Expansion isotropization during the inflationary era

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A solution of Einstein's field equations for a vacuum with flat, anisotropic three-space and a nonvanishing cosmological constant is derived. It represents a generalization of the flat de Sitter solution to the anisotropic case. This solution describes a universe model entering the Guth inflationary era at $t_1 = 10^{-35}$ sec as an anisotropic Kasner universe, and terminating this era at $t_2 = 1.3 \times 10^{-33}$ sec as an isotropic de Sitter universe. The mean expansion anisotropy has decreased by a factor of order 10^{-168} during the inflationary era. The properties of this solution imply that the transition to an inflationary era is not prevented by any amount of anisotropy. The mean expansion isotropization of the most general homogeneous universe models with anisotropic curved three-space is estimated, and the anisotropy is now found to decrease by a factor of order 10^{-112} during the Guth inflationary era.

I. INTRODUCTION

After the discovery of the cosmic background radiation and measurements of its isotropy, anisotropic cosmological models were investigated in order to define a class of initial conditions that could lead to the observed properties of the Universe. One found that a class of anisotropic universe models does exist, which develops towards isotropic Friedmann universes due to a frictionlike action produced by the radiation and matter contents of the universe models.[1,2] But it was shown by Collins and Hawking[3] that this class represents a space of initial conditions of measure zero in the space of all possible initial conditions. These investigations were, however, restricted to space-times with a vanishing cosmological constant.

More recently, Fabbri[4] investigated the evolution of anisotropy in a low-density universe with a nonvanishing cosmological constant, and found that the observed degree of isotropy is obtained whatever the primordial anisotropy may have been, if the sum of the cosmological constant and the matter density is close to the critical value.

Because of the appearance of the inflationary universe models,[5–8] the possibility has appeared that the Universe became isotropic already during this era. A preliminary investigation of this possibility was undertaken by Barrow and Turner.[9] Their investigation was based on one generalized Friedmann equation for the scale factor, including a term which represents an "anisotropy energy density." They concluded that a universe with a large amount of anisotropy will not undergo the inflationary phase. A universe with only moderate anisotropy will undergo inflation and will be rapidly isotropized. Steigman and Turner[10] and also Demianski[11] later showed that according to the new inflationary models, large anisotropy in the universe does not prevent inflation. Wald[12] recently investigated the asymptotic behavior of initially expanding homogeneous cosmological models with a positive cosmological constant. He showed that such models of all Bianchi types except IX exponentially evolve toward the de Sitter solution, with time scale $(3/\Lambda)^{1/2}$, and that the behavior of type-IX universes is similar, provided that there is not a positive spatial curvature that is too large. Futamase[13] has investigated the effect of space-time anisotropy upon the shape of the potential $V(\phi)$, associated with a Higgs field ϕ. He found that space-time anisotropy has an effect similar to that of temperature on the effective potential $V(\phi)$, and that the effect of anisotropy will not change the essence of the improved inflationary scenario. Futamase did only consider the case when the anisotropic energy density is less than the Planck energy density. The effect of anisotropy on $V(\phi)$ may be important if the transition occurs near the Planck energy. An analysis of this problem is, however, outside the scope of the present work.

In a paper which appeared after this work was completed, Gonzalez and Jones[14] investigated the role of primordial space-time anisotropy in two inflationary scenarios, Planck time (Linde[15,16]) and the grand-unified-theory (GUT) inflation. They find that in the GUT picture inflation can occur notwithstanding the presence of anisotropy. However, the degree of inflation achieved is not sufficient to contain the whole of the actual universe in one coherent GUT time bubble if the initial (pre-GUT) anisotropy were very large. In Planck-time inflation, on the other hand, the actual universe was encompassed within one Planck bubble even for extreme primordial anisotropy. Furthermore, it was found that anisotropy aids inflation in the sense that the most anisotropic models inflate the most and so ultimately have the smallest final anisotropy.

The transition of an anisotropic universe into an inflationary era, and the evolution of anisotropy during this era, will be investigated in the following way. In Sec. II Einstein's field equations will be solved for an anisotropic space-time with flat three-space and a nonvanishing cosmological constant. The solution represents an anisotropic generalization of the de Sitter space-time. This solution is used in Sec. III to investigate if a large amount of anisotropy will prevent the transition to an inflationary era. The behavior of the solution is shown to imply that such a transition will happen at a point of time which depends upon the magnitude of the cosmological constant, and which does not depend upon the amount of anisotropy. The decrease of anisotropy for this solution during the inflationary era is calculated.

In Sec. IV the investigation is extended to space-times with curved three-space. The decrease of the average expansion anisotropy for the most general homogeneous universe models during the inflationary era is estimated in Sec. V. The results are summarized in Sec. VI.

II. ANISOTROPIC GENERALIZATION OF THE DE SITTER UNIVERSE

In Secs. II and III we shall consider anisotropic world models with flat three-space, or Bianchi type-I spaces. The line element has the form

$$ds^2=dt^2-[R_i(t)dx^i]^2, \quad i=1,2,3 . \tag{1}$$

The directional Hubble factors H_i are defined by

$$H_i=\dot{R}_i/R_i . \tag{2}$$

The average scale factor is given by

$$R=(R_1R_2R_3)^{1/3} . \tag{3}$$

A "volume scale factor" will also be useful,

$$V=R^3 . \tag{4}$$

The average Hubble factor is

$$H=\tfrac{1}{3}(H_1+H_2+H_3) . \tag{5}$$

Equations (2)–(5) give

$$H=\tfrac{1}{3}(\ln V)^{\cdot}=(\ln R)^{\cdot} . \tag{6}$$

The field equations for a vacuum with a nonvanishing cosmological constant may be written

$$R^{\mu}{}_{\nu}=\Lambda\delta^{\mu}{}_{\nu} . \tag{7}$$

With the line element (1) the field equations may be given the form[17]

$$(\ln V)^{\cdot\cdot}+H_1{}^2+H_2{}^2+H_3{}^2=\Lambda , \tag{8}$$

$$(VH_1)^{\cdot}=(VH_2)^{\cdot}=(VH_3)^{\cdot}=\Lambda V . \tag{9}$$

Equations (9) give

$$\Delta H_i=H_i-H=K_i/V, \quad \sum_{i=1}^{3} K_i=0 . \tag{10}$$

The constants K_i are proportional to the anisotropy of the Hubble parameters. The average expansion anisotropy is defined by[18]

$$A=\tfrac{1}{3}\sum_{i=1}^{3}(\Delta H_i/H)^2 . \tag{11}$$

Using Eq. (10) we get

$$A=a^2/H^2V^2, \quad a=\tfrac{1}{3}\sum_{i=1}^{3}K_i{}^2 . \tag{12}$$

The constant a^2 is proportional to the average expansion anisotropy. Summarizing Eqs. (9) and using Eq. (6) gives

$$\dot{H}=\Lambda-3H^2 . \tag{13}$$

Substituting Eqs. (10) into Eqs. (8) and using Eq. (13) leads to

$$(\dot{R}/R)^2=H_0{}^2+a^2/2R^6, \quad H_0=(\Lambda/3)^{1/2} . \tag{14}$$

Putting this into Eq. (12) gives

$$A=2(1-H_0{}^2/H^2) . \tag{15}$$

For later comparison two special cases will be noted before the equations are integrated in the general case. If the cosmological constant vanishes, $\Lambda=0$, integration of Eqs. (13) and (14) with $H(0)=\infty$ and $V(0)=0$ give

$$H=1/3t, \quad V=(3\sqrt{2}a/2)t . \tag{16}$$

Substituting this into Eq. (10) and integrating gives

$$R_i=R_{i0}t^{p_i}, \quad p_i=\tfrac{1}{3}(1+\sqrt{2}K_i/a) . \tag{17}$$

From Eqs. (10) and (12) it follows that

$$\sum_{i=1}^{3}p_i=\sum_{i=1}^{3}p_i{}^2=1 . \tag{18}$$

This is the Kasner solution of Einstein's field equation, which describes an empty anisotropic universe with a vanishing cosmological constant. From Eq. (15) it is seen that a Kasner universe has a constant average expansion anisotropy, $A=2$. A plane-symmetric Kasner universe has $p_1=p_2=\frac{2}{3}$, $p_3=-\frac{1}{3}$.

An isotropic universe has $a=0$. In this case Eqs. (14) and (10) give

$$R_1=R_2=R_3=e^{H_0t} . \tag{19}$$

This is the flat de Sitter solution which describes an empty isotropic universe with a nonvanishing cosmological constant.

We now return to the general case. Integration of Eq. (13) gives the mean Hubble factor for this solution,

$$H=H_0\coth(3H_0t) . \tag{20}$$

Integration of Eq. (14) gives[19]

$$V=(a\sqrt{2}/2H_0)\sinh(3H_0t) . \tag{21}$$

Equations (10) may now be integrated. By suitable adjustment of the spatial coordinates, the integration constants R_{i0} may be chosen so that the usual asymptotic form for large t is obtained. The result is

$$R_i=2^{2/3}\sinh^{p_i}(\tfrac{3}{2}H_0t)\cosh^{2/3-p_i}(\tfrac{3}{2}H_0t) , \tag{22}$$

where the constants p_i are given in Eq. (17) and satisfy Eq. (18). This solution represents the anisotropic generalization of the de Sitter universe. It represents a special case of a solution found by Saunders, describing a Bianchi type-I space-time with a nonvanishing cosmological constant, which also contains dust.[20] From Eqs. (15) and (20) it follows that the average expansion anisotropy of the solution (22) is given by

$$A=2/\cosh^2(3H_0t)\ . \tag{23}$$

For small H_0t the general solution takes the form (17). In this limit the general solution has the same form as the Kasner solution. For large H_0t the general solution approaches the isotropic de Sitter solution. Thus the universe described by the solution (22) develops from a Kasner universe with arbitrary anisotropy to the isotropic de Sitter universe.

The plane-symmetric special case of the solution (22) is represented by $p_1=p_2=\frac{2}{3}$, $p_3=-\frac{1}{3}$, which gives[21]

$$\begin{aligned} R_1=R_2&=2^{2/3}\sinh^{2/3}(\tfrac{3}{2}H_0t)\ ,\\ R_3&=2^{2/3}\mathrm{csch}^{1/3}(\tfrac{3}{2}H_0t)\cosh(\tfrac{3}{2}H_0t)\ . \end{aligned} \tag{24}$$

These expressions show that the plane-symmetric de Sitter universe develops from a cigar singularity of Kasner type toward an asymptotically isotropic de Sitter universe.

III. DOES LARGE ANISOTROPY PREVENT TRANSITION INTO AN INFLATIONARY ERA?

Equation (14) has been used by Barrow and Turner[9] to show that large anisotropy prevents transition into an inflationary era according to Guth's original inflationary scenario. In this section I show that their argument is not valid. Although Steigman and Turner,[10] and later Demianski,[11] have shown that according to the new inflationary scenario transition into an inflationary era is not prevented by large anisotropy, the argument of Barrow and Turner has recently been repeated by Hakim,[8] so its nonvalidity should be pointed out.

The characteristic feature of the inflationary era is that space-time expands exponentially owing to the repulsive gravitation of a dominating vacuum energy. This vacuum energy is due to Higgs fields, which produce a large cosmological constant. Before the inflationary era there is a radiation-dominated period. While the radiation density ρ diminishes during the expansion, the vacuum energy density ρ_0 remains constant. In the isotropic universe models $\rho=\rho_0$ at a point of time $t_1=10^{-35}$ sec, and the universe enters the vacuum-dominated inflationary era.

We now consider the situation that the universe is an-

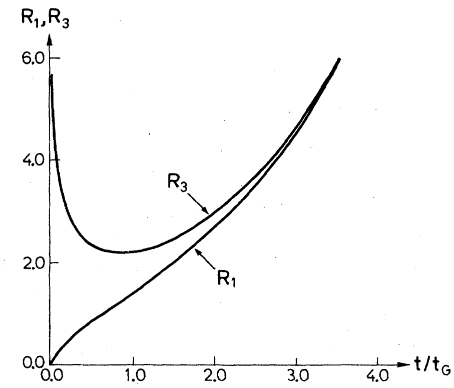

FIG. 2. Expansion factors for a plane-symmetric generalized de Sitter universe.

isotropy dominated at the point of time when the radiation energy density becomes less than the vacuum energy density. Then Eq. (14) is valid with $a^2/2R^6 \gg H_0^2$. The argument of Barrow and Turner is that the universe shows the Kasner anisotropic behavior $R\propto t^{1/3}$ when the anisotropy dominates (when $a^2 \gg 2H_0^2R^2$), which prevents transition into the de Sitter phase. According to this argument the inflationary era will not happen if the constant a^2 is too large.

What invalidates this argument is that according to the solution (21) [or Eq. (4) of Barrow and Turner[9]] the mean volume expansion parameter, $V=R^3$, is proportional to the anisotropy parameter a. Inserting the solution (21) into Eq. (14) gives

$$(\dot R/R)^2=H_0^2+H_0^2\mathrm{csch}^2(3H_0t)\ . \tag{25}$$

Thus the domination of the anisotropy term does not depend upon the magnitude of a. In other words the magnitude of the anisotropy parameter a does not affect the time scale of the transition to the isotropic de Sitter solution. The time scale is determined solely by the value of the vacuum energy density, which determines the constant H_0. This value is a result coming out of the GU theories. Given this value of H_0, a GUT time t_G is defined by $t_G=(2H_0)^{-1}$. From the GU theories it follows

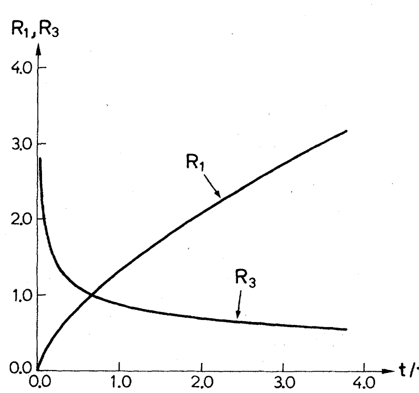

FIG. 1. Expansion factors for a plane-symmetric Kasner universe.

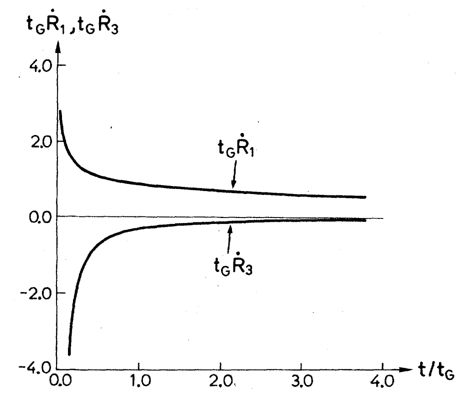

FIG. 3. Expansion (and contraction) velocities for a plane-symmetric Kasner universe.

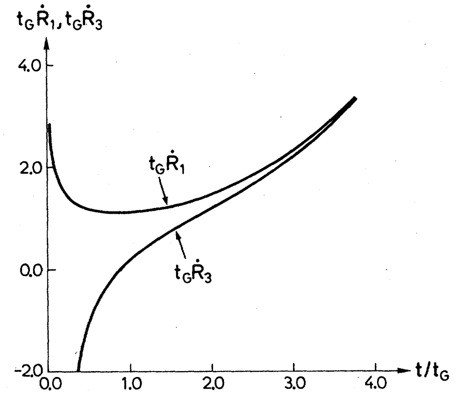

FIG. 4. Expansion (and contraction) velocities for a plane-symmetric generalized de Sitter universe.

that $t_G = 10^{-35}$ sec. In the isotropic case the point of time t_1 for transition into the inflationary era equals t_G. An anisotropy-dominated universe, described by Eq. (22), will show a transition from a Kasner era to an isotropically, exponentially expanding era at the same point of time. Thus large anisotropy will not prevent transition into an inflationary era.

It should be pointed out that the feature of Eq. (25) being independent of the anisotropy parameter may be a consequence of the limitation to Bianchi type-I cosmological models in Secs. II and III. The most general anisotropic cosmological models will be investigated in Sec. V.

As an illustration the development of a plane-symmetric Kasner universe and a plane-symmetric generalized de Sitter universe are shown in Figs. 1—4. The Kasner universe develops from a cigar singularity, through an isotropic moment to a pancake singularity.

The plane-symmetric de Sitter universe develops from a cigar singularity of Kasner type toward an asymptotically isotropic de Sitter universe. Initially the expansion law for the geometric mean expansion factor is Kasner type, $R \propto t^{1/3}$, but at about the GUT time there is a transition to exponential expansion.

The general anisotropic solution (22) describes the first moments of the inflationary era, when the radiation density has ceased to be important for the evolution of space-time, but the expansion is still anisotropic.

According to Guth's model the inflationary era lasts from $t_G = 1.0\times10^{-35}$ sec to $t_2 = 1.3\times10^{-33}$ sec, which increases the flatness of the Universe by a factor of order 10^{56}. During this time the average expansion anisotropy, as given in Eq. (23) (see Fig. 5), decreases by

$$A(t_2)/A(t_G) = e^{-3t_2/t_G} \approx 10^{-168} . \tag{26}$$

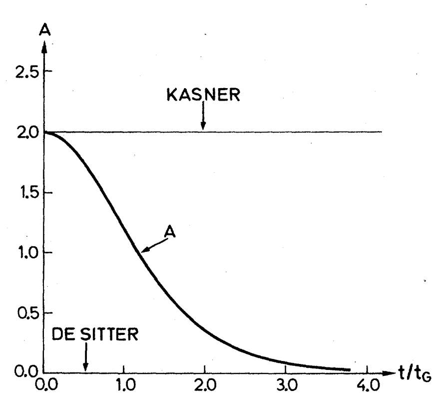

FIG. 5. The average expansion anisotropy for plane-symmetric Kasner and de Sitter universes.

IV. ISOTROPIZATION OF UNIVERSE MODELS WITH SPATIAL CURVATURE

The Bianchi type-I universes have a vanishing spatial curvature. Now, one of the main results of the inflationary scenario is that it shows how a spatially curved universe develops towards a flat universe during the inflationary era. The Universe is expected to have a curved spatial geometry while entering the vacuum-dominated inflationary era.

To see how curvature affects the dynamical evolution of a vacuum-dominated universe, the de Sitter solution with hyperbolic three-space will first be briefly considered. It is given by[22]

$$R = H_0{}^{-1}\sinh(H_0 t) , \tag{27}$$

which represents Milne's kinematical universe, as transformed to a general relativistic universe model of the hyperbolic Friedmann type. When $t \to \infty$ the solution (27) approaches the form (19). Thus an isotropic universe with hyperbolic three-space develops from a Milne universe with curved three-space toward a flat de Sitter universe. The spatial curvature, $K \propto R^{-2}$, decreases exponentially with time for $H_0 t > 1$.

The most general homogeneous universe models are the Bianchi spaces of types VII, VIII, and IX. The evolution of anisotropy in a restricted class of such models with a nonvanishing cosmological constant was investigated by Fabbri.[4] The evolution of the directional expansion anisotropy is given by

$$\Delta\dot{H}_i = -3H\Delta H_i + \mu_i R^* , \tag{28}$$

where the coefficients μ_i are of order unity for the most general models, and R^* is the curvature of three-space. For a plane-symmetric space, $R_1 = R_2$, of Bianchi type-VII$_h$ the spatial curvature is[23]

$$R^* = -6k^2/R_3{}^2 , \tag{29}$$

where k is a constant. In this case integration of Eq. (28) gives

$$\Delta H_i = V^{-1}\left[K_i - 6\mu_i k^2 \int_0^t (V/R_3{}^2)dt\right], \quad \sum_{i=1}^{3} \mu_i = 0 , \tag{30}$$

which generalizes Eq. (10).

The curvature-driven part of the directional anisotropy is

$$(\Delta H_i/H)_c = -(18k^2\mu_i/\dot{V})\int_0^t (V/R_3{}^2)dt \ . \quad (31)$$

A curvature-driven mean expansion anisotropy is defined by

$$A_c = \tfrac{1}{3}\sum_{i=1}^{3} (\Delta H_i/H)_c{}^2 \ . \quad (32)$$

In order to get an estimate of the decrease of this expansion anisotropy during the inflationary era, we make use of the flat-space solution, given in Eqs. (21) and (22). This leads to

$$A_c(t_2)/A_c(t_G) \approx e^{-2t_2/t_G} \approx 10^{-112} \ . \quad (33)$$

The decrease of anisotropy is not so great in a universe with curved three-space as in a flat universe, but the anisotropy of the expansion velocity still decreases very considerably during the inflationary era.

V. "MIXMASTER DESCRIPTION" OF EXPANSION ISOTROPIZATION

In order to make connection with a wider part of the literature that has treated anisotropic cosmological models, I will give a short discussion of the problem using the form of the line element which has become usual in connection with the so-called mixmaster universe models.[24]

Bianchi spaces of arbitrary type may be described by a line element of the form

$$ds^2 = dt^2 - e^{2\alpha}(e^{2\beta})_{ij}\Omega^i\Omega^j \ , \quad (34)$$

where Ω^i are one-forms that are not exact in general, and β is a symmetric traceless matrix. The angle averaged Hubble parameter is $H = \dot{\alpha}$. The main contribution to the anisotropy of the expansion rate is described by the tensor

$$\sigma_{ij} = (e^{\beta})^{\cdot}_{ki}(e^{-\beta})_{kj} + (e^{\beta})^{\cdot}_{kj}(e^{-\beta})_{ki} \ . \quad (35)$$

The shear energy density is defined by

$$\rho_\sigma = \tfrac{1}{2}\sigma_{ij}\sigma^{ij} \ . \quad (36)$$

This energy density is governed by the equation[25]

$$(\rho_\sigma{}^{1/2})^{\cdot} = -3\dot{\alpha}\rho_\sigma{}^{1/2} + \psi R^* \ , \quad (37)$$

where ψ is a factor of order unity. Integration gives

$$\rho_\sigma{}^{1/2} = C e^{-3\alpha} + e^{-3\alpha}\int_0^t \psi e^{3\alpha} R^* dt \ , \quad (38)$$

which is a generalization of Eq. (30).

In an era dominated by the vacuum energy density it is a good approximation to write $R^* \propto e^{-2\alpha}$. Then[23]

$$e^{\alpha} \propto \sinh^{2/3}(\tfrac{3}{2}H_0 t) \ .$$

Thus for $t > H_0{}^{-1}$ the volume expansion is exponential, $\exp(\alpha) \propto \exp(H_0 t)$. From Eq. (31) we may define a flat-space anisotropy energy density ρ_F and a curvature-driven anisotropy energy density ρ_c by

$$\rho_F = C^2 e^{-6\alpha} \ ,$$
$$\rho_c = e^{-6\alpha}\left[\int_0^t \psi e^{3\alpha} R^* dt\right]^2 \quad (39)$$

so that

$$\rho_\sigma = \rho_F + \rho_c + 2(\rho_F\rho_c)^{1/2} \ .$$

During the exponentially expanding era we get

$$\rho_F \propto e^{-3t/t_G}, \ \ \rho_c \propto e^{-2t/t_G} \ . \quad (40)$$

Thus ρ_F decreases by 10^{-168} and ρ_c by 19^{-112} during the inflationary era.

It was shown by Salucci and Fabbri[26] that the evolution of expansion anisotropy in a universe filled with a perfect fluid endowed with vorticity is not affected significantly by the matter velocity field. Thus the results deduced above are valid for the most general anisotropic, homogeneous universe models.

VI. CONCLUSION

There is an infinity of possible initial conditions for the expansion pattern of our Universe. Only one of these represents an isotropic expansion. Thus there is *a priori* a vanishing probability that the Universe started with an isotropic initial expansion. As stated by Misner, Thorne, and Wheeler,[24] the fundamental cosmological question then, is why the isotropic Friedmann metrics should be a more accurate approximation to the real Universe than an anisotropic Kasner metric.

The present article gives an answer to this question. An anisotropic generalization of the de Sitter universe has been discussed, which includes infinitely many initial expansion conditions. According to this model the Universe is initially in a Kasner era. It enters the vacuum-dominated inflationary era in an arbitrarily anisotropic phase. During this era the average expansion anisotropy decreases by 10^{-168}, and the Universe develops into a de Sitter phase. The Universe comes out of the inflationary era with isotropic expansion, and the phase transition at the end of this era leads the Universe into an isotropic, radiation-dominated Friedmann era.

The kinematical properties of the generalized de Sitter solution shows that an arbitrarily great anisotropy at the GUT time will not prevent the transition into an inflationary era. The point of time for this transition and the time scale with which it happens are independent of the state of anisotropy of the universe, and depends only upon the magnitude of the vacuum energy density.

The anisotropic generalization of the de Sitter universe investigated in Sec. II is still peculiar in that it has a flat three-space. When the curvature of three-space is included into the analysis, the equations contain a curvature-driven component of the expansion anisotropy. Even if the decrease of anisotropy during the inflationary era is less, when this is taken into account, than in the case of a flat three-space, the decrease of anisotropy is still extremely great. For the most general homogeneous cosmological models the average expansion anisotropy decreases by a factor of order 10^{-112} during the Guth inflationary era.

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