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Path-integral quantum cosmology. I. Vacuum Bianchi type I

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The Feynman path-integral formalism is applied to study the quantum mechanics of the vacuum Bianchi type-I cosmological model. A measure is obtained for the functional integral representing the minisuperspace transition amplitude which allows the performance of the Arnowitt-Deser-Misner (ADM) reduction within the path integral. This requires a careful analysis of the roles played by the original coordinate time and the intrinsic minisuperspace time. The role of the imposed Feynman boundary conditions is discussed. The integrals are all evaluated to obtain the transition amplitude in closed form. An alternative scheme for evaluation of the transition amplitude is presented to avoid problems related to the nonpolynomial nature of the ADM Hamiltonian.

I. INTRODUCTION

Grand unified theories (GUT's) of the strong, weak, and electromagnetic interactions may be able to account for the observed baryon asymmetry of the Universe.[1] The simplest GUT model [minimal SU(5)] conflicts with searches for proton decay, however.[2] Straightforward analysis of the GUT phase transition within the standard Friedmann-Roberston-Walker cosmological models yields too many magnetic monopoles.[3] The failure of the most naive models suggests that details of both the unification and cosmological models may be important. In attempts to build more successful models and cosmological scenarios, one may argue that the closeness of the GUT scale ($M_{\rm GUT}\sim 10^{14-19}$ GeV/c^2) to the Planck scale ($M_P\sim 10^{19}$ GeV/c^2) should not be ignored. Even the "new inflationary model" which solves several long-standing cosmological problems in addition to the monopole problem contains several *ad hoc* features which might be affected if gravity were included.[4] The relationship between the classical general theory of relativity (GTR) and a theory of quantum gravity (QG) is unclear.[5] The nonrenormalizability of quantization of weak-field gravity coupled to matter suggests (as for the Fermi model of weak interactions) that GTR is a low-energy effective theory of a quite different QG.[6] Several proposals have been advanced for the nature of QG. These include superstrings[7] and supergravity,[8] higher-derivative Lagrangians,[9] generalized geometry,[10] and generalized topology[11] theories.

It may be, however, that QG requires a generalization of the notion of quantization[12] and/or renormalizability[13] applied to an almost[14] standard Einstein-Hilbert Lagrangian. At another level, this (modified) GTR Lagrangian has been analyzed both classically[15–17] and quantum mechanically[18,19] as a constrained dynamical system. Since the constraints are in fact the generators of general coordinate transformations[20] (the invariance group of GTR), this analysis can be couched in the language of gauge theories—most usefully in the functional-integral [or Feynman path-integral (FPI)] formalism[21–26] which allows the most extensive use of the known classical action.

It has long been recognized that the greatest obstacle to the usual canonical quantization methods has been the Hamiltonian constraint, the generator of time coordinate transformations. Difficulties arise both due to the arbitrary choice of time in GTR compared to the c-number role of time in quantum mechanics (or equivalently its role in the definition of positive frequency in quantum field theory) and due to the fact that the Hamiltonian constraint is not linear in the canonical momentum conjugate to reasonable time variables leading to nonpolynomial dynamical Hamiltonians after the reduction of constraints.[27]

In this paper, we examine the role of the Hamiltonian constraint in QG by applying the FPI formalism to the dynamical degrees of freedom of a spatially homogeneous cosmological model.[28] Similar quantum cosmology (QC) analyses have been explored to illustrate various approaches to QG.[12,18,19,24] The relationship of QC to QG is unknown since the dynamical degrees of freedom classically absent by symmetry are neglected in QC. At the classical level, it can be shown[29] that generic cosmologies are "velocity dominated" near the classical singularity—i.e., spatial derivatives can be neglected compared to time derivatives.

As a simple but nontrivial model we consider here the vacuum Bianchi type-I model—the well-known Kasner model.[30] This model is described by three minisuperspace[19] variables related to the volume and anisotropic expansion scales. The dynamics is nontrivial (at the quantum level) because the dynamical variables must obey the Hamiltonian constraint equation. In this model, this constraint is just the usual Hubble's equation with the volume expansion rate (Hubble parameter) proportional to the anisotropy energy density. In the Hamiltonian form of the Einstein-Hilbert action, the role of the Hamiltonian (called super-Hamiltonian) is played by the scalar product of the lapse-shift (Lagrange multipliers) and Hamiltonian momentum constraint "four-vectors." In QC, the spatial integrations are performed prior to variation of the action. In the model considered here the momentum constraint vanishes identically and the resultant QC action yields Einstein's equations for the model.[31] The modification of the Einstein-Hilbert action suggested by Gibbons and Hawking[14] has been used to eliminate total-derivative (boundary) terms which might otherwise be expected to play a role in the quantum analysis.

Classically, using the Dirac method,[15,32] one obtains the equations of motion by variation of the super-Hamiltonian with respect to all degrees of freedom. The constraints act as supplementary equations. At the end, functions of the degrees of freedom may play the role of coordinate conditions. In the procedure of Arnowitt, Deser, and Misner[16] (ADM), functions of the dynamical variables become coordinate conditions while their conjugate variables are obtained by solving the constraints. The remaining degrees of freedom are regarded as truly dynamical. The ADM Hamiltonian is obtained as the negative of the momentum conjugate to the chosen time variable. Its variation then yields the equations of motion for the dynamical degrees of freedom. We show in Sec. II that the invariance of the QC action under a change of the time coordinate leads to the disappearance of the original time parameter from the problem.[33] This allows the imposition of a canonical gauge (use of a canonical variable to parametrized minisuperspace trajectories) without conflict between the role of the chosen canonical variable and that of the original coordinate time.[34] The use of this gauge in the FPI quantization of the model according to standard techniques for constrained systems[35] leads to the ADM reduction at the level of the path integral. The FPI is evaluated in the Hamiltonian formulation using the broken-path approximation.[35] A measure is constructed which performs a canonical transformation in the part of phase space orthogonal to the dynamical degrees of freedom between the original nondynamical degree of freedom and a set of conjugate variables which are, respectively, the Hamiltonian constraint and the canonical gauge condition.[36] The Jacobian of the transformation (which would be the Faddeev-Popov determinant in a quantum field theory) is the Poisson bracket between the Hamiltonian constraint and the gauge condition and is not unity. Integrating out the nondynamical degree of freedom with this measure yields the FPI which would be the starting point for the ADM quantization using FPI formalism.

To define the functional integral, Feynman boundary conditions are imposed in Sec. III. These require the positive (negative)[37] square root to be taken when solving the constraint for the ADM Hamiltonian for evolution into the future (past).[36] The classical behavior and canonical quantization of the degrees of freedom of this model are given in Sec. IV. The minisuperspace transition amplitudes are given for both Dirac and ADM quantization. The Dirac quantization imposes the Hamiltonian constraint on the wave function.[32] In its operator form, this becomes the Wheeler-DeWitt equation.[18] The ADM quantization uses the ADM Hamiltonian in the Schrödinger equation with the role of time played by the canonical variable in the gauge condition.[19]

In Sec. V the explicit evaluation of the path integral is made beginning with the ADM starting point obtained in Sec. III. The remaining integrations are performed to yield the transition amplitude in closed form in agreement with that obtained using the canonical ADM quantization in Sec. IV. Finally, in Sec. VI, the transition amplitude with the measure from Sec. II is recomputed by performing integrations in a different order. Both the dynamical and nondynamical Gaussian integrations over momenta are performed first. Then the integration over the lapse [which in Sec. II yields (neglecting boundary conditions) a δ function of the constraint] is performed to yield the (infinitesimal) Lagrangian form of the transition amplitude. This method has been used for more complicated models in which the ADM Hamiltonian is nonpolynomial.[38] Problems associated with the square root may be either avoided or postponed. Conclusions are presented in Sec. VII.

II. CONSTRUCTION OF THE PATH INTEGRAL

The general coordinate invariance of the Einstein-Hilbert action leads to the appearance of constraints in its Hamiltonian formulation.[15,16] These constraints are in fact the generators of coordinate transformations.[20] It is well known[35] that construction of the FPI for this action requires a procedure (e.g., that of Faddeev and Popov[39] or DeWitt[40]) to avoid overcounting paths which are equivalent under a coordinate transformation. Heuristically, the Faddeev-Popov procedure is equivalent to factorization of the $6\times\infty^3$-dimensional phase space of the field into dynamical and nondynamical phase subspaces. The dynamical phase subspace contains the degrees of freedom of the three-geometry.[41] The integration over the nondynamical phase subspace is the integral over the gauge group. The imposition of gauge conditions and the inclusion of the Faddeev-Popov determinant in the measure are required for this factorization to allow the nondynamical field variables to be canonically transformed to gauge and constraint variables.[42]

This heuristic picture which becomes subtle[43] and nonunique[40] in nonrenormalizable field theories is straightforward and accurate in the quantum cosmology "approximation."[44] The spatial dependence is removed at the classical level (thus, of course, violating the uncertainty principle) to cast this problem into the form of quantum mechanics of a constrained system. We further specialize to systems[30] in which the only remaining coordinate freedom lies in specification of the lapse function which defines the spacing in time between spacelike hypersurfaces. The foliation itself has been fixed by the homogeneity requirement. The homogeneous cosmologies are described by a finite number of degrees of freedom—e.g., volume scale factor, anisotropy matrix, and amplitudes for spatially homogeneous nongravitational fields. The classical evolution of the system follows a trajectory in the minisuperspace whose axes are the canonical configuration variables.

The primary quantum gravity issues which remain in the truncated problem are factor ordering and the invariance under change of time coordinate generated by the still nontrivial Hamiltonian constraint. Here we shall concentrate on the latter issue. The factor ordering will be implicit in the Hamiltonian form of the path integral once a choice of canonical variables has been made.[45]

The standard method for path-integral quantization of constrained systems[35] requires the imposition of an auxiliary (e.g., gauge) condition in the measure which at each time step (in the "broken path" evaluation of the path integral) picks a single point on each trajectory of the flow generated by the constraints. The use of this method and choice of gauge conditions have been the subject of recent discussion.[46] In fact, it has been claimed (e.g., by Teitelboim[22]) that the gauge condition to be used here is not allowed. The difficulties can be traced to the fact that the Hamiltonian constraint generates the time evolution as well as the time coordinate transformation. It appears that fixing the gauge to select a point on the constraint-generated trajectory will eliminate the dynamical evolution of the system.[47] Furthermore, a choice of a canonical gauge which in effect identifies a canonical variable as the time will be inconsistent since the range of the canonical variable will be fixed (in the definition of the transition amplitude) but that of the coordinate time will not.[22]

These objections to the use of a canonical gauge to fix a point on each constraint-generated trajectory are not applicable here. They result from the manner in which the classical reduction of constraints is usually stated. For a typical minisuperspace model, one identifies a canonical variable with time (e.g., volume or trace of extrinsic curvature) or equivalently fixes the lapse either after solving the equations of motion (Dirac) or as the first step in the ADM procedure. Actually, a choice of lapse is never required to construct the minisuperspace trajectory—i.e., the coordinate invariance really means that the original time parameter may be completely eliminated from the description of the system since the minisuperspace trajectory must be independent of its parametrization. At the classical level, then, the identification of a canonical variable with time really means that a canonical variable has been selected to label points on the minisuperspace trajectory since the original time parameter has disappeared.

A similar elimination of the original time parameter occurs in the path integral. The minisuperspace models considered here and elsewhere[38] are characterized by degrees of freedom $p_A=(p_0,p_J)$, $q_A=(q_0,q_J)$, where $A=0,1,\ldots,N-1$ and $J=1,\ldots,N-1$ for a system with N degrees of freedom. The minisuperspace action will generally have the Hamiltonian form

$$S=\int_{[i]}^{[f]} dt\left[\sum_A p_A\frac{dq_A}{dt}-\alpha\mathscr{H}[p_A,q_A]\right], \tag{2.1}$$

where α is a rescaled lapse function and the Hamiltonian constraint is

$$2\mathscr{H}=-{p_0}^2+\sum_J {p_J}^2+V(q_A). \tag{2.2}$$

The rescaling of the lapse and choice of q_A have been made to yield the kinetic term in Eq. (2.2). This process also prescribes a factor ordering. The signature is a generic property of minisuperspace.[48] The potential $V(q_A)$ is model dependent. The symbols $[i]$ and $[f]$ denote, respectively, initial and final minisuperspace configurations. Thus $[i]\equiv(q_0^i,q_1^i,\ldots,q_{N-1}^i)$, etc. It is assumed that $\mathscr{H}$ has no explicit dependence on the coordinate time t. In deriving Eq. (2.1) from the Einstein-Hilbert action, a total time derivative has been discarded. Since this term may influence the quantum mechanics of the model, it is preferable to regard Eq. (2.1) to be a consequence of the Einstein-Hilbert action as modified by Gibbons and Hawking[14] to eliminate this boundary term.

The path integral may be expressed formally as the minisuperspace transition amplitude

$$\langle [f]\,|\,[i]\rangle=\int_{\text{paths}}\mathscr{D}[p_A,q_A,\alpha]e^{iS}. \tag{2.3}$$

To evaluate (2.3), we require an explicit form for the measure and boundary[49] (or equivalent) conditions to define integrals. The integral is to be taken over all possible paths between $[i]$ and $[f]$ in minisuperspace. If we imagine each path to be parametrized by t, we see that the gauge freedom means that paths which are merely reparametrizations of other paths are gauge equivalent and thus overcounted in the path integral. [Note that Eq. (2.3) must be independent of t_i and t_f so that t translations are included in the group of transformations under which the paths are equivalent.]

Since $\alpha\mathscr{H}$ which plays the role of Hamiltonian in Eq. (2.1) is t independent, the "broken path" representation of the path integral may be used.[35] Here (schematically) we replace a sum path by a path with a t-ordered product of sums over points on each path labeled by each t value. This yields the same overcounting of paths which are equivalent under reparametrization. To eliminate the overcounting in this representation, we apply the standard methods used for constrained systems.[35] A gauge condition is imposed to select a single point on each trajectory generated by the constraint $\mathscr{H}$ (considering only those trajectories which lie in the minisuper phase subspace in which $\mathscr{H}=0$).[50] (In fact, one may use the flow generated by any function of the canonical variables multiplied by the constraint which vanishes when $\mathscr{H}=0$.) Clearly, $\mathscr{H}$ generates the classical minisuperspace trajectories for all possible t parametrizations. The role of the gauge condition will be to select a single point on each trajectory to correspond to each t value—i.e., to fix the t parametrization. Before the gauge is imposed, the invariance under t reparametrization allows a given value of t to be associated with an arbitrary point (subject to differentiability and ordering) on each trajectory. This is illustrated in Fig. 1. A gauge condition which fixes the point on each trajectory corresponding to fixed t is, for example, that a fixed t corresponds to a specific value of g_0. (See Fig. 1.) The precise implementation of this gauge is as follows: In the

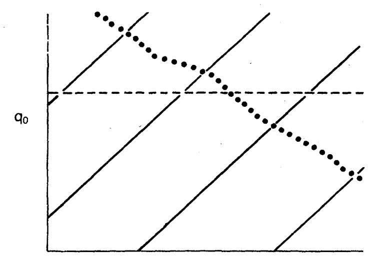

FIG. 1. The role of the gauge condition. On a graph of the timelike canonical variable q_0 vs a schematic projection of the other degrees of freedom q_J, the minisuperspace trajectories are indicated schematically by the solid lines. The dashed and dotted curves are among an infinite number of curves which can be drawn to indicate points on the trajectories labeled by a given fixed t value. The dashed line is the choice of trajectory parametrization made in this paper.

"broken path," label each t value by an ordered number k where $1 \le k \le n$. (Ultimately the limit $n \to \infty$ is taken.) Define

$$\Delta \equiv (q_0^f - q_0^i)/n \; . \tag{2.4}$$

At each t value impose the gauge condition

$$\mathscr{G}_k \equiv q_0^k - q_0^i - k\Delta = 0 \; . \tag{2.5}$$

Equation (2.5) causes q_0 to label the minisuperspace trajectories and thus to play the role of time. No explicit relationship between q_0 and t is required so the conflict implied in Ref. 34 does not occur.

Given the gauge condition (2.5), we can begin to construct the path space measure $\mathscr{D}(p_A, q_A, \alpha)$. The analysis in Ref. 35 suggests the choice

$$\mathscr{D}(p_A,q_A,\alpha) = (2\pi)^{-(N-1)n} \times \prod_{j=1}^{n}\prod_{k=1}^{n-1} dp_0^j dq_0^j \delta(q_0^f - q_0^n)\frac{d\alpha'^j}{2\pi} \times \prod_{J=1}^{N-1} dp_J^j dq_J^k \times \delta(\mathscr{G}_k)\,|\,\{\mathscr{H}_k, \mathscr{G}_k\}_{\rm PB}\,| \; , \tag{2.6}$$

where $\alpha' \equiv \alpha\delta t$, the first Dirac δ function imposes the same number of p_0 and q_0 integrations, and

$$\{\mathscr{H}_k, \mathscr{G}_k\}_{\rm PB} = \sum_{A=0}^{N-1}\left[\frac{\partial \mathscr{H}_k}{\partial q_A}\frac{\partial \mathscr{G}_k}{\partial p_A} - \frac{\partial \mathscr{H}_k}{\partial p_A}\frac{\partial \mathscr{G}_k}{\partial q_A}\right] = p_0^k \tag{2.7}$$

is the standard Poisson bracket. In this representation,

$$e^{iS} = \exp\left[i\sum_{k=1}^{n}\left[\sum_A p_A^k(q_A^k - q_A^{k-1}) - \alpha'^k \mathscr{H}_k(p_A^k, q_A^{k-1})\right]\right] , \tag{2.8}$$

where we require $q_A^0 \equiv q_A^i, q_A^n \equiv q_A^f$. The notation $\alpha' = \alpha\delta t$ in Eqs. (2.6) and (2.8) shows that all t dependence has disappeared to make the path integral manifestly t independent as required. No restriction on allowed t values at the end points is ever made.

Substitution of Eqs. (2.6) and (2.8) in the path integral (2.3) allows immediate identification of the integrations over α' as representations of $\delta(\mathscr{H}_k)$. Thus, at each k value, the $A=0$ minisuperphase subspace integrations have the form

$$I_0 \equiv \int dp_0 dq_0 \delta(\mathscr{G})\delta(\mathscr{H})\,|\,\{\mathscr{H},\mathscr{G}\}_{\rm PB}\,|\exp(ip_0\delta q_0) \; , \tag{2.9}$$

where $\delta q_0^k = q_0^k - q_0^{k-1}$. Since the Poisson bracket is the appropriate Jacobian, I_0 can be reexpressed[51] as

$$I_0 = \int d\mathscr{G}\, d\mathscr{H}\, \delta(\mathscr{G})\delta(\mathscr{H})\exp(ip_0\delta q_0) = \exp(ip_0\delta q_0)\,|_{\,\mathscr{G}=0,\mathscr{H}=0} \tag{2.10}$$

for p_0 and q_0 evaluated such that $\mathscr{G}=0$ and $\mathscr{H}=0$. The gauge condition requires

$$\delta q_0^k = \Delta \tag{2.11}$$

for all k. The evaluation of p_0 at $\mathscr{H}=0$ requires a prescription for the sign of the square root. Straightforward evaluation of the p_0 integral in Eq. (2.9) yields

$$I_0 = 2\cos[\,|H_{\rm ADM}(p_J,q_J)|\,\Delta] \; , \tag{2.12}$$

where

$$H_{\rm ADM} \equiv -p_0 = \pm\left[\sum_{J=1}^{N-1} p_J{}^2 + V(q_A)\right]^{1/2} \tag{2.13}$$

is the ADM (dynamical) Hamiltonian [obtained by solving the constraint (2.2) for p_0]. Equation (2.12) should be regarded as the result obtained using the principal part of $\delta(\mathscr{H})$.

III. BOUNDARY CONDITIONS AND THE ADM QUANTIZATION

Classically, the variation of $H_{\rm ADM}$ yields the minisuperspace trajectory $\{q_J(q_0)\}$ for $J=1,\ldots,N-1$. If q_0 is a "good" trajectory label (i.e., time coordinate), it will vary monotonically along the trajectory. If the initial and final minisuperspace trajectories are given, the sign of $H_{\rm ADM}$ fixes the direction of motion of the system point on the trajectory. $H_{\rm ADM} > (<) 0$ is appropriate for $q_0^f > (<) q_0^i$. Since $H_{\rm ADM}$ formally plays the role of an energy, the correlation of its sign with the q_0 direction may be regarded to be equivalent to Feynman boundary conditions.[52]

These boundary conditions may be imposed on the path integral via the usual $i\epsilon$ prescription which we use to redefine $\delta(\mathscr{H})$ in Eq. (2.9). Rather than the Fourier transform representation which yields the principal part, we use (with k suppressed)

$$\delta(\mathscr{H}) \equiv \frac{1}{2\pi}\int_0^\infty d\alpha' \exp\left[i\frac{\alpha'}{2}\left[p_0{}^2 - \sum_J p_J{}^2 - V + i\epsilon\right]\right] , \tag{3.1}$$

where $\epsilon > 0$. Evaluation of Eq. (3.1) and performance of the p_0 integration by the calculus of residues yields rather than Eq. (2.12)

$$I_0 = \begin{cases} \exp(-i \mid H_{\mathrm{ADM}} \mid \Delta), & \Delta > 0 , \\ \exp(+i \mid H_{\mathrm{ADM}} \mid \Delta), & \Delta < 0 . \end{cases} \tag{3.2}$$

Using this result, the path integral (2.3) becomes

$$\begin{aligned} \langle [f] \mid [i] \rangle^{\pm} &= \lim_{n\to\infty} (2\pi)^{-(N-1)n} \\ &\times \int \prod_k \prod_J dp_J^k dq_J^k \delta(q_J^f - q_J^n) \\ &\times \exp\left[i \sum_k \left[\sum_J p_J^k \delta q_J^k \right] \mp i \sum_k \mid H^k_{\mathrm{ADM}}(p_J, q_A) \mid \Delta \right] , \end{aligned} \tag{3.3}$$

where the upper (lower) sign is taken for $\Delta > (<) 0$ and $\delta q_J^k \equiv q_J^k - q_J^{k-1}$ with $q_J^0 = q_J^i$ at q_0^i and $q_J^n = q_J^f$ at q_0^f. Equation (3.3) represents a reasonable definition of the path integral for a minisuperspace model in which the ADM reduction has been performed at the classical level. [For Δ sufficiently small, $\exp(-iH_{\mathrm{ADM}}\Delta)$ should be a good approximation to the evolution operator even though H_{ADM} is, in general, q_0 dependent.]

IV. THE VACUUM BIANCHI TYPE-I MODEL

The formalism of Secs. II and III are applied to the vacuum Bianchi type-I (Kasner) cosmology. The classical and quantum analysis has been given by Misner.[19] The spatially homogeneous, spatially flat, anisotropically expanding metric is expressed in terms of a volume expansion parameter $\Omega(t)$, an anisotropy matrix $\beta_{ij}(t)$ where

$$\beta_{ij} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) , \tag{4.1}$$

and an arbitrary function $N(t)$ which allows change of time coordinate as

$$ds^2 = -N^2 dt^2 + e^{2\Omega}(e^{2\beta})_{ij} dx^i dx^j \tag{4.2}$$

for $i,j = 1,2,3$ and repeated indices summed. The variables have been chosen to yield an Einstein-Hilbert action in the form of Eq. (2.1) where

$$q_0 \equiv \Omega, \quad p_0 \equiv p_\Omega ,$$

$$\mathbf{q} \equiv (q_1, q_2) = (\beta_+, \beta_-) \equiv \boldsymbol{\beta}, \quad \mathbf{p} \equiv (p_1, p_2) = (p_+, p_-) ,$$

where p_Ω and $\mathbf{p}$ are conjugates of Ω and $\boldsymbol{\beta}$, and $\alpha \equiv 4N\pi/3V\sqrt{g}$ has been defined to absorb inconvenient factors with g the determinant of the spatial metric and V the integral over the spatial coordinates. One may impose T^3 topology to yield finite spatial volume. The action is defined to absorb a total time derivative proportional to dp_Ω/dt. The Hamiltonian constraint in the form of Eq. (2.2) with $V(q_A) = 0$ is

$$\mathcal{H} = (p^2 - p_\Omega{}^2)/2 , \tag{4.3}$$

where $p^2 \equiv \mathbf{p}\cdot\mathbf{p} = p_+{}^2 + p_-{}^2$.

The classical equations of motion and their solution may be obtained by either variation of $\alpha\mathcal{H}$ or by variation of $H_{\mathrm{ADM}} = \pm p$ found by solving Eq. (4.3) for $-p_\Omega$ and expressing the trajectories in minisuperspace as functions of Ω. In the $\boldsymbol{\beta}$-Ω configuration space, the trajectories are the straight lines

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 \pm (\mathbf{p}/p)\Omega , \tag{4.4}$$

where $\boldsymbol{\beta}_0$ and $\mathbf{p}$ are arbitrary constants and the upper (lower) sign is chosen for Ω increasing (decreasing). (The sign change is equivalent to $\Omega \to -\Omega$. Since $-\infty < \Omega < +\infty$, this transformation is just time reversal.) Equation (4.4) completely describes the classical dynamics of the system. If the spacetime metric (4.2) is desired, one specifies either $\Omega(t)$ or equivalently α and substitutes in the variation of $\alpha\mathcal{H}$ with respect to p_Ω. It is easily shown that Eq. (4.4) is in fact the Kasner solution.

Canonical quantization in minisuperspace for the Bianchi type-I vacuum model has been discussed by Misner.[19] It is, in fact, just that for a massless (relativistic) particle in two spatial dimensions.[53] The wave function $\Psi(\boldsymbol{\beta}, \Omega)$ is the amplitude that the system have the coordinates $(\boldsymbol{\beta}, \Omega)$ in minisuperspace. The question of interpretation of this wave function is nontrivial since there is no external observer.[54] We shall assume a superobserver residing in superspace with the usual interpretation. The wave function may be assumed to satisfy either the Wheeler-DeWitt equation[18]

$$\mathcal{H}\Psi = \left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta_+{}^2} - \frac{\partial^2}{\partial \beta_-{}^2} \right] \Psi(\boldsymbol{\beta}, \Omega) = 0 , \tag{4.5}$$

where $\mathcal{H}$ has now become an operator with the usual canonical commutation relations (for $\hbar = 1$) between $(\boldsymbol{\beta}, \Omega)$ and their conjugate momenta $(\mathbf{p}, p_\Omega)$ imposed, or the ADM quantization[16]

$$\left[i \frac{\partial}{\partial \Omega} - H^{\pm}_{\mathrm{ADM}} \right] \Psi(\boldsymbol{\beta}, \Omega) = 0 , \tag{4.6}$$

where the upper (lower) sign is chosen for Ω increasing (decreasing) and the ADM Hamiltonian operator may be defined by its momentum-space representation. In general, these two quantization procedures are inequivalent. Boundary conditions must be imposed to completely specify the solutions to Eqs. (4.5) and (4.6). In the absence of particular boundary conditions[55] and to compare with the path integral, we compute the minisuperspace transition amplitude $\langle \boldsymbol{\beta}^f, \Omega^f \mid \boldsymbol{\beta}^i, \Omega^i \rangle$ which is defined to be[56] the solution to Eq. (4.5) or (4.6) subject to (in either case)

$$\langle \boldsymbol{\beta}^f, \Omega^i \mid \boldsymbol{\beta}^i, \Omega^i \rangle = \delta(\boldsymbol{\beta}^f - \boldsymbol{\beta}^i) . \tag{4.7}$$

We shall use the subscript D (ADM) to indicate the solution to the Dirac (ADM) quantization Eq. (4.5) [(4.6)]. It is easiest to solve Eq. (4.6) in momentum space (with $H^{\pm}_{\mathrm{ADM}} = \pm p$) and then to Fourier transform. Clearly,

$$\langle \boldsymbol{\beta}^f, \Omega^i \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm}_{\mathrm{ADM}} = \frac{1}{4\pi^2} \int d^2p \exp[i\mathbf{p}\cdot(\boldsymbol{\beta}^f - \boldsymbol{\beta}^i) \mp ip(\Omega^f - \Omega^i)] \tag{4.8}$$

solves Eq. (4.6) subject to Eq. (4.7) where the upper (lower) sign is taken for $\Omega^f > (<) \Omega^i$. The integrations may be performed explicitly to yield

$$\langle \boldsymbol{\beta}^f, \Omega^i \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm}_{\mathrm{ADM}} = \frac{i \mid \Omega^f - \Omega^i \mid}{2\pi[\mid \boldsymbol{\beta}^f - \boldsymbol{\beta}^i \mid^2 - (\Omega^f - \Omega^i)^2]^{3/2}} . \tag{4.9}$$

Since H_{ADM} for this model is independent of Ω (but not in general),

$$\mathscr{H} = \left[i\frac{\partial}{\partial \Omega} - H^{+}_{\mathrm{ADM}} \right] \left[-i\frac{\partial}{\partial \Omega} + H^{-}_{\mathrm{ADM}} \right] \tag{4.10}$$

with the order of the two right-hand side factors arbitrary since they commute. Thus any superposition of

$$\langle \boldsymbol{\beta}^f, \Omega^i \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{+}_{\mathrm{ADM}}$$

and

$$\langle \boldsymbol{\beta}^f, \Omega^i \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{-}_{\mathrm{ADM}}$$

with δ-function normalization will satisfy Eqs. (4.5) and (4.7). We note that the transition amplitude (4.9) is sharply peaked about the classical solution (4.4).

V. PATH-INTEGRAL EVALUATION OF THE TRANSITION AMPLITUDE

The vacuum Bianchi type-I variables may be used in the procedure described in Secs. II and III. The gauge condition (2.5) becomes

$$\mathscr{G}_k = \Omega^k - \Omega^i - k\Delta = 0 , \tag{5.1}$$

where $\Delta = (\Omega^f - \Omega^i)/n$. Finally, we obtain for Eq. (3.3)

$$\langle \boldsymbol{\beta}^f, \Omega^f \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm} = \lim_{n \to \infty} (2\pi)^{-2n} \int \prod_k d^2p^k \, d^2\beta^k \, \delta(\boldsymbol{\beta}^f - \boldsymbol{\beta}^i) \exp \left[i \sum_k [\mathbf{p}^k \cdot (\boldsymbol{\beta}^k - \boldsymbol{\beta}^{k-1}) \mp p^k \Delta] \right] \tag{5.2}$$

after performing the Ω, p_Ω, and α integrations. The momentum integrals are all of the form of Eq. (4.8) so that we find

$$\langle \boldsymbol{\beta}^f, \Omega^f \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm} = \lim_{n \to \infty} (2\pi)^{-n} (\pm i\Delta)^n [\mid \boldsymbol{\beta}^n - \boldsymbol{\beta}^{n-1} \mid^2 - \Delta^2]^{-3/2} \delta(\boldsymbol{\beta}^f - \boldsymbol{\beta}^i) \int \prod_k d^2\beta^k [\mid \boldsymbol{\beta}^k - \boldsymbol{\beta}^{k-1} \mid^2 - \Delta^2]^{-3/2} . \tag{5.3}$$

The extra factor arises from the extra $\int d^2p^n$. Each $\int d^2\beta^k$ involves two factors of the integrand product. The $\boldsymbol{\beta}$ integrations are most easily performed by changing variables to $\boldsymbol{\chi}^k = \boldsymbol{\beta}^k - \boldsymbol{\beta}^{k-1}$. The Jacobian of the transformation is unity. Eq. (5.3) can then be put in the form

$$\langle \boldsymbol{\beta}^f, \Omega^f \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm} = \lim_{n \to \infty} \frac{(\pm i\Delta)^n}{(2\pi)^{n+2}} \int d^2\lambda \, e^{i\boldsymbol{\lambda} \cdot (\boldsymbol{\beta}_f - \boldsymbol{\beta}_i)} \int \prod_k d^2\chi^k \, e^{-i\boldsymbol{\lambda} \cdot \boldsymbol{\chi}_k} [(\chi^k)^2 - \Delta^2]^{-3/2} . \tag{5.4}$$

Replacing the factor $[(\chi^k)^2 - \Delta^2]^{-3/2}$ by its transform, performing the angular integration, and using the orthogonality over argument of zero-order Bessel functions yields

$$\langle \boldsymbol{\beta}^f, \Omega^f \mid \boldsymbol{\beta}^i, \Omega^i \rangle^{\pm} = \frac{1}{4\pi^2} \int d^2\lambda \exp[i\boldsymbol{\lambda} \cdot (\boldsymbol{\beta}^f - \boldsymbol{\beta}^i) \mp i\lambda(\Omega^f - \Omega^i)] \tag{5.5}$$

which is precisely (4.8) so that (4.9) follows immediately.

VI. EVALUATION OF THE PATH INTEGRAL BY GAUSSIAN INTEGRATION

In more complicated models to be discussed elsewhere,[38] the explicit reduction described in Secs. II and III may be prevented by the nonpolynomial character of $H^{\pm}_{\mathrm{ADM}}$. In these cases, it is instructive to start from the form (2.3) for the path integral using the measure (2.6) obtained in Sec. II and to compute all the integrals over momenta before integrating over α. We shall regularize the integrals by inserting the Feynman boundary conditions as in Sec. III. The still undefined Gaussian momentum integrals will be evaluated by the method of stationary phase.

The terms of interest are n factors of the form

$$(2\pi)^{-3} \int d^2p \, dp_\Omega d\alpha' \mid p_\Omega \mid \times \exp\{ i [\mathbf{p} \cdot \delta\boldsymbol{\beta} + p_\Omega \Delta - (\alpha'/2)(p^2 - p_\Omega{}^2 - i\epsilon)] \} , \tag{6.1}$$

where $\delta\boldsymbol{\beta} \equiv \boldsymbol{\beta}^k - \boldsymbol{\beta}^{k-1}$. We require $\epsilon > 0, \alpha' > 0$. The method of stationary phase used to perform the momentum integrations yields

$$(2\pi)^{-3/2} \mid \Delta \mid e^{-i\pi/4} \int_0^\infty d\alpha' \alpha'^{-5/2} \times \exp \left[\frac{i}{2\alpha'}(\delta\beta^2 - \Delta^2) - \frac{\alpha'\epsilon}{2} \right] , \tag{6.2}$$

where $\delta\beta \equiv \mid \delta\boldsymbol{\beta} \mid$. This integral may be evaluated to yield (for $\epsilon \to 0$)

$$\frac{i}{2\pi} \mid \Delta \mid (\delta\beta^2 - \Delta^2)^{-3/2} . \tag{6.3}$$

Finally, inclusion of all n factors (with appropriate superscripts) of (6.3) in the original expression yields the previous result (5.3) for the form of the final integrations over $\boldsymbol{\beta}$.

VII. CONCLUSIONS

The Feynman path-integral formalism has been applied to the quantization of the dynamical degrees of freedom of the vacuum Bianchi type-I (Kasner) cosmological model. For this model, explicit evaluation in the Hamiltonian formalism using the broken-path approximation has been performed for all required integrals to obtain the transition amplitude between two minisuperspace configurations.

The action used for the gravitational degrees of freedom included the modification proposed by Gibbons and Hawking to allow arbitrary variations at the end points by eliminating total time derivative terms. In contradistinction to the arguments of Teitelboim and others, it is shown that a canonical gauge is permitted. It was shown that this gauge uses a canonical variable to parametrize the minisuperspace paths but requires no relationship between this variable and the completely hidden coordinate time. We have shown that, in this gauge, the ADM reduction of the original (β,Ω) degrees of freedom to the dynamical β degrees of freedom may be performed at the level of the path integral.

Standard methods for treatment of constraints in the path integral or appeals to consistency yield extra functions of the canonical variables in the measure. In the general quantum-mechanical case of N degrees of freedom with M constraints and M auxiliary (gauge) conditions, one extra factor is the determinant of the $M\times M$ matrix of Poisson brackets between the constraints and the gauge conditions. Although it has not been done here, this determinant could be represented by ghost degrees of freedom in the usual way.[57]

Thus we have shown that with the correct measure the canonical quantization transition amplitude can be reproduced using the path-integral formalism. The advantage to this approach, however, lies in its relatively straightforward generalization to cosmological models containing perfect-fluid sources, cosmological constants, spatially homogeneous scalar fields, and spatial curvature. Details of these models will be presented elsewhere.[38] For these models, it will be useful to follow the ordering of Sec. VI—to perform the Gaussian integrals over momenta before the integration over the Lagrange multiplier. For these models, we shall show that the simple model discussed here forms the zeroth-order solution for a perturbation scheme.[58]

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[46]See Ref. 34.

[47]This argument was brought to our attention by R. Geroch.

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[49]A proposal to choose boundary conditions has been given in Ref. 24.

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