Nuclear matter in the crystal soliton bag model

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A model for nuclear matter is introduced as consisting of an infinite number of bags placed on a spatial cubic lattice. Using the soliton bag model of Friedberg and Lee in the self-consistent mean-field approximation we study the properties of the system as a function of the lattice constant. At low densities the hadronic matter is well described by the solutions of isolated nucleons. With decreasing lattice constant the energies of the quarks spread out into bands and the quark wave functions of different bags start to overlap. At a certain critical density an abrupt phase transition to a uniform quark distribution occurs. The model yields a critical density of the order of the normal nuclear density which shows that the model cannot adequately describe the repulsive part of the nucleon-nucleon interaction at small relative distances.

I. INTRODUCTION

It is generally assumed that nuclear matter undergoes a phase transition to an unconfined quark plasma, when the nuclear density is raised sufficiently. There is no proof, however, since even the nature of confinement in QCD is not fully understood. Theories of this phase transition usually describe each of the two phases within separate models rather than using a single consistent model for the whole range of densities.[1]

In the following we utilize the soliton bag model of Friedberg and Lee[2] as a unified framework, allowing us to treat both phases on an equal basis. This model has been used for the same purpose by Baym, Blaizot, and Friman,[3] but the actual construction of nuclear matter has not been attempted. Those authors studied the thermodynamics of uniform phases only.

In this paper we explicitly take into account the localized structure within the hadronic phase by treating nuclear matter as a collection of bags filled with three quarks each. In order to study such matter we assume a crystal symmetry for the quark and soliton fields by arranging the nucleons in a simple cubic spatial lattice. This static model for nuclear matter, which has been described briefly in Ref. 4, is justified for higher nuclear densities and allows us to describe the phase transition from granulated nuclear matter into a continuous one.

Recently two other approaches to the problem of phase transition have been published applying also the crystal lattice model. Kutschera, Pethick, and Ravenhall[5] studied this approach within the Skyrme model and Banerjee, Glendenning, and Soni[6] used the chiral bag model. Both groups of authors have used the Wigner-Seitz approximation, replacing the corresponding fields in a lattice cell by spherically symmetric ones. This approximation is not employed in this paper, where we solve the self-consistent problem directly by using the wave functions appropriate for the simple cubic lattice.

The Lagrangian density of the model is given by

$$\mathscr{L}=\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-g\sigma)\psi+\tfrac{1}{2}\partial^{\mu}\sigma\partial_{\mu}\sigma-U(\sigma)\ , \tag{1}$$

where ψ is the quark field, and σ is a scalar field representing the collective nonperturbative effects of QCD. The color and flavor indices are omitted. The summation over these degrees of freedom, however, has to be performed in all the following formulas. The quark-σ coupling constant is denoted as g, and $U(\sigma)$ is a nonlinear self-interaction function, which is taken to be a fourth-order polynomial

$$U(\sigma)=B+\tfrac{1}{2}a\sigma^2+\frac{1}{3!}b\sigma^3+\frac{1}{4!}c\sigma^4\ . \tag{2}$$

With a suitable adjustment of the parameters, the function has its absolute minimum at $\sigma=\sigma_{\rm vac}$ and a relative minimum or an inflection point at $\sigma=0$. The constant $B=U(0)$ may be identified with the MIT bag constant and is chosen so that $U(\sigma_{\rm vac})$ vanishes. The remaining four free parameters a, b, c, and g can be fixed by nucleon data.

We restrict the calculations to static solutions in the mean-field approximation, where the field operator σ is replaced by a time-independent classical field. The quark field operators are expanded in a complete set of basis functions

$$\psi=\sum_k c_k\psi_k\ , \tag{3}$$

where c_k are fermion annihilation operators. The mean-field approximation then leads to the following coupled set of equations:[2,4]

$$(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+g\beta\sigma)\psi_k=\epsilon_k\psi_k\ , \tag{4}$$

$$-\nabla^2\sigma+\frac{\partial U}{\partial\sigma}=-g\sum_{k\ \rm occ}\psi_k^{\dagger}\beta\psi_k\ . \tag{5}$$

The total energy of the system is given by

$$E=\sum_{k\ \mathrm{occ}}\epsilon_k+\int[\tfrac{1}{2}(\nabla\sigma)^2+U(\sigma)]d^3r\ . \tag{6}$$

Spherically symmetric solutions of these equations can be used to describe single nucleons as done by various authors.[2,4,7–10] This has not led, however, to a "best" set of the parameters. Quite a variety of parameter sets gives a satisfactory description of single nucleons.[7]

II. CRYSTAL SOLITON BAG MODEL

In the following we consider nuclear matter as consisting of an infinite number of bags arranged on a regular periodic lattice. For simplicity we choose a simple cubic (sc) lattice with a lattice constant d. The scalar field is taken to be periodic

$$\sigma(\mathbf{r})=\sigma(\mathbf{r}+\mathbf{R})\ , \tag{7}$$

where $\mathbf{R}$ is an arbitrary lattice vector. We also require that σ contains the reflectional and discrete rotational symmetries of the cubic lattice. For the translational symmetry (7) we can write the solutions of the Dirac equation (4) in the form of Bloch waves,

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}\phi_{\mathbf{k}}(\mathbf{r})\ , \tag{8}$$

where $\phi_{\mathbf{k}}$ has the translational symmetry as in (7), but not necessarily the other symmetries of σ. The "crystal momentum" $\mathbf{k}$ is a continuous vector, which can be restricted to the first Brillouin zone. The $\phi_{\mathbf{k}}$ satisfy the Dirac equation,[4]

$$[\boldsymbol{\alpha}\cdot(-i\nabla+\mathbf{k})+g\beta\sigma(\mathbf{r})]\phi_{\mathbf{k}}(\mathbf{r})=\epsilon_{\mathbf{k}}\phi_{\mathbf{k}}(\mathbf{r})\ , \tag{9}$$

where the $\epsilon_{\mathbf{k}}$ display the familiar band spectrum of a crystal. In practice it is not feasible to perform the calculations for all required values and orientations of the crystal momentum $\mathbf{k}$. Therefore, as an approximation we assume spherically symmetric energy surfaces and perform the calculations for a special direction of $\mathbf{k}$ only. In order to keep the calculations of the energies simple, we choose a [100] direction, which still has a high symmetry. In the sums over crystal momenta [Eqs. (5) and (6)] we make the replacements

$$\epsilon_{\mathbf{k}}\approx\epsilon_{k\hat{\mathbf{z}}},\quad \psi^\dagger_{\mathbf{k}}\beta\psi_{\mathbf{k}}\approx\overline{\psi^\dagger_{k\hat{\mathbf{z}}}\beta\psi_{k\hat{\mathbf{z}}}}\ , \tag{10}$$

where the overbar denotes averaging over the six possible orientations of the [100] direction.

In the case of symmetric nuclear matter with only up and down quarks the spin-flavor-color degeneracy of each state is 12-fold, namely, 2 for spin, 2 for flavor, and 3 for color. Since each band provides one state for every unit cell, and we need to occupy the cells with three quarks each, only one-quarter of the first band will be filled in the ground state at zero temperature. A filling factor $f=\frac{1}{4}$ of the band implies a Fermi momentum given by

$$k_F=\left[f\frac{6}{\pi}\right]^{1/3}\frac{\pi}{d}=0.78\frac{\pi}{d}\ . \tag{11}$$

The smallest distance between the center of the first Brillouin zone and its boundary is π/d. Thus the Fermi surface does not get too close to the zone boundaries, where the energy surfaces will be deformed most. The filling factor $f=\frac{1}{4}$ is the minimum demanded by the Pauli principle. If one requires each localized state to be of the form of a nucleon (n or p, spin up or down), the factor could be greater. If the lattice were filled, for example, with Δ^{++} in $m=\frac{3}{2}$ states ($m=$quantum number of spin projection), this would require dilute filling of the entire band, i.e., $f=1.0$, in order to satisfy the Pauli principle. Uniform nuclear matter ($n,p,m=\pm\frac{1}{2}$) clearly requires less of the band to satisfy Pauli exclusion. In this case the appropriate filling factor lies between $\frac{1}{4}$ and 1.

It is difficult to address this question of the filling in the context of Bloch states, because they extend over all space. One should perform a transformation to localized basis functions, the so-called Wannier functions. This problem is presently studied by Birse *et al.*[11] in the framework of the Wigner-Seitz approximation by transforming from partially filled Bloch states to incomplete Wannier states clustered to form a color-singlet three-quark nucleon state at each cell.

In the actual calculations we expand σ and $\phi_{\mathbf{k}}$ in Fourier series

$$\sigma(\mathbf{r})=\sum_{\mathbf{K}}\sigma_{\mathbf{K}}e^{i\mathbf{K}\cdot\mathbf{r}},\quad \phi_{\mathbf{k}}(\mathbf{r})=\sum_{\mathbf{K}}\phi_{\mathbf{kK}}e^{i\mathbf{K}\cdot\mathbf{r}}\ , \tag{12}$$

where the sums run over all reciprocal lattice vectors. Inserting these expansions into Eq. (9) we obtain a matrix eigenvalue problem

$$\boldsymbol{\alpha}\cdot(\mathbf{K}+\mathbf{k})\phi_{\mathbf{kK}}+g\beta\sum_{\mathbf{K}'}\sigma_{\mathbf{K}-\mathbf{K}'}\phi_{\mathbf{kK}'}=\epsilon_{\mathbf{k}}\phi_{\mathbf{kK}}\ . \tag{13}$$

The expansion coefficients are not completely independent because of symmetry. The field σ possesses the full cubic symmetry, while the solutions $\phi_{\mathbf{k}}$ can be classified according to particular irreducible representations of the group of the crystal momentum.[12] This group contains all the operations belonging to the point group of the sc lattice which transform $\mathbf{k}$ into itself. As trial wave functions we used symmetrized combinations of plane waves (SCPW), which are similar to those introduced in Ref. 13. For the lowest-lying band we choose a SCPW, where the upper components transform according to the irreducible representation Δ_1 (Refs. 12 and 13).

By solving (13) for several values of $\mathbf{k}=k\hat{\mathbf{z}}$ we obtain the scalar density

$$\sum_{\mathbf{k}\ \mathrm{occ}}\psi^\dagger_{\mathbf{k}}\beta\psi_{\mathbf{k}}\approx\frac{3}{2\pi^2 f}\int_0^{k_F}dk\,k^2\overline{\psi^\dagger_{k\hat{\mathbf{z}}}\beta\psi_{k\hat{\mathbf{z}}}}\ , \tag{14}$$

where the wave functions are normalized to one quark per unit volume and the overbar denotes averaging over the six possible orientations of the [100] direction as in Eq. (10).

Now we proceed with Eq. (5). Since σ and the scalar density (14) can be written in a Fourier series [see Eq. (12)], the field equation (5) will be reduced to a system of coupled algebraic equations

$$(K^2+a)\sigma_{\mathbf{K}}+\frac{b}{2}\sum_{\mathbf{K}'}\sigma_{\mathbf{K}'}\sigma_{\mathbf{K}-\mathbf{K}'}+\frac{c}{3!}\sum_{\mathbf{K}',\mathbf{K}''}\sigma_{\mathbf{K}'}\sigma_{\mathbf{K}''}\sigma_{\mathbf{K}-\mathbf{K}'-\mathbf{K}''}+g\left[\sum_{\mathbf{k}\ \mathrm{occ}}\psi^\dagger_{\mathbf{k}}\beta\psi_{\mathbf{k}}\right]_{\mathbf{K}}=0\ . \tag{15}$$

This nonlinear system is solved using the Newton-Raphson iteration method. Equations (13) and (15) are solved alternately until self-consistency is reached.

III. RESULTS

We present results for two different sets of parameters ($\tilde{B}=\hbar c B$):

(i) $a=0$, $b=-105.14\ \mathrm{fm}^{-1}$, $c=1000$,
$g=9.037$, $\tilde{B}=27.12\ \mathrm{MeV/fm}^3$,

(ii) $a=0$, $b=-7482.4\ \mathrm{fm}^{-1}$, $c=200\,000$,
$g=19.357$, $\tilde{B}=86.97\ \mathrm{MeV/fm}^3$.

Both sets have been taken from a variety of parameter sets investigated by Horn,[7] which all lead to the correct mean nucleon-Δ mass and proton rms charge radius of single bags, when recoil corrections are included. The parameter set (i) is best suited for our calculations, because it produces soft-bag surfaces and, therefore, does not require too many terms in the Fourier expansions (12). The parameter set (ii) belongs to a stiffer bag with smaller surface thickness than that of the bag produced with the parameter set (i).

For large lattice constants we should recover the results of single, separated, spherically symmetric bags. This is quite well obtained with the parameters (i) of the soft bag, but can only be approximately achieved for the parameter set (ii).

Figure 1 shows plots of the quark density $\psi^\dagger\psi$ and the soliton field $g\sigma$ along a [100] axis of the crystal. In Fig. 2 we display $\psi^\dagger\psi$ in a (100) plane of a lattice cell. With decreasing lattice constant we observe two effects.

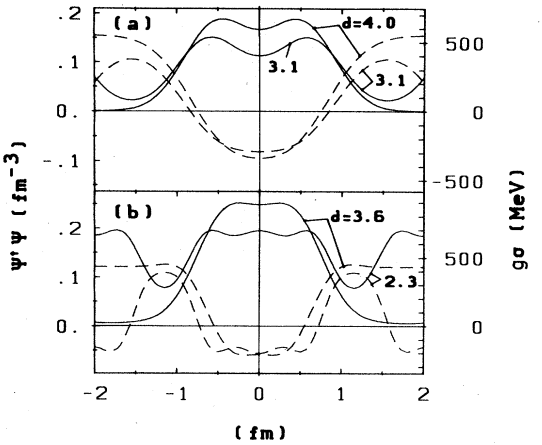

FIG. 1. The quark density $\psi^\dagger\psi$ (solid curves, left scale) and the soliton field $g\sigma$ (dashed curves, right scale) along a [100] axis for two parameter sets. (a) parameter set (i) and $d=3.1$, 4.0 fm, (b) parameter set (ii) and $d=2.3$, 3.6 fm.

(a) When the quark distributions begin to overlap, the bag sizes are increased. An increase of the bag size with growing density has also been found by Jändel and Peters[9] who interpreted this effect as an explanation of the EMC (European Muon Collaboration) effect.

(b) At a critical value $d_c\approx 3$ fm and $d_c\approx 2.1$ fm for (i) and (ii), respectively, the coupling between the quark and σ fields starts to destroy the barriers between the bags, and the localized soliton solutions cease to exist. Then a uniform quark density is formed. The "phase transition"

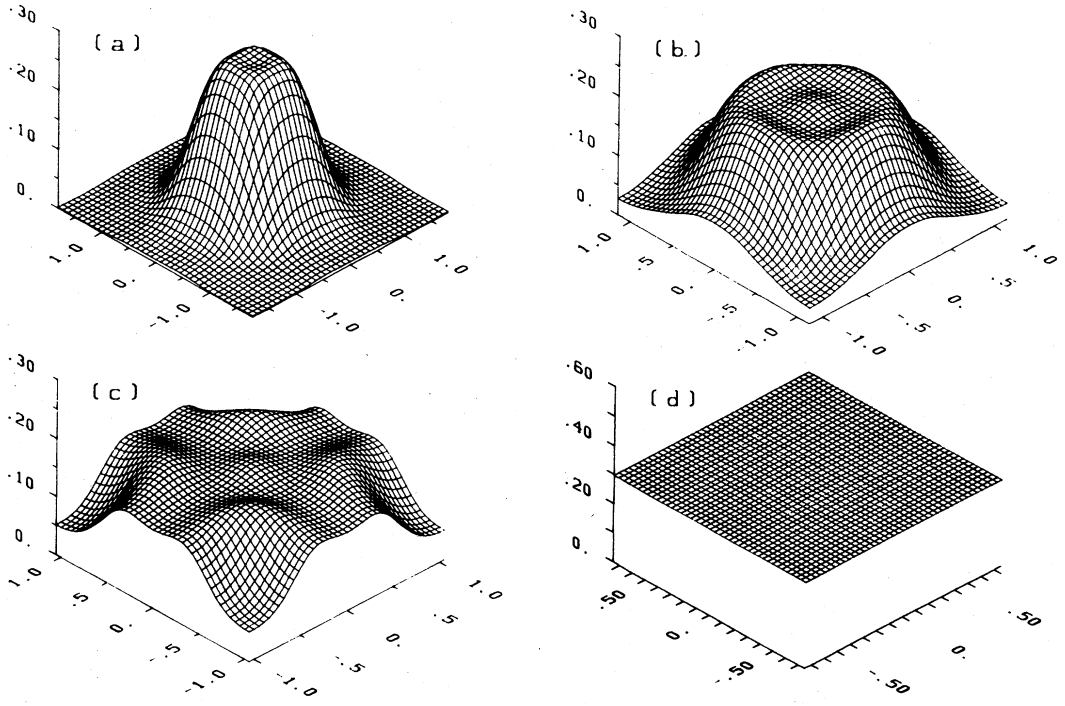

FIG. 2. The quark density in a (100) plane for the parameter set (ii) and lattice constants $d=3.6$ fm (a), 2.3 fm (b), 2.1 fm (c), and 1.5 fm (d). The units are fm^{-3} and fm for the density and length scales, respectively.

to uniform matter and fields occurs below nuclear matter density at $\rho/\rho_0 \approx 0.7$ for the parameter set (ii), where $\rho_0 = 0.15$ nucleons/fm^3 is the density of normal nuclear matter.

Slightly below the transition point ($d > d_c$) the quark eigenvalues for $k = 0$ and $k = k_F$ differ by 10 MeV for parameters (i) and 50 MeV for (ii). The energy $\epsilon(k=0) \approx 250$ MeV is the same for both parameter sets.

The transition occurs suddenly as a function of d, as can be recognized from Fig. 3 which shows the total energy and the energies of the quark and σ fields per unit cell. Quark and σ field energies are given by the first and second terms in Eq. (6), respectively. The dashed curves represent the energy for uniform solutions of Eqs. (4) and (5). Branch 1 describes a free massless ($\sigma = 0$) quark plasma, whereas branch 2 belongs to free massive quarks ($\sigma \approx \sigma_{\text{vac}}$).

One readily observes a slight decrease in energy of the soliton solutions with decreasing lattice spacing, which implies a negative pressure. This can be attributed to the fact that the model Lagrangian (1) does not contain any repulsive vector fields, which have to be included in the next step. We expect that repulsive fields shift the phase transition to higher densities.

In Fig. 4 we compare our results with calculations of Birse *et al.*[11] carried out in the Wigner-Seitz approximation for the parameter set (i). The total energy per nucleon is plotted as a function of the lattice constant d and the Wigner-Seitz radius R, respectively. Both these parameters are related in such a way that the volume of a unit cell is equal to the volume of the Wigner-Seitz sphere:

$$d^3 = \frac{4\pi}{3} R^3 . \tag{16}$$

For larger lattice constants d both methods agree. The transition to uniform distributions occurs at about the same value of d in both calculations. The energy values obtained with the Wigner-Seitz approximation are about 20 MeV/A above our values for $d = 3$ fm, because this approximation does not describe the nonspherical character of the fields in the unit cell at small lattice constants. Our solution gives a higher quark density in the direction of the nearest neighbors.

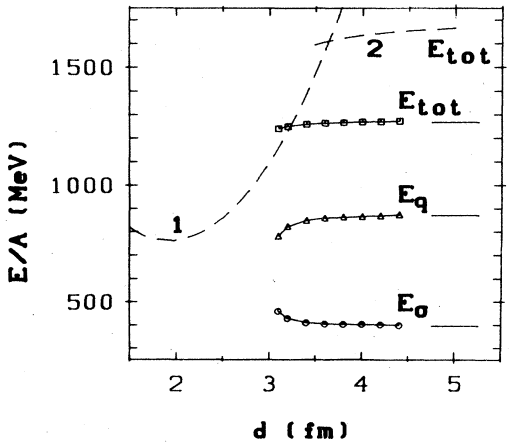

FIG. 3. Energy per unit cell as a function of the lattice constant for the parameter set (i). For the soliton solutions the energies for the quark and σ fields are shown separately. On the right-hand side the corresponding energies for a single bag are shown for comparison. The dashed curves are the energies for uniform fields: branch 1 describes a free massless quark plasma ($\sigma = 0$) and branch 2 a free massive quark plasma ($\sigma \approx \sigma_{\text{vac}}$).

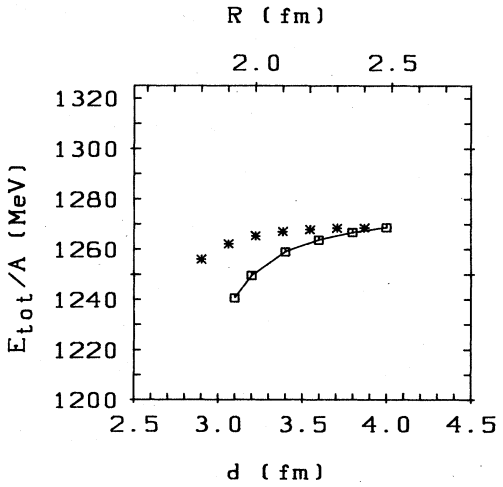

FIG. 4. Total energy per unit cell as a function of the lattice constant d and the Wigner-Seitz radius R for the parameter set (i). Open squares connected by a line: present results (shown also in Fig. 3); asterisks: Wigner-Seitz approximation calculated by Birse *et al.* (Ref. 11).

IV. CONCLUSIONS

In conclusion the crystal soliton bag model can describe the localized structure of nuclear matter and its transition to uniform quark matter. The present model, which is self-consistently treated, predicts a transition occurring at a too low nuclear density.

In order that the localized soliton solutions can persist also at higher densities, the following extensions and developments of the model should be considered as important future steps.

The main deficiency of the present model is the omission of the one-gluon-exchange processes, which act repulsively between the bags and may shift the transition to higher densities. Therefore, gluon-exchange processes should properly be incorporated in the model.

Recoil corrections, which are not yet included, but should be taken into account, have the effect of reducing the static rms radius of single bags by approximately 15% (Refs. 4 and 14). The presently applied sc lattice does not give the best arrangement of nucleons at high densities and may be replaced by a face-centered-cubic (fcc) lattice with a packing fraction larger by 42%. The combined effects of the recoil corrections and the fcc lattice may lead to an increase of the transition density by a factor 2.3, thereby shifting the transition point for the parameter set (ii) to $1.6\rho_0$.

The parameters of the model may be treated as depen-

dent on the nucleon density. Density-dependent parameters would phenomenologically describe possible higher-order coupling terms between the quark and σ fields, bag-bag correlations, and bag motion (phonons). If properly chosen, they can lead to higher transition densities. Also parameter sets with a larger bag constant may yield higher transition densities, as has been observed by Gagnon[15] for the MIT bag model. In addition the filling parameter f in Eq. (11) may be chosen $0.25 < f < 1$. As Wigner-Seitz cell calculations have indicated,[11] the total energy can be repulsive with decreasing cell size if f is set greater than $\frac{1}{4}$. Recently, Wendel and Hilf[16] have shown that the total energy can also be repulsive if special boundary conditions are used in the Wigner-Seitz approximation.

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