^{1/2})^{(-k-k')}\partial_0\phi^{(k)}\partial_0\phi^{(k')} . \tag{4.9}$$

The functional measure for these scalar modes should be given by[9]

$$\left[\prod_k d\phi^{(k)}\right]=\sum_x[\det_{k,k'}(g^{00}g^{1/2})^{(k+k')}]^{1/2}\prod_k d\phi^{(k)} . \tag{4.10}$$

Determining whether the expression in (4.10) is equal to (4.7) appears somewhat difficult because of the complications involved in transforming (4.7) into an eigenmode basis. Therefore, for now let us merely check the equivalence of the two expressions in their zero-mode sector, i.e., show that the dimensionally reduced theory has the correct zero-mode functional factors (later on, we shall demonstrate their exact equality).

To check this equality in purely zero-mode functional factors, we simply apply the process of dimensional reduction to those fields in the Lagrangian which give rise to the functional-measure factors, in this case the metric tensor, retaining only the purely zero-mode components. The functional measure in (4.7) assumes the form

$$[d\tilde{\phi}]=\prod_{x,k}(g^{00})^{1/2}g^{1/4}d\phi^{(k)} , \tag{4.11}$$

where we have suppressed the zero-mode indices of our metric field. Likewise the remaining portion of (4.10) is

$$\left[\prod_k d\phi^{(k)}\right]=\prod_x\left[\prod_k(g^{00}g^{1/2})^{1/2}\right]\left[\prod_k d\phi^{(k)}\right] =\prod_{x,k}(g^{00})^{1/2}g^{1/4}d\phi^{(k)}=[d\tilde{\phi}] , \tag{4.12}$$

and the two functional measures are equal, implying the quantum consistency of a scalar Kaluza-Klein theory, at least with regard to the purely zero-mode portion of its functional factor.

The reason for this equality is very simple. If we simply discard all but the zero-mode portion of our metric field in (4.6) and (4.8), it is easy to see that the crucial requirement for our $(n+1)$-dimensional functional measure to factorize properly is that the functional-measure factor for a massless scalar field theory in $n+1$ dimensions be equal to the functional-measure factor for a massless scalar field theory in n dimensions and also equal to the functional-measure factor for a massive scalar field theory in n dimensions. That is, if

$$[d\phi]^{(n)}_{\phi;M=0}=\prod_x M^{(n)}_{\phi;M=0}d\phi , \qquad [d\phi]^{(n)}_{\phi;M\neq0}=\prod_x M^{(n)}_{\phi;M\neq0}d\phi , \tag{4.13}$$

then our consistency requirement is that

$$M^{(n+1)}_{\phi;M=0}=M^{(n)}_{\phi;M=0}=M^{(n)}_{\phi;M\neq0} . \tag{4.14}$$

Since $M^{(n)}_{\phi;M=0}=M^{(n)}_{\phi;M\neq0}=(g^{00})^{1/2}g^{1/4}$ for all dimensions n, this consistency condition is satisfied. Thus the functional-measure factor for our $(n+1)$-dimensional massless scalar field factorizes into the product of the functional-measure factors for our massless zero-mode and massive higher-mode scalar field theories in n dimensions.

Simply iterating the above procedure demonstrates the quantum consistency of a toroidally compactified $(n+m)$-dimensional scalar field theory coupled to quantized gravitation.

Similar relations demonstrate the quantum consistency (in the zero-mode metric sector) of a compactified massless vector field coupled to gravitation. The canonical functional measures for massless and massive vector fields in n dimensions are given by[9]

$$M^{(n)}_{A_\mu;M=0}=(g^{00})^{(n-2)/2}g^{(n-3)/4} , \qquad M^{(n)}_{A_\mu;M\neq0}=(g^{00})^{(n-1)/2}g^{(n-2)/4} , \tag{4.15}$$

implying that

$$M^{(n+1)}_{A_\mu;M=0}=(M^{(n)}_{A_\mu;M=0})(M^{(n)}_{\phi;M=0})=M^{(n)}_{A_\mu;M\neq0} . \tag{4.16}$$

These relations ensure that the zero-mode portion of the n-dimensional metric field factor in the functional measure for a massless $(n+1)$-dimensional vector field factorizes into the product of the factors for the functional measures of each of the n-dimensional modes. (The zero-mode portion of the action contains separate massless and scalar-vector fields, while all the nonzero modes consist of massive vector fields, produced by a Higgs mechanism, just as in the case of Kaluza-Klein gravitational nonzero modes.[4]) The zero-mode portion of the $(n+1,n+1)$ component of the $(n+1)$-metric tensor (the σ of Sec. II) also factorizes correctly. Iterating these results for the quantum consistency of the compactifications of vector and scalar fields demonstrates the quantum consistency of all toroidally compactified massless vector fields in $n+m$ dimensions.

Now let us turn to the slightly more complicated process of checking the quantum consistency (with regard to purely zero-mode functional factors) of a compactified gravitational field. Using the parametrization of the $(n+1)$-dimensional metric tensor derived in Sec. II, our full action has the form

$$S=\int d^n x\, dy \left[-\frac{1}{2\widetilde{G}}(-\widetilde{g})^{1/2}\widetilde{R} \right]$$
$$=\int d^n x\, dy \left[-\frac{1}{2\widetilde{G}}(-g)^{1/2}R-\frac{1}{8\widetilde{G}}\sigma(-g)^{1/2}F_{\mu\nu}F^{\mu\nu}-\frac{1}{8\widetilde{G}}\left[\frac{n-2}{n-1}\right](-g)^{1/2}\frac{\partial_\mu\sigma\partial^\mu\sigma}{\sigma^2} + \text{terms linear or lower in } \mu \text{ derivatives} \right]. \tag{4.17}$$

Now since we are only interested in checking the equality of purely zero-mode functional-measure factors, we may assume that all fields except those acted upon by derivatives are purely zero mode, i.e., are reduced to their zero-mode components. By using the techniques in Ref. 9, we find that the purely zero-mode functional-measure factors for each mode field are given by

$$\begin{aligned}
[d\sigma^{(k)}] &= \prod_x (g^{00})^{1/2} g^{1/4}\sigma^{-1} d\sigma^{(k)} ,\\
[dA_\mu^{(k)}] &= \prod_x (g^{00})^{(n-2)/2} g^{(n-3)/4}\sigma^{(n-1)/2} dA_\mu^{(k)} d\overline{\eta}^{(k)} d\eta^{(k)} ,\\
[dg_{\mu\nu}^{(k)}] &= \prod_x (g^{00})^{n(n-3)/4} g^{(n^2-5n-4)/8} dg_{\mu\nu}^{(k)} d\overline{\eta}_\mu^{(k)} d\eta^{\nu(k)} .
\end{aligned} \tag{4.18}$$

(Actually, just as in the vector case, the nonzero modes of σ and A_μ are eaten by the gauge components of the nonzero $g_{\mu\nu}$ modes, which become massive, in a Kaluza-Klein version of the Higgs mechanism as pointed out in Ref. 4; but this has no effect on the functional-measure factors.) On the other hand, the zero-mode canonical functional-measure factor for the $(n+1)$-dimensional gravitational field is given by

$$\begin{aligned}
[d\widetilde{g}_{MN}] &= \prod_{x,y} (\widetilde{g}^{00})^{(n+1)(n-2)/4} \widetilde{g}^{(n^2-3n-8)/8} d\widetilde{g}_{MN} d\overline{\eta}_M d\overline{\eta}^N\\
&= \prod_{x,y} \sigma^{(n-3)/2} (g^{00})^{(n+1)(n-2)/4} g^{(n^2-3n-8)/8} dg_{\mu\nu} dA_\mu d\sigma\, d\overline{\eta}_\mu d\eta^\nu d\overline{\eta}\, d\eta ,
\end{aligned} \tag{4.19}$$

where the second line incorporates the functional-measure factors obtained from the Jacobian of our change of functional variables (with only the zero-mode portion of the Jacobian being retained). Now the functional-measure factor in (4.19) is equal to the product of the functional-measure factors in (4.18), implying the quantum consistency of the compactification of an $(n+1)$-dimensional gravitational field.

Once again, combining and iterating the above results for gravitational, vector, and scalar fields demonstrates the quantum consistency of toroidally compactified $(n+m)$-dimensional gravitation.

It is interesting to note that the above consistency results are partly a consequence of the very strong similarity between massless integer-spin fields in $n+1$ dimensions and massive fields of the same spin in n dimensions. For scalar, vector, and spin-2 fields, the functional-measure factors in the two cases are identical, as are the number of physical polarizations, i.e., on-shell states. This appears to be a general consequence of the structure of the Lagrangian and the form of the canonical functional measure for integer-spin fields.

It is important to point out that the above quantum consistency proof for a compactified gravitational field is not a trivial result, nor is it an automatic consequence of any functional measure we might choose. It is a direct consequence of the form of the canonical functional measure for gravitation, and if we had instead chosen to use any of the other functional measures for gravitation which are given in the literature, we would have discovered the quantum inconsistency of Kaluza-Klein theories based on a compactified higher-dimensional gravitational field. For example, the gravitational functional measure suggested by Fujikawa[13] has the form

$$[dg_{\mu\nu}]^{(n)}_{\text{Fujikawa}}=\prod_x g^{(n^2-5n-8)/8} dg_{\mu\nu} d\overline{\eta}_\mu d\eta^\nu , \tag{4.20}$$

and is derived by naively assuming the absence of any nontrivial point permutation Jacobian under the Becchi-Rouet-Stora-Tyutin (BRST) extension of a general coordinate transformation. Even if we removed the g^{00}

functional-measure factors from our vector and scalar functional measures, this Fujikawa measure would still yield discrepancies in the functional factors of σ after compactification, resulting in the quantum inconsistency of Kaluza-Klein theories. The gravitational functional measure sketched out by DeWitt[14]

$$[dg_{\mu\nu}]^{(n)}_{\rm DeWitt}=\prod_x g^{(n^2-3n-4)/8}dg_{\mu\nu}\ , \tag{4.21}$$

would also produce discrepancies in the functional measure of Kaluza-Klein compactifications of gravitation. It is important to emphasize that these discrepancies are in the zero-mode sectors of the compactified theories, and hence would be present at low and medium energies; they cannot be argued away as being offset by the new physics entering at Planck-mass energy scales.

V. THE CANONICAL FUNCTIONAL MEASURE AND FIELD REDEFINITION

We have just seen that the choice of the canonical functional measure for gravitation appears to result in the quantum consistency (at least with regard to purely zero-mode functional factors) of toroidally compactified Kaluza-Klein theories, while using, for example, the Fujikawa or DeWitt gravitational functional measures does not. We have also seen that the canonical functional measures for scalar and vector field theories result in the quantum consistency of their toroidal compactifications as well.

This is no accident. As we are about to see, these results follow as special cases of the general transformation properties of the canonical functional measure under field redefinitions, which formally ensure the quantum consistency of any two theories connected by field redefinition. This general conclusion will also establish the quantum consistency of Kaluza-Klein theories with regard to nonzero-mode functional-measure factors and nontoroidal compactifications as well. In this section, capital latin letters will represent completely general field indices.

Consider a Lagrangian $\tilde{\mathscr{L}}$ containing quantum fields $\tilde{Q}_A$ (which may be either bosonic or fermionic). If these quantum fields $\tilde{Q}_A$ have integer spin and are physical (i.e., propagating), the canonical functional measure for this theory has the form

$$[d\tilde{Q}_A]=\prod_x\left[\mathrm{sdet}\left[\frac{\delta^2\tilde{\mathscr{L}}}{\delta(\partial_0\tilde{Q}_A)\delta(\partial_0\tilde{Q}_B)}\right]\right]^{1/2}d\tilde{Q}_A\ , \tag{5.1}$$

with sdet being the superdeterminant. Now suppose that we choose to write our theory in a new form by using the field redefinition $\tilde{Q}_A=\tilde{Q}_A[Q_B]$. The new Lagrangian for our theory is simply defined by $\mathscr{L}[Q_B]\equiv\tilde{\mathscr{L}}[\tilde{Q}_A[Q_B]]$ (actually, as noted previously, this naive change of variables procedure is not quite right, and extra terms must be added to the effective action at two loops and higher), and the canonical functional measure for our new theory is given by

$$[dQ_A]=\prod_x\left[\mathrm{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\right]\right]^{1/2}dQ_A\ . \tag{5.2}$$

Since the Lagrangian of our two theories connected by field redefinitions are defined to be identical, our two quantum theories are identical, i.e., have the same generating functionals

$$Z=\int[dQ_A]\exp\left[i\int dx\,\mathscr{L}[Q_A]\right]\ , \tag{5.3}$$

if and only if the two functional measures (5.1) and (5.2) are identical. Such a quantum consistency condition is automatically satisfied by the canonical functional measure.

This is very easy to show. If our quantum field redefinition is to be well defined, it must be nonsingular, implying that the superdeterminant of its Jacobian is nonzero, and the change of functional variables may be inverted to yield $Q_B=Q_B[\tilde{Q}_A]$. Therefore, using the chain rule, the functional-measure factor in (5.1) can be rewritten as

$$\begin{aligned}\mathrm{sdet}\left[\frac{\delta^2\tilde{\mathscr{L}}}{\delta(\partial_0\tilde{Q}_A)\delta(\partial_0\tilde{Q}_B)}\right]&=\mathrm{sdet}\left[\left[\frac{\delta Q_C}{\delta\tilde{Q}_A}\right]\left[\frac{\delta^2\mathscr{L}}{\delta(\partial_0 Q_C)\delta(\partial_0 Q_D)}\right]\left[\frac{\delta Q_D}{\delta\tilde{Q}_B}\right]\right]\\&=\mathrm{sdet}\left[\frac{\delta Q_C}{\delta\tilde{Q}_A}\right]\mathrm{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta(\partial_0 Q_C)\delta(\partial_0 Q_D)}\right]\mathrm{sdet}\left[\frac{\delta Q_D}{\delta\tilde{Q}_B}\right]\ .\end{aligned} \tag{5.4}$$

On the other hand, the remaining piece of the functional measure in (5.1) transforms as

$$d\tilde{Q}_A=\mathrm{sdet}\left[\frac{\delta\tilde{Q}_A}{\delta Q_B}\right]dQ_B\ , \tag{5.5}$$

with the new measure factor produced being the superdeterminant of the field redefinition Jacobian. Combining these two results, we find that

$$\begin{aligned}[d\tilde{Q}_A]&=\prod_x\left[\mathrm{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\right]\right]^{1/2}\\&\quad\times\mathrm{sdet}\left[\frac{\delta Q_C}{\delta\tilde{Q}_B}\right]\mathrm{sdet}\left[\frac{\delta\tilde{Q}_E}{\delta Q_F}\right]dQ_A\\&=\prod_x\left[\mathrm{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta(\partial_0 Q_A)\delta(\partial_0 Q_B)}\right]\right]^{1/2}dQ_A\\&=[dQ_A]\ ,\end{aligned} \tag{5.6}$$

with the two extra Jacobian factors exactly canceling out.

This automatic consistency of the canonical functional measure under change of field variables ensures that any two field theories connected by field redefinitions both have the same functional measure and are hence identical on the quantum level. Among other results, this formally establishes the quantum consistency of the Higgs mechanism, the background field method, the Kaluza-Klein *Ansatz*, nonlinear σ models, and many other standard procedures in modern quantum field theory which either implicitly or explicitly rely upon field redefinition.

Exactly similar arguments may be used to demonstrate the quantum consistency of the canonical functional measures under field redefinitions for the cases of half-integer-spin fields and nonpropagating auxiliary fields. For these cases, the relevant identities are[9]

$$\begin{aligned}[d\tilde{Q}_A] &= \prod_x \left[\text{sdet}\left[\frac{\delta^2\tilde{\mathscr{L}}}{\delta\tilde{Q}_A\delta(\partial_0\tilde{Q}_B)}\right]\right]^{1/2} d\tilde{Q}_A \\ &= \prod_x \left[\text{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta Q_A\delta(\partial_0 Q_B)}\right]\right]^{1/2} dQ_A \\ &= [dQ_A]\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}[d\tilde{Q}_A] &= \prod_x \left[\text{sdet}\left[\frac{\delta^2\tilde{\mathscr{L}}}{\delta\tilde{Q}_A\delta\tilde{Q}_B}\right]\right]^{1/2} d\tilde{Q}_A \\ &= \prod_x \left[\text{sdet}\left[\frac{\delta^2\mathscr{L}}{\delta Q_A\delta Q_B}\right]\right]^{1/2} dQ_A \\ &= [dQ_A] \, .\end{aligned} \tag{5.8}$$

Taken together, these results strongly suggest that the canonical functional measure, besides being the most elegant and simplest to derive, is also the correct functional measure for a quantum field theory.[9]

Although the above manipulations are purely formal and abstract, they are buttressed by the special case of the quantum consistency of various toroidally compactified Kaluza-Klein theories which was worked out and checked at length in Sec. IV above. Furthermore, purely formal arguments are necessitated by the absence of any completely satisfactory means of regulating the Feynman path integral, and as noted by DeWitt,[14] they tend to acquire a consistency and logic of their own.

VI. STABILIZING THE KALUZA-KLEIN CASIMIR EFFECT

As previously mentioned, realistic Kaluza-Klein theories require that the compact manifold have an extremely small size. This is because the coupling constants g for the gauge forces produced by the compactification are each given by the ratio of $2\pi\sqrt{2G}$ to a particular root-mean-square circumference of the compact manifold.[3] In order for Kaluza-Klein theories to yield any of the coupling constants observed in nature, the circumferences of the compact manifold must be no more than 1 or 2 orders of magnitude longer than the Planck length.

A plausible explanation for the extremely small size of the compact manifold follows as one of the most interesting results of an analysis of the quantum dynamics of Kaluza-Klein theories. As Appelquist and Chodos first demonstrated,[4] the one-loop effective potential for a Kaluza-Klein theory of gravity exhibits the Casimir effect, causing any compact manifold to shrink in size. In particular, for a $(4+1)$-dimensional compactification, they obtained the expression

$$V_{\text{eff}}(\sigma_c) = \frac{\Lambda_0^5}{8\pi} + \frac{5\beta}{(2\pi\sigma_c{}^{1/3}R_5)^5} \, , \tag{6.1}$$

with $L = 2\pi\sigma_c{}^{1/3}R_5$ being the effective circumference of the compact dimension, $\beta = -0.394$, and Λ_0 being our momentum-space cutoff. The first term of this potential has the form of a large induced cosmological constant, while the second term represents an attractive potential, causing the size of the compact manifold to shrink down to a minimal value. This is merely a form of the well-known Casimir effect, caused in this case by the vacuum fluctuations between the two "plates" $y=0$ and $y=2\pi R_5$.

While this result is desirable in some ways, it does present certain difficulties. The large induced cosmological constant term is endemic to all theories of quantized gravitation, and must simply be "renormalized" to the observed value of zero. However, the attractive potential itself also presents a problem since its value is minimized only when the radius of the compact manifold has shrunk completely to zero. This may simply be an artifact of our one-loop quantum gravitational calculation, and it is possible that higher-loop terms in the effective potential would serve to stabilize the theory at a finite-size radius. But such higher-loop contributions would only become non-negligible at energy scales for which the full theory of quantum gravity becomes important, and this raises severe problems. First, the resulting size for the stabilized compact radii would presumably be equal to or smaller than the Planck length, resulting in gauge coupling constants much too large to correspond to those observed in nature. More importantly, our entire Kaluza-Klein analysis would probably break down at such energy scales. We do not yet possess a full theory of quantum gravity, and the additional Planck-mass-suppressed terms in the effective action expansion of full quantum gravitation are completely unknown to us; these additional terms would contribute significantly to Kaluza-Klein theories at Planck-mass energy scales and above. Therefore, our Kaluza-Klein picture should only be taken seriously at length scales for which the higher-loop quantum effects can be ignored.

The instability of the one-loop effective action appears to be a problem endemic to Kaluza-Klein theories, whether based on toroidal compactification or not.[4,5] Attempts[6] have been made to cure this instability through the addition of large numbers of scalar or spinor matter fields to the theory. The one-loop contributions of these additional fields can stabilize the effective potential at a compact circumference greater than the Planck length. But the sheer number of these additional fields which must be "put in by hand" to balance the one-loop attractive gravitational contribution is prohibitively huge, ranging in the hundreds or thousands. (This fact has led to Mark Rubin's aphoristic observation that "one graviton is worth ten thousand scalars.") The need for so many extra matter fields completely negates the principles of simplicity and elegance which were the chief motivations for Kaluza-Klein theory in the first place.

However, all the above derivations of the one-loop effective potential completely ignore contributions from the functional measure, and it is possible that such contributions may serve to stabilize the Casimir effect at a circumference longer than the Planck length. (Appelquist and Chodos[4] examine the measure only to dismiss it, partly because of the conflicting functional measures suggested by Fradkin and Vilkovisky, 't Hooft, and DeWitt;[14,15] this dismissal is justified by the use of dimensional or ζ-function regularizations, which eliminate any measure factors. Yasuda[16] claims to show that no terms from the functional measure appear in the effective potential for quantum gravity, but his analysis is based on the use of Fujikawa's gravitational functional measure, which we have seen above is probably not correct. In most of the remaining papers on the Kaluza-Klein Casimir effect, the functional measure is never even mentioned.) This possibility is quite easy to understand. Our quantum generating functional (or partition function) has the form

$$Z = \int [d\tilde{g}_{MN}] \exp(iS[\tilde{g}_{MN}]) \ . \qquad (6.2)$$

If the functional measure contains positive powers of the dilation scalar σ which parametrizes the size of our compact manifold, this will reduce the relative weighting for those total field configurations in which σ is small, partly off setting any contrary effect from the action itself.

This intuitive argument can be made precise by calculating the additional terms in the one-loop effective potential which derive from the functional measure. Since the functional measure is independent of $\hbar$, it enters the effective action as a one-loop effect; this can equally be seen by using measure ghost fields to bring the functional measure into the effective action,[4] and by noting that the resulting diagrams (with no external ghosts) enter at one loop. For the definition of the one-loop effective potential used in Ref. 4, we have

$$Z[\sigma_c] = \exp\left[-V_{\text{eff}}(\sigma_c)\sigma_c^{-1/3} \int d^4x\, dy \right] . \qquad (6.3)$$

Now if we use the canonical functional measure for gravitation in 4+1 dimensions which was given in (4.1), we can follow the exact procedure of Ref. 4 to calculate the contribution of this measure factor to the one-loop effective potential. Just as in Ref. 4, the zero-mode fields and ghosts do not contribute (their apparent contribution is exactly canceled by their measure factor contribution). The nonzero field modes contribute a factor of $\exp\left[6 \sum_n \sum_k \ln\sigma_c\right]$ to $Z[\sigma_c]$, partly coming from the initial measure and partly from a change of variables Jacobian factor. Finally, the nonzero ghost modes contribute a factor of $\exp\left[-\frac{2}{3}\sum_n \sum_k \ln\sigma_c\right]$ to $Z[\sigma_c]$. Thus, the total additional factor contributed through measure effects is

$$\exp\left[16 \sum_n \sum_k \ln(\sigma_c^{1/3}) \right] . \qquad (6.4)$$

We can freely multiply this factor by a numerical constant if we wish (since such a constant can always be absorbed into the normalization of Z) yielding

$$\exp\left[16 \sum_n \sum_k \ln(2\pi\sigma_c^{1/3} R_5) \right] .$$

Next, we can use the relation $\left[\int d^4x\right] d^4k/(2\pi)^4 = 1$ to rewrite our factor as

$$\exp\left[16 \sum_n \frac{1}{2\pi R_5} \int \frac{d^4k}{(2\pi)^4} \ln(2\pi\sigma_c^{1/3} R_5) \int d^4x\, dy \right] , \qquad (6.5)$$

which corresponds to an additional term in our effective potential of

$$\Delta V_{\text{eff}}(\sigma_c) = -16 \sum_n \frac{\sigma_c^{1/3}}{2\pi R_5} \int \frac{d^4k}{(2\pi)^4} \ln(2\pi\sigma_c^{1/3} R_5) \ . \qquad (6.6)$$

Now if we cut off our momentum at $\Lambda = \sigma_c^{-1/6}\Lambda_0$, with Λ_0 being the cutoff in our standard coordinate system ($\sigma_c = 1$), we have[4]

$$\int \frac{d^4k}{(2\pi)^4} = \frac{\Lambda^4}{32\pi^2} = \frac{\Lambda_0^4}{32\pi^2} \sigma_c^{-2/3} , \qquad (6.7)$$

and if we cutoff our infinite mode sum at mode numbers N whose masses are equal to this energy scale, $N = 2\pi R_5 \sigma_c^{1/3} \Lambda_0$, we obtain

$$\begin{aligned}\Delta V_{\text{eff}}(\sigma_c) &= (-16)(4\pi R_5 \sigma_c^{1/3}\Lambda_0) \left[\frac{\sigma_c^{1/3}}{2\pi R_5} \right] \\ &\quad \times \left[\frac{\Lambda_0^4 \sigma_c^{-2/3}}{32\pi^2} \right] \ln(2\pi\sigma_c^{1/3} R_5) \\ &= -\frac{\Lambda_0^5}{\pi^2} \ln(2\pi\sigma_c^{1/3} R_5) \ . \end{aligned} \qquad (6.8)$$

Therefore, the total one-loop effective potential, including both the new term and the terms previously derived in Ref. 4, is

$$V_{\text{eff}}(\sigma_c) = \frac{\Lambda_0^5}{8\pi} + \frac{5\beta}{L^5} - \frac{\Lambda_0^5}{\pi^2} \ln L \ , \qquad (6.9)$$

with $\beta = -0.394$ and with $L = 2\pi\sigma_c^{1/3} R_5$ being the effective circumference of the compact manifold. This potential has its minimum at

$$L = (-25\pi^2\beta)^{1/5} \frac{1}{\Lambda_0} \approx 2.50 \frac{1}{\Lambda_0} \ . \qquad (6.10)$$

Our Casimir effect does stabilize, but the stability point is cutoff dependent.

Such a cutoff-dependent result is not as bad as it might seem. Since we lack a full theory of quantum gravity, we must cut off all our calculations around the Planck mass anyway in order to avoid having to deal with the higher-mass-dimension terms in the effective action expansion of quantum gravity. Furthermore, Hawking has speculated that the formation of quantum black holes might provide a natural Planck mass cutoff for quantum gravity.[17] In any event, it is encouraging that the above calculation yields a stability length which is (somewhat) longer than our cutoff length scale; if it had been shorter, our approximation would have been inconsistent and the result completely untrustworthy. Even for the above stability length, our entire calculation should not be taken too seriously. This is because our gravitational functional measure was only derived for length scales long compared to the Planck length, and we are extending its use to scales of comparable length.

If we were to choose our cutoff at $\Lambda_0=(8\pi G)^{-1/2}$, the (rationalized) Planck mass, which is the natural mass scale for quantum gravity, the value of the gauge-coupling fine-structure constant $\alpha(g)=g^2/4\pi$ which results is almost exactly one. This is not too close to the realistic value of, e.g., $\alpha(g)\approx 0.02$ for our known forces at the grand-unified-theory (GUT) scale, but such a wide discrepancy is not surprising given the toy model nature of our $(4+1)$-dimensional theory and our speculative choice of a gravitational cutoff energy. Calculations based on a more realistic choice of compact manifold might yield larger values for the numerical coefficient in (6.10), and hence smaller coupling constants for theory. Still, the coupling constant obtained by the simple analysis above is generally more realistic than those obtained in Ref. 6 through the addition of 1000 extra species of spinor matter fields into the theory. This is highly encouraging.

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[1]Th. Kaluza, Sitzungsber. Preuss Akad. Wiss. Phys. Math. K1, 966 (1921); O. Klein, Z. Phys. **37**, 895 (1926).

[2]E. Witten, Nucl. Phys. **B186**, 412 (1981).

[3]S. Weinberg, Phys. Lett. **125B**, 265 (1983).

[4]T. Appelquist and A. Chodos, Phys. Rev. D **28**, 772 (1983); T. Appelquist, A. Chodos, and E. Myers, Phys. Lett. **127B**, 51 (1983).

[5]N. A. Voronov and Ya. I. Kogan, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 262 (1983) [JETP Lett. **38**, 311 (1983)]; M. A. Rubin and B. D. Roth, Phys. Lett. **127B**, 55 (1983); Nucl. Phys. **B226**, 444 (1983); T. Inami and O. Yasuda, Phys. Lett. **133B**, 180 (1983); A. Chodos and E. Myers, Ann. Phys. (N.Y.) **156**, 412 (1984); A. Chodos, Report No. YTP-84-05 (unpublished); D. Rohrlich, Phys. Rev. D **29**, 330 (1984); W. Ishizuka and Y. Kikuchi, Phys. Lett. **139B**, 35 (1984); G. Gilbert, B. McClain, and M. A. Rubin, *ibid.* **142B**, 28 (1984); A. Chodos and E. Myers, Phys. Rev. D **31**, 3064 (1985).

[6]P. Candelas and S. Weinberg, Nucl. Phys. **B237**, 397 (1984).

[7]S. Coleman, in *Laws of Hadronic Matter,* proceedings of the 1973 International School of Subnuclear Physics, Erice, Italy, edited by A. Zichichi (Academic, New York, 1975).

[8]S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977); J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).

[9]R. K. Unz, Report No. SLAC-PUB-3656 (unpublished).

[10]G. 't Hooft, Diagrammer, 1973 Louvain lectures.

[11]J.-L. Gervais and A. Jevicki, Nucl. Phys. **B110**, 93 (1976).

[12]K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); **44**, 1733 (1980); Phys. Rev. D **23**, 2262 (1981).

[13]K. Fujikawa, Nucl. Phys. **B226**, 437 (1983); **B245**, 436 (1984).

[14]B. S. DeWitt, in *Recent Developments in Gravitation, Cargèse, 1978,* edited by M. Lévy and S. Deser (Plenum, New York, 1979); in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, New York, 1979).

[15]E. S. Fradkin and G. A. Vilkovisky, Phys. Rev. D **8**, 4241 (1973); G. 't Hooft, in *Recent Developments in Gravitation, Cargèse, 1978* (Ref. 14).

[16]O. Yasuda, Phys. Lett. **137B**, 52 (1984).

[17]S. W. Hawking (unpublished).