Path-integral quantum cosmology. II. Bianchi type I with volume-dependent source

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The Feynman path-integral formalism is applied to the degrees of freedom of the fluid-filled Kasner (and related) cosmological models. The measure constructed for the vacuum analysis is used to effect the Arnowitt-Deser-Misner reduction to dynamical degrees of freedom within the functional integral. The Hankel transform of the minisuperspace transition amplitude is obtained. A perturbation expansion about the classical vacuum solution is presented.

I. INTRODUCTION

In the preceding paper[1] (paper I), we discussed the motivation to study the role of the Hamiltonian constraint in the quantization of gravity via the quantum cosmology approximation. In paper I, the Feynman path-integral (FPI) formalism was applied to the degrees of freedom of the vacuum Bianchi type-I (Kasner) cosmological model.[2] The reduction scheme of Arnowitt, Deser, and Misner[3] (ADM) was performed at the level of the functional integral. A closed-form expression for the minisuperspace transition amplitude was obtained. Here we report the generalization of these results to the fluid-filled Kasner (and related) models. The fluid pressure is assumed to be isotropic. The measure constructed in paper I to factor out the phase space of the nondynamical degree of freedom remains valid and is used here. The major complication caused by the introduction of a volume-dependent source is the nonpolynomial character of the ADM Hamiltonian. A procedure developed in paper I is used, however, to reduce the functional integral to a single ordinary integral which is just a Hankel transform.

In Sec. II the classical problem is described with regard to subsequent calculations. The ADM procedure is used to obtain an exact classical solution for the dynamical degrees of freedom. As is well known,[2] the fluid-filled Kasner models are anisotropy dominated near the singularity and Friedmann-Robertson-Walker fluid dominated far from it.

In Sec. III the path-integral quantization and ADM reduction are described. In Sec. IV the original formulation of the FPI with the measure of Sec. III is evaluated by first performing the Gaussian integrations over all momenta. The closed-form expression is obtained and evaluated in the limits of negligible and dominant sources. Finally, in the Appendix, an expansion of the Lagrangian transition amplitude about the classical vacuum solution is constructed. It is shown to agree with the appropriate limit of the closed-form amplitude. Conclusions are presented in Sec. V.

II. THE CLASSICAL SOLUTION

The metric is again (paper I) the Bianchi type-I cosmology

$$ds^2 = -N^2dt^2 + e^{2\Omega}(e^{2\beta})_{ij}dx^i dx^j \,, \tag{2.1}$$

where N, Ω, and β_{ij} are functions of t only, the spatial volume is $g^{1/2}V$, and

$$\beta_{ij} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \,. \tag{2.2}$$

The classical Einstein equations may be obtained from the action

$$S = \int_{[i]}^{[f]} dt \left[\mathbf{p}\cdot\frac{d\boldsymbol{\beta}}{dt} + p_\Omega \frac{d\Omega}{dt} - \alpha\mathscr{H} \right] , \tag{2.3}$$

where $\mathbf{p}$ and p_Ω are, respectively, conjugate to $\boldsymbol{\beta} \equiv (\beta_+, \beta_-)$ and Ω (for $p \equiv |\mathbf{p}|$),

$$2\mathscr{H} = p^2 - p_\Omega{}^2 + F(\Omega) \,, \tag{2.4}$$

and $[i]$ and $[f]$ denote, respectively, the initial and final minisuperspace configurations. The lapse N has been rescaled to absorb inconvenient factors so that $\alpha \equiv 4N\pi/3g^{1/2}V$. These models contain a volume-dependent source term $F(\Omega)$ which takes on the values $\mu e^{3\Omega}$, $\Gamma e^{2\Omega}$, and $\Lambda e^{6\Omega}$ for dust, radiation, and the cosmological constant, respectively. Isotropic scalar curvature would have the form $-ke^{4\Omega}$ for $k>0$ ($k<0$) spatially closed (open). Of course, $F(\Omega)$ may be any combination of these special cases.

To compare most easily with the quantum formalism, we shall obtain the classical solution by means of the ADM reduction. We follow paper I, eliminate t dependence to express the dynamics in terms of Ω, and solve Eq. (2.4) for $-p_\Omega = H_{\mathrm{ADM}}(\mathbf{p}, \boldsymbol{\beta}, F)$. Thus we find

$$H^{\pm}_{\mathrm{ADM}} = \pm[p^2 + F(\Omega)]^{1/2} \,. \tag{2.5}$$

The significance of the $\pm$ is identical to that in paper I so that we shall restrict discussion to the positive-"energy" case with $\Omega_f > \Omega_i$. The negative-energy case may be put in by hand at any time. Equation (2.5) represents a significant complication over the vacuum model because the extraction of the square root cannot be explicitly performed and H_{ADM} is no longer conserved with respect to Ω time. The solution to Hamilton's equations from (2.5) can be found, however, since $\mathbf{p}$ is still a constant of the motion. The general solution is

$$\boldsymbol{\beta}=\boldsymbol{\beta}_0+\int_{\Omega_0}^{\Omega}\frac{\mathbf{p}\,d\Omega'}{[p^2+F(\Omega')]^{1/2}}\ . \tag{2.6}$$

If $F(\Omega)=ae^{b\Omega}$, the integrations may be performed exactly, but the result is not particularly illuminating. Equation (2.6) reduces to the Kasner solution as $\Omega\to-\infty$. The anisotropy $\boldsymbol{\beta}$ decays exponentially as $\Omega\to\infty$. If $a<0$ (i.e., the closed model), the range of Ω is finite since $\boldsymbol{\beta}$ must be real. The relationship $\boldsymbol{\beta}(\Omega)$ is single valued (given the phase of the square root). It is only when $\boldsymbol{\beta}(t)$ and $\Omega(t)$ are required (e.g., for t the comoving proper time) that a single parametrization $\Omega(t)$ cannot be used.

III. PATH-INTEGRAL QUANTIZATION

As discussed in paper I, the transition amplitude between points $(\boldsymbol{\beta}^i,\Omega^i)$ and $(\boldsymbol{\beta}^f,\Omega^f)$ in minisuperspace is given by

$$\langle[f]\,|\,[i]\rangle\equiv\langle\boldsymbol{\beta}^f,\Omega^f\,|\,\boldsymbol{\beta}^i,\Omega^i\rangle=\int\mathscr{D}(\mathbf{p},p_\Omega,\boldsymbol{\beta},\Omega,\alpha)e^{iS}\ , \tag{3.1}$$

where S is the action (2.3). The integration is over all paths between the initial and final states. The ADM reduction can be performed in the path integral if the measure is given by (see paper I)

$$\mathscr{D}(\mathbf{p},p_\Omega,\boldsymbol{\beta},\Omega,\alpha)=\lim_{n\to\infty}(2\pi)^{-3n}\prod_{k=1}^{n}d^2p^k\,dp_\Omega^k\,d^2\beta^k\,d\Omega^k\,d\alpha'^k\,|p_\Omega^k\,|\,\delta(\Omega^k-\Omega^i-k\Delta)\,\delta(\Omega^n-\Omega^f)\,\delta(\boldsymbol{\beta}^n-\boldsymbol{\beta}^f)\ , \tag{3.2}$$

where $\Delta\equiv(\Omega^f-\Omega^i)/n$ and $\alpha'\equiv\alpha\delta t$. In this broken-path approximation the action (2.3) takes the form[4]

$$S=\sum_{k=1}^{n}\left[\mathbf{p}^k\cdot(\boldsymbol{\beta}^k-\boldsymbol{\beta}^{k-1})+p_\Omega^k(\Omega^k-\Omega^{k-1})\right.$$
$$\left.-\frac{\alpha'^k}{2}[(p^k)^2-(p_\Omega^k)^2+F(\Omega^{k-1})]\right]\ . \tag{3.3}$$

In Eqs. (3.2) and (3.3), k refers to the value at the kth time step (see discussion in paper I) so that $k=0$ ($k=n$) is the initial (final) value. As shown in paper I,[5] the imposition of Feynman boundary conditions on the path integral (3.1) requires $\alpha'>0$ to be the range of integration. The factors in the measure (3.2) may be understood as follows.

(1) Inspection of Eqs. (3.2) and (3.3) shows that for each k the integration over α' is the Fourier transform of a δ function of the constraint. (This is actually the principal part. Imposition of Feynman boundary conditions modifies the value, but not the interpretation. This is discussed in paper I.)

(2) The factors $\delta(\Omega^k-\Omega^i-k\Delta)$ define a canonical gauge causing the Ω degree of freedom to be identified as nondynamical. If

$$Q=\mathscr{G}^k\equiv\Omega^k-\Omega^i-k\Delta$$

and $P=\mathscr{H}^k$ then

$$1=\int dQ\,dP\,\delta(Q)\delta(P)=\int d\Omega\,dp_\Omega\,J(Q,P,\Omega,p_\Omega)$$
$$\times\delta[Q(\Omega,p_\Omega)]\,\delta[P(\Omega,p_\Omega)] \tag{3.4}$$

at each k. The Jacobian J of the canonical transformation is just $|\{Q,P\}_{\Omega,p_\Omega}|$ where $\{\ ,\ \}$ denotes the usual Poisson bracket.

(3) The term $|p_\Omega|$ is just the Poisson bracket between the constraint $\mathscr{H}$ and the gauge condition $\mathscr{G}$ and is just the Jacobian of the requisite canonical transformation. The choice of canonical gauge $\mathscr{G}=0$ ensures that the source term $F(\Omega)$ does not contribute to the Jacobian. This Jacobian becomes the Fadeev-Popov determinant in the full field theory.[4]

(4) The remaining δ functions must be included to have the same number of q and p integrations for each minisuperspace variable.

It was shown in paper I that the measure (3.2) allows the nondynamical part of minisuper phase space to be factored out so that the transition amplitude (3.1) becomes

$$\langle[f]\,|\,[i]\rangle=\int\mathscr{D}(\mathbf{p},\boldsymbol{\beta})e^{iS_{\rm ADM}} \tag{3.5}$$

for

$$\mathscr{D}(\mathbf{p},\boldsymbol{\beta})=\lim_{n\to\infty}(2\pi)^{-2n}\prod_{k=1}^{n}d^2p_k\prod_{k=1}^{n-1}d^2\beta_k \tag{3.6}$$

and

$$S_{\rm ADM}=\int_{\Omega_1}^{\Omega_f}d\Omega\left[\mathbf{p}\cdot\frac{d\boldsymbol{\beta}}{d\Omega}-H_{\rm ADM}(\mathbf{p},\boldsymbol{\beta},\Omega)\right] \tag{3.7}$$

with $H_{\rm ADM}$ given by Eq. (2.5). At this stage, however, one is confronted with the nonpolynomial character of the ADM Hamiltonian.[6] In paper I, it was shown that the transition amplitude could be constructed directly from Eqs. (3.1)—(3.3) by first performing all the integrations over momenta and only then integrating over α'. This allows postponement of the taking of the square root while still keeping track of the constrained dynamics via the measure (3.2). In Sec. IV, we shall show that this procedure yields the (Hankel transform of the) transition amplitude in closed form.

IV. EVALUATION OF THE TRANSITION AMPLITUDE

The Gaussian integrations over momenta are identical to those of paper I (Sec. V). The transition amplitude becomes

$$\langle[f]\,|\,[i]\rangle=\int\mathscr{D}(\boldsymbol{\beta},\Omega,\alpha')e^{iS_L} \tag{4.1}$$

for

$$\mathscr{D}(\boldsymbol{\beta},\Omega,\alpha')=\lim_{n\to\infty}(i/2\pi)^{3n/2}\prod_k d\Omega^k d^2\beta^k d\alpha'^k(\alpha'^k)^{-5/2}\Delta \times\delta(\Omega^k-\Omega^i-k\Delta) \quad (4.2)$$

and

$$S_L=\sum_k\{[(\delta\beta^k)^2-\Delta^2]/2\alpha'^k-\alpha'^k F(\Omega^{k-1})/2\}\ , \quad (4.3)$$

where S_L is the (infinitesimal) Lagrangian form of the minisuperspace action and

$$(\delta\beta^k)^2\equiv|\boldsymbol{\beta}^k-\boldsymbol{\beta}^{k-1}|^2 .$$

Integration over α'^k (with the restriction $0\le\alpha'\le\infty$ imposed by Feynman boundary conditions) yields

$$\langle[f]\,|\,[i]\rangle=\lim_{n\to\infty}(i/2\pi)^{3n/2}2^n\int\prod_k d^2\beta^k\Delta\{F(\Omega^{k-1})/[\Delta^2-(\delta\beta^k)^2]\}^{3/4}K_{3/2}(\{F(\Omega^{k-1})[(\delta\beta^k)^2-\Delta^2]\}^{1/2})\ , \quad (4.4)$$

where $K_\nu(x)$ is a modified Bessel function of the second kind.

Here we may note that [especially since $K_{3/2}(x)$ may be expressed in closed form in terms of elementary functions] Eq. (4.4) lends itself to expansion about the infinitesimal vacuum solution characterized (see paper I) by $\epsilon\equiv(\delta\beta^2-\Delta^2)^{1/2}=0$. The results of this expansion are given in the Appendix. In this model, however, it is possible to proceed further without approximation. To perform the integrations over $\boldsymbol{\beta}^k$ define a new variable $\boldsymbol{\chi}^k\equiv\boldsymbol{\beta}^k-\boldsymbol{\beta}^{k-1}$ and include in the measure a factor of unity of the form $\int d^2\beta\,\delta(\boldsymbol{\beta}^n-\boldsymbol{\beta}^f)$ so that

$$\int\prod_{k=1}^{n-1}d^2\beta^k=\frac{1}{4\pi^2}\int d^2\lambda\prod_{k=1}^{n}d^2\chi^k\exp\left[i\boldsymbol{\lambda}\cdot\left[\boldsymbol{\beta}^f-\boldsymbol{\beta}^i-\sum_{k=1}^{n}\boldsymbol{\chi}^k\right]\right] . \quad (4.5)$$

The kth integral in Eq. (4.4) takes the form (suppressing k)

$$l_\chi\equiv4\pi[iF(\Omega)]^{3/4}\int_0^\infty\chi\,d\chi\,J_0(\lambda\chi)[\chi^2-\Delta^2]^{-3/4} \times K_{3/2}([F(\Omega)(\chi^2-\Delta^2)]^{1/2}) . \quad (4.6)$$

Evaluation of this integral yields

$$l_\chi=(2\pi)^{3/2}i^{1/2}\Delta^{-1}\exp\{-i\Delta[\lambda^2+F(\Omega)]^{1/2}\} . \quad (4.7)$$

The product over k causes the argument of the exponential to become an integral over Ω. Combining everything and performing the angular part of the λ integration yields (for $\Delta>0$ required by the positive-frequency choice for $H_{\rm ADM}$)

$$\langle[f]\,|\,[i]\rangle=\frac{1}{2\pi}\int_0^\infty\lambda\,d\lambda\,J_0(\lambda|\boldsymbol{\beta}^f-\boldsymbol{\beta}^i|) \times\exp\left[-i\int_{\Omega_i}^{\Omega_f}d\Omega[\lambda^2+F(\Omega)]^{1/2}\right] . \quad (4.8)$$

[If $F(\Omega)=ae^{b\Omega}$, the integration in Eq. (4.8) may be performed.] Equation (4.8) is recognized to be a Hankel transform.

It is clear (see paper I) that the vacuum transition amplitude results for $F(\Omega)=0$. In fact, Eq. (4.8) may be rewritten as a perturbation expansion in $F(\Omega)$. We find

$$\langle[f]\,|\,[i]\rangle=\langle[f]\,|\,[i]\rangle_{F=0}+\frac{\int_{\Omega_i}^{\Omega_f}F(\Omega)d\Omega}{(4\pi)[\,|\boldsymbol{\beta}^f-\boldsymbol{\beta}^i|^2-(\Omega^f-\Omega^i)^2]^{1/2}}\ , \quad (4.9)$$

where

$$\langle[f]\,|\,[i]\rangle_{F=0}=\frac{i(\Omega^f-\Omega^i)}{(2\pi)[\,|\boldsymbol{\beta}^f-\boldsymbol{\beta}^i|^2-(\Omega^f-\Omega^i)^2]^{3/2}} \quad (4.10)$$

is the vacuum transition amplitude obtained in paper I.

Equation (4.9) represents the expansion of Eq. (4.8) to linear order in F/λ. To continue the expansion yields divergent integrals. We note that the perfect-fluid form of $F(\Omega)$ decays away exponentially near the classical singularity $\Omega=-\infty$ so that in this regime $F(\Omega)\ll\lambda^2$ is meaningful even as $\lambda\to0$. It is also possible to explore the limit $\Omega\to\infty$ where for a perfect fluid $F(\Omega)\gg\lambda^2$. The dominant term in Eq. (4.8) for $|\boldsymbol{\beta}^f-\boldsymbol{\beta}^i|\to0$ is

$$\langle[f]\,|\,[i]\rangle=\mathscr{N}\exp\left[-i\int_{\Omega_i}^{\Omega_f}d\Omega\,F^{1/2}(\Omega)\right]\ , \quad (4.11)$$

where $\mathscr{N}$ is an (infinite) normalization factor from the λ integral. Equation (4.11) is the expected transition amplitude for the isotropic model.

V. CONCLUSIONS

The formalism of path-integral ADM reduction applied in paper I to the vacuum Bianchi type-I cosmology has been applied here to a type-I model containing a volume-dependent source term. This source term may, for example, be interpreted to be a perfect fluid, cosmological constant, isotropic spatial curvature, or a combination thereof. Even though the ADM Hamiltonian is nonpolynomial, the calculation may be carried sufficiently far to express the amplitude for transition from an initial to a final point in minisuperspace as the Hankel transform of a known function. The quantum behavior again reflects the classical behavior as has been determined by an analysis of the transition amplitude in the small- and large-volume limits. The model behaves as the vacuum solution near the classical singularity and as an isotropic universe far from it. Construction of physically interesting quantities from the transition amplitude will be given elsewhere.[7] Finally, it was shown that the closed-form result evaluated near the singularity can be obtained through a perturbation expansion of the infinitesimal (Lagrangian) transition amplitude about the classical vacuum solution. It suggests that the perturbation expansion may be useful in more complicated models where a closed-form transition amplitude cannot be obtained.

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APPENDIX: EVALUATION OF THE TRANSITION AMPLITUDE AS A PERTURBATION ABOUT THE CLASSICAL VACUUM SOLUTION

After expression (4.4), we define a parameter ϵ which we regard to be small for all Ω values. It is then possible to expand about the classical vacuum solution $\epsilon=0$. The decaying exponential in the modified Bessel function will kill off terms with powers of ϵ greater than zero. Thus, a consistent approximation requires that only terms with negative powers of ϵ be considered. In this approximation (4.4) becomes

$$\langle [f] \mid [i] \rangle = \lim_{n\to\infty} (i/2\pi)^{3n/2} (i\Delta)^n \times \int \prod_k d^2\beta^k (\epsilon^k)^{-3} \left[1-\frac{F(\Omega^{k-1})}{2}(\epsilon^k)^2\right] . \tag{A1}$$

A transformation to the variables χ^k is performed as described in Sec. IV. Performance of the χ integrations yields

$$\langle [f] \mid [i] \rangle = \lim_{n\to\infty} (2\pi)^{-n/2} (2\Delta)^n \frac{(i)^{3n/2}}{2\pi} \times \int d^2\lambda\, e^{i\boldsymbol{\lambda}\cdot(\boldsymbol{\beta}_f-\boldsymbol{\beta}_i)} \times \prod_k \{[1/i\Delta - F(\Omega^k)/2\lambda] e^{-i\lambda\Delta}\} . \tag{A2}$$

The approximation requires only that terms linear in $F(\Omega)$ be kept. Thus (A2) becomes

$$\langle [f] \mid [i] \rangle = \lim_{n\to\infty} \frac{i}{2\pi} \times \int_0^{2\pi} \lambda\, d\lambda\, J_0(\lambda\,|\boldsymbol{\beta}^f-\boldsymbol{\beta}^i|)\, e^{-i\lambda(\Omega_f-\Omega_i)} \times \left[1-(i/2\lambda)\int_{\Omega_i}^{\Omega_f} F(\Omega)d\Omega\right] . \tag{A3}$$

Integration yields Eq. (4.8) showing that the perturbation about the vacuum solution may be performed at an earlier stage in the calculation than was done in Sec. IV.

[1]B. K. Berger and C. N. Vogeli, preceding paper, Phys. Rev. D **32**, 2477 (1985); hereafter referred to as paper I.

[2]For a discussion of this and related models see M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University, Princeton, New Jersey, 1975).

[3]R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **113**, 745 (1959); **116**, 1322 (1959); **117**, 1595 (1960).

[4]See, for example, C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[5]See also C. Teitelboim, Phys. Lett. **96B**, 77 (1980); Phys. Rev. D **25**, 3159 (1982).

[6]See, for example, K. Kuchar, in *Quantum Gravity 2: A Second Oxford Symposium,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981).

[7]B. K. Berger (unpublished).