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Fermion-monopole system reexamined. IV

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The chiral properties of the fermion-monopole system are investigated in detail. For one species of massless Dirac fermions, we show that the chiral symmetry is spontaneously broken, even in the presence of anomalies. The symmetry breaking is accompanied by a Nambu-Goldstone mode in the physical sector, which essentially coincides with the bosonic fields employed by Rubakov and Callan. A discrepancy noted previously for a Higgs-boson mass is also resolved. For the case of two species, we find two classes of degenerate solutions corresponding to whether or not the nonanomalous chiral symmetry is broken. A comparison of the two solutions in the presence of a mass term indicates that monopoles are unlikely to catalyze proton decay at strong-interaction rates.

I. INTRODUCTION

Since its discovery in 1949, the chiral anomaly[1-8] has persistently been revealing unexpected features, not the least being its connection with some important theorems in mathematics.[9-11] Similarly, chiral-symmetry breaking[12-18] has assumed a major role in elementary particle physics, since its transcription from BCS theory in the 1960's. Their pertinence to the fermion-monopole system has also been widely recognized, and there already exists considerable literature on this subject,[19-48] particularly with respect to proton-decay catalysis.[28-49]

However, it is well known that both issues can develop quite delicate aspects, so we have decided to perform a detailed study. Needless to say, much has already been anticipated in the literature in one form or another. However, our analysis has also led to the distinct possibility that monopoles do not catalyze proton decay at strong-interaction rates, so we believe a systematic reexamination is justified. Essentially, we find that the fermion-monopole system possesses a Dirac phase, which is degenerate with the Rubakov-Callan phase if the fermions are massless, and is expected to have a lower energy if the fermions are massive.

Our results also bring to the fore an important aspect of anomaly theory (although the point is well known to the experts): Chiral symmetry for massless fermions is spontaneously broken, even in the presence of anomalies. Furthermore, a new feature is that, unlike the case of the Schwinger model,[50] the CP^n model,[51] or (presumably) QCD, the symmetry breaking is accompanied by a Nambu-Goldstone mode in the physical sector, showing that the U(1) problem[15-19] is indeed a nontrivial dynamical issue.

The organization of the paper is as follows. Section II deals with the Abelian case, beginning with a brief review of the chiral anomaly in the absence of monopoles. The necessary modifications in a background monopole (dyon) field are derived, paying particular attention to the singularity at the origin. As suggested by Nair,[36] we find that the scalar field of Rubakov[28] and Callan[31] may be identified as a Nambu-Goldstone mode associated with the breaking of chiral symmetry. The essential aspects are shown to be unchanged if the Coulomb self-energy of the fermion is included in addition to the background field.

Section III deals with the non-Abelian case with isodoublet (Dirac) fermions. After a brief review of the chiral anomaly, such as in Sec. II, we first consider the semiclassical ground state,[52] where chirality is unbroken for massless fermions. The state is shown to be unstable in the presence of a four-Fermi interaction induced by the dyon degree of freedom.[22,23] Two proposals as to the nature of the true ground state are then discussed. One is that of our previous paper[23] (II), where the boundary conditions (BC's) for the fermions are effectively modified from the semiclassical result, and the other is the approach of Callan,[31] where BC's for the bosons are modified after a bosonization of the original Hamiltonian.

For one doublet, we find that the two proposals lead to essentially equivalent results, confirming the result of Polchinski.[43] In particular, this allows us to resolve a discrepancy noted previously concerning fermion-number conjugation and the Jackiw-Rebbi zero mode.[52] We then turn to the case of two doublets, which may be regarded as the prototype of catalysis. Here, we find that the two proposals lead to a different conclusion with respect to the nonanomalous chiral symmetry, as expected from an analysis of conservation laws.[32] If the fermions are massless, we may explicitly construct the approximate ground states with both proposals, which we find to be degenerate in energy. Adding a mass term to resolve the degeneracy, we find strong indications that the Dirac phase is to be favored, although an explicit construction of the ground state eludes us for Callan's proposal. The result is consistent with the observation of Polchinski[43] that the chiral perturbation series around the Rubakov-Callan ground state is infrared divergent.

Section IV is devoted to a discussion of the various arguments put forward in favor of catalysis, as well as some open questions. We also speculate on the possible significance of our results beyond the fermion-monopole system.

Appendix A is devoted to a partial-wave analysis of the anomaly. Appendix B provides a brief review of the relation between the renormalized and unrenormalized versions of the chiral anomaly. Appendix C compares the results of this paper with the existing literature on some technical issues.

II. THE ABELIAN CASE

Let us first recall the relevant aspects of the chiral anomaly for massless QED in 3 + 1 dimensions. At the tree level, the chiral current $j_{\mu 5}=\overline{\Psi}\gamma_\mu\gamma_5\Psi$ is invariant under the gauge transformation

$$\delta_h A_\mu=\partial_\mu h \ , \quad \delta_h\Psi=-ieh\Psi \ , \tag{2.1}$$

and satisfies the conservation law $\partial_\mu j_{\mu 5}=0$.

However, the gauge invariance and the conservation law become incompatible in the presence of quantum corrections, as represented by the triangle diagram. Since $j_{\mu 5}$ is an external operator, some latitude exists in its renormalization, and the corrections are dependent on the precise prescription adopted. If the prescription maintains gauge invariance, the current obtained $j_{\mu 5}^{\text{inv}}$ is no longer conserved,

$$\delta_h j_{\mu 5}^{\text{inv}}=0 \ , \quad \partial_\mu j_{\mu 5}^{\text{inv}}=\frac{e^2}{8\pi^2}[F_{\mu\nu}\widetilde{F}_{\mu\nu}] \ . \tag{2.2}$$

On the other hand, if the prescription maintains the conservation law, the resulting current $j_{\mu 5}^{\text{sym}}$ is no longer gauge invariant,

$$\delta_h j_{\mu 5}^{\text{sym}}=-\frac{e^2}{4\pi^2}\widetilde{F}_{\mu\nu}\partial_\nu h \ , \quad \partial_\mu j_{\mu 5}^{\text{sym}}=0 \ . \tag{2.3}$$

(All quantities are renormalized unless specified to the contrary. See Appendix B for details.)

Despite its noninvariance, however, it is $j_{\mu 5}^{\text{sym}}$ which is of interest for chiral-symmetry breaking, since it is the one which determines the operator chiralities n_O according to

$$\int d\mathbf{x}[j_{05}^{\text{sym}}(\mathbf{x},t),\ O(\mathbf{y},t)]=n_O O(\mathbf{y},t) \ . \tag{2.4}$$

In particular,

$$\int d\mathbf{x}[j_{05}^{\text{sym}}(\mathbf{x},t),\ \Psi(\mathbf{y},t)]=-\gamma_5\Psi(\mathbf{y},t) \ , \tag{2.5}$$

$$\int d\mathbf{x}[j_{05}^{\text{sym}}(\mathbf{x},t),\ F_{\mu\nu}(\mathbf{y},t)]=0 \ . \tag{2.6}$$

We note that Eq. (2.6) would not be true if $j_{\mu 5}^{\text{sym}}$ is replaced by $j_{\mu 5}^{\text{inv}}$, owing to anomalous commutators. To lowest order

$$\begin{aligned}&[j_{05}^{\text{inv}}(\mathbf{x},t),\ \mathbf{B}(\mathbf{y},t)]=0 \ ,\\ &[j_{05}^{\text{inv}}(\mathbf{x},t),\ \mathbf{E}(\mathbf{y},t)]=\frac{ie}{2\pi^2}\mathbf{B}(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y}) \ .\end{aligned} \tag{2.7}$$

Also,

$$[j_0(\mathbf{x},t),\ j_{05}^{\text{inv}}(\mathbf{y},t)]=-\frac{ie}{2\pi^2}\mathbf{B}(\mathbf{x},t)\cdot\nabla_x\delta(\mathbf{x}-\mathbf{y}) \ , \tag{2.8}$$

which is expected to be valid to all orders.

For a gauge-invariant operator O^{inv}, the chirality n_O is also invariant, since

$$\begin{aligned}\delta_h\int d\mathbf{x}[j_{05}^{\text{sym}}(\mathbf{x},t),\ O^{\text{inv}}(\mathbf{y},t)]&=\frac{e^2}{4\pi^2}\int d\mathbf{x}[\mathbf{B}(\mathbf{x},t),\ O^{\text{inv}}(\mathbf{y},t)]\cdot\nabla_x h(\mathbf{x},t)\\ &=\lim_{R\to\infty}\frac{e^2}{4\pi^2}\int_{|\mathbf{x}|=R}d\mathbf{S}\cdot[\mathbf{B}(\mathbf{x},t),\ O^{\text{inv}}(\mathbf{y},t)]h(\mathbf{x},t)=0 \ ,\end{aligned} \tag{2.9}$$

where we have used locality for gauge-invariant operators. In particular, the integrated charge itself (if it exists)

$$Q_5^{\text{sym}}=\int d\mathbf{x}\, j_{05}^{\text{sym}}(\mathbf{x}) \tag{2.10}$$

is invariant under proper gauge transformations with $\Lambda(\infty)=0$. Hence, we may meaningfully speak of a spontaneous breakdown of the anomalous chiral symmetry, if a gauge-invariant operator with nonzero chirality acquires a nonvanishing vacuum expectation value.[53]

So far, we have discussed the ordinary case of QED without magnetic monopoles. If magnetic monopoles are included, the discussion above requires modification. One obvious place is (2.9) where we have used the Bianchi identity $\nabla\cdot\mathbf{B}=0$. A less obvious place is (2.8), which may be seen as follows. In the absence of monopoles, we may equally take the commutator as

$$[j_0(\mathbf{x},t),\ j_{05}^{\text{inv}}(\mathbf{y},t)]=-\frac{ie}{2\pi^2}\mathbf{B}(\mathbf{y},t)\cdot\nabla_x\delta(\mathbf{x}-\mathbf{y}) \tag{2.11}$$

since

$$\begin{aligned}\mathbf{B}(\mathbf{y},t)\cdot\nabla_x\delta(\mathbf{x}-\mathbf{y})&=\nabla_x\cdot[\mathbf{B}(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y})]\\ &=\nabla_x\cdot[\mathbf{B}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})]\\ &=\mathbf{B}(\mathbf{x},t)\cdot\nabla_x\delta(\mathbf{x}-\mathbf{y}) \ .\end{aligned} \tag{2.12}$$

However, (2.8) and (2.11) are no longer equivalent in the presence of a monopole, which leads to an ambiguity in extrapolating from the monopole-free case.

To determine the appropriate modifications to Eqs. (2.2)–(2.11), let us first consider the simplest case of a Dirac electron moving in the background field of a purely magnetic monopole.[23,24] (For simplicity, we take the monopole to be of unit strength.[54]) In this case, there is yet no distinction between

$$j_{\mu 5}^{\text{inv}}(x)=\lim_{x'\to x}\tfrac{1}{2}[\overline{\Psi}(x),\gamma_\mu\gamma_5\Psi(x')]\exp\left[ie\int_x^{x'}dx_\mu A_\mu\right] \tag{2.13}$$

and

$$j_{\mu 5}^{\rm sym}(x) = :\bar{\Psi}(x)\gamma_\mu\gamma_5\Psi(x): \tag{2.14}$$

since the anomalous divergence vanishes in a purely magnetic field.[55]

This statement may sound puzzling, since $j_{\mu 5}^{\rm inv}$ and $j_{\mu 5}^{\rm sym}$ are supposed to transform differently under gauge transformations. The puzzle is resolved however if we recall that the gauge field is taken as a fixed background, so that the ground state also must transform under (time-independent) gauge transformations. With a new background field $A'_\mu = A_\mu + \partial_\mu h$, the ground state is given by

$$|\ \rangle' = e^{iQ[h]}|\ \rangle , \tag{2.15}$$

with

$$Q[h] = \int d\mathbf{x}\, h(\mathbf{x}) j_0(\mathbf{x}) . \tag{2.16}$$

Furthermore, in this sense, it is the definition (2.14) for $j_{\mu 5}^{\rm sym}$ which is covariant, since

$$e^{iQ[h]} j_{\mu 5}^{\rm sym} e^{-iQ[h]}$$

is properly normal ordered with respect to the transformed ground state (2.15).

On the other hand, (2.8) suggests that the anomalous commutator is nonvanishing, even in the absence of an electric field. Again, this may seem surprising, in view of the close connection between the anomalous commutator and the anomalous divergence for the triangle diagram. However, the connection was derived in the presence of a photon where both electric and magnetic fields must be present; here, we are dealing with a static situation where electric and magnetic fields can exist separately.

Let us consider the commutator in more detail. Since we are working with a background field, the Schwinger term can only be a c number, and it is sufficient to evaluate

$$\langle [j_0(\mathbf{x},t),\ j_{05}(\mathbf{y},t)] \rangle .$$

As suggested by Callan, and explicitly shown in Appendix A, it is sufficient to consider only the lowest partial wave

$$\Psi(\mathbf{x},t) \to \chi(r,t)\eta(\Omega)/r , \tag{2.17}$$

$$i\frac{\partial}{\partial t}\chi(r,t) = -i\gamma_5\frac{\partial}{\partial r}\chi(r,t) , \tag{2.18}$$

$$\chi = \begin{bmatrix} R \\ L \end{bmatrix} , \quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \gamma_0 = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{2.19}$$

The vector and axial-vector currents then reduce to

$$j_0(\mathbf{x},t) \to \frac{e}{4\pi r^2}\rho_V(r,t) , \quad j_k(\mathbf{x},t) \to \frac{e\hat{\mathbf{x}}_k}{4\pi r^2}\rho_A(r,t) , \tag{2.20}$$

$$j_{05}(\mathbf{x},t) \to \frac{e}{4\pi r^2}\rho_A(r,t) , \quad j_{k5}(\mathbf{x},t) \to \frac{e\hat{\mathbf{x}}_k}{4\pi r^2}\rho_V(r,t) , \tag{2.21}$$

where

$$\rho_V(r,t) = :\chi^\dagger(r,t)\chi(r,t): ,$$
$$\rho_A(r,t) = :\chi^\dagger(r,t)\gamma_5\chi(r,t): . \tag{2.22}$$

As noted by Rubakov and Callan, the kinematical relation between the two currents is reminiscent of the Schwinger model.[50]

It turns out, however, that the normal ordering in (2.22) requires more specification. If we take the c-number solutions of

$$-i\gamma_5\frac{d}{dr}u_k(r) = ku_k(r) , \quad -i\gamma_5\frac{d}{dr}v_k(r) = -kv_k(r) \quad (k>0) , \tag{2.23}$$

the BC (Refs. 22–25 and 56)

$$R(0) = ie^{i\theta}L(0) \tag{2.24}$$

must be imposed to ensure that we have a complete orthonormal set $\{u_{k\theta}, v_{k\theta}\}$. Therefore, there exists a one-parameter family of ground states $|\theta\rangle$ defined by

$$\chi(r,t) = \frac{1}{\pi}\int_0^\infty dk [b_{k\theta}u_{k\theta}(r)e^{-ikt} + d_{k\theta}^\dagger v_{k\theta}(r)e^{ikt}] ,$$
$$b_{k\theta}|\theta\rangle = d_{k\theta}|\theta\rangle = 0 . \tag{2.25}$$

Using the formulas

$$\frac{i}{\pi}\int_0^\infty dx\, u_{k\theta}(r)u_{k\theta}^\dagger(r')e^{-ikt} = -\frac{1}{2\pi}\frac{(t-i0)1 + (r-r')\gamma_5}{(t-i0)^2-(r-r')^2} - \frac{1}{2\pi}\frac{(t-i0)\gamma_5 + (r+r')1}{(t-i0)^2-(r+r')^2}\gamma_0 e^{-i\theta\gamma_5} , \tag{2.26}$$

$$\frac{i}{\pi}\int_0^\infty dx\, v_{k\theta}(r)v_{k\theta}^\dagger(r')e^{ikt} = \frac{1}{2\pi}\frac{(t+i0)1 + (r-r')\gamma_5}{(t+i0)^2-(r-r')^2} + \frac{1}{2\pi}\frac{(t+i0)\gamma_5 + (r+r')1}{(t+i0)^2-(r+r')^2}\gamma_0 e^{-i\theta\gamma_5} , \tag{2.27}$$

and

$$\delta(x)\frac{\mathscr{P}}{x} = -\tfrac{1}{2}\delta'(x) , \tag{2.28}$$

we find

$$\langle\theta|[\rho_V(r,t),\ \rho_A(r',t)]|\theta\rangle = -\frac{i}{\pi}\delta'(r-r') + \frac{i}{\pi}\delta'(r+r') . \tag{2.29}$$

There are two points which should be noted with the result. One is that we cannot drop the second term from the right-hand side: $\delta(r+r')$ can be set equal to zero on a half-line, but $\delta'(r+r')$ cannot. The other is that when we integrate the formula with test functions $f(r)$ and $g(r')$, at least one of them must vanish at infinity, owing to the slow falloff $\mathbf{B} = O(r^{-2})$.

To demonstrate these points, let us introduce a regularized δ function $\delta^{\rm reg}$ such that

$$\int_{-\infty}^\infty dx\, \delta^{\rm reg}(x) = 1 ,$$
$$\delta^{\rm reg}(x) = \delta^{\rm reg}(-x) , \quad \delta^{\rm reg}(\pm\infty) = 0 . \tag{2.30}$$

Then

$$\int_0^R dr \int_0^{R'} dr' f(r)g(r')[\delta^{\text{reg}'}(r-r')-\delta^{\text{reg}'}(r+r')] = f(R)g(R)\theta(R'-R) - \int_0^\infty dr\, \theta(R-r)\theta(R'-r)f'(r)g(r) \tag{2.31}$$

and therefore

$$[Q[f],\, Q_5[g]] = \frac{ie}{\pi}\int_0^\infty dr\, f'(r)g(r)\ ,\quad f(\infty)g(\infty)=0\ . \tag{2.32}$$

Taking $f=1$, we find that

$$[Q,\, Q_5[g]]=0\ ,\quad g(\infty)=0\ ; \tag{2.33}$$

i.e., electric charge is a chiral singlet. This is to be expected, since in the presence of massless fermions we should have

$$0=\langle\theta+2\alpha\,|\,Q\,|\,\theta+2\alpha\rangle = \lim_{g\to 1}\langle\theta\,|\,e^{-i\alpha Q_5[g]}Qe^{i\alpha Q_5[g]}\,|\,\theta\rangle = \langle\theta\,|\,Q\,|\,\theta\rangle\ . \tag{2.34}$$

On the other hand, taking $g=1$, we find

$$[Q[f],\, Q_5] = -\frac{ie}{\pi}f(0)\ ,\quad f(\infty)=0\ ; \tag{2.35}$$

i.e., the axial charge is no longer gauge invariant. Nevertheless, since it changes only by a c number, the chirality of a gauge-invariant operator remains well defined. From (2.26) and (2.27), it is, in fact, easy to show that

$$[Q_5, \overline{\Psi}(1\pm\gamma_5)\Psi] = \mp 2\overline{\Psi}(1\pm\gamma_5)\Psi\ , \tag{2.36}$$

$$\langle\theta\,|\,\tfrac{1}{2}[\overline{\Psi}(\mathbf{x},t),(1\pm\gamma_5)\Psi(\mathbf{x},t)]\,|\,\theta\rangle = -\frac{1}{8\pi^2 r^3}e^{\pm i\theta}\ . \tag{2.37}$$

(As before, only the lowest partial wave is involved here, since the higher partial waves conserve chirality.)

Equations (2.36) and (2.37) indicate that, in a sense, the explicit chirality breaking (2.24) at the first-quantized level can be regarded as a spontaneous breaking of chiral symmetry at the second-quantized level. Other aspects which also lend support to this view were already discussed[23] in (I): the degeneracy of the ground states $|\theta\rangle$, their physical equivalence, and unitary inequivalence.

Here, we wish to investigate a point which was not discussed adequately: namely, the existence of Nambu-Goldstone modes. To this end, we introduce the bosonic representation[50,57–60]

$$\chi(r,t) = \left[\frac{c\mu}{2\pi}\right]^{1/2} \begin{bmatrix} i{:}\exp\left[i\sqrt{\pi}\left[\int_r^\infty ds\, e^{-\mu s/2}\dot\phi(s,t)+\phi(r,t)+\dfrac{\theta}{2\sqrt{\pi}}\right]\right]{:} \\ {:}\exp\left[i\sqrt{\pi}\left[\int_r^\infty ds\, e^{-\mu s/2}\dot\phi(s,t)-\phi(r,t)-\dfrac{\theta}{2\sqrt{\pi}}\right]\right]{:} \end{bmatrix}, \tag{2.38}$$

where μ is an infinitesimal mass and $c=\ln\Gamma'(1)$. If $\phi(r,t)$ is a massless free field on the half-line

$$\ddot\phi(r,t)-\phi''(r,t)=0\quad (r>0)\ , \tag{2.39}$$

obeying the BC

$$\phi(0,t)=0\ , \tag{2.40}$$

it is easily seen that $\chi(r,t)$ formally satisfies Eqs. (2.18) and (2.24).

However, since the notion of a field operator at a point is actually ill defined, the proper procedure would be to calculate the propagators. Using the standard formulas

$$e^{A+B}=e^A e^B e^{-[A,B]/2}\ ,\quad [A,B]=c \text{ number}\ , \tag{2.41}$$

$${:}\exp A{:}=\exp(A^{(-)})\exp(A^{(+)})\ , \tag{2.42}$$

and the two-point function

$$4\pi\langle\phi(r,t)\phi(r',0)\rangle = 4\pi[\phi^{(+)}(r,t),\ \phi^{(-)}(r',0)] = \ln i(t-i0+r+r')+\ln i(t-i0-r-r')-\ln i(t-i0+r-r')-\ln i(t-i0-r+r')\ , \tag{2.43}$$

we may check that (2.38) in fact does reproduce (2.26) and (2.27). Similarly, we find

$$\rho_V(r,t) = -\dot\phi(r,t)/\sqrt{\pi}\ ,\qquad \rho_A(r,t)=\phi'(r,t)/\sqrt{\pi}\ , \tag{2.44}$$

in accordance with the Schwinger term (2.29).

Comparison of (2.50) with (2.21) shows that $\phi(r,t)$ has gradient coupling to the axial current; i.e., it may be identified as the Nambu-Goldstone mode associated with the breaking of chiral symmetry (2.37). We should note that the BC (2.40) is of some importance in this respect. First of all, it guarantees that the two-point function (2.43) is infrared finite, so that the standard theorems[61] against Nambu-Goldstone modes in 1 + 1 dimensions do not apply. Furthermore, it implies that the transformation $\phi(r,t)\to\phi(r,t)+\theta$ corresponding to a global chiral rotation cannot be implemented in the same Hilbert space. Finally, the BC also guarantees that $\Phi(\mathbf{x},t)=\phi(r,t)/r$ is a free field in the (3 + 1)-dimensional sense; in particular, $\partial_\mu\Phi$ is conserved even at the origin.

It may seem bizarre that fermions should form a Nambu-Goldstone boson when there is obviously no interaction between them. The physics, however, is quite simple. Owing to the Lorentz force, the charged fermion is trapped around the radial lines of force emanating from the monopole, but is free to move along it. Hence the situation is effectively 1 + 1 dimensional, in which case a massless fermion-antifermion pair is indistinguishable from a massless boson.[5,62]

To investigate this point more closely, we may Fourier analyze the scalar field

$$\phi(r,t)=\frac{1}{\pi}\int_0^\infty \frac{dq}{\sqrt{q}}\sin qr(a_q e^{-iqt}+a_q^\dagger e^{iqt}) , \qquad (2.45)$$

where $a_q^\dagger$ and a_q are the usual creation and annihilation operators normalized to

$$[a_q,a_{q'}^\dagger]=\pi\delta(q-q') . \qquad (2.46)$$

We then find from (2.44) that

$$a_q=\frac{1}{\sqrt{\pi q}}\left[\int_0^q dk\, d_k b_{q-k}+\int_0^\infty dk(b_k^\dagger b_{k+q}-d_k^\dagger d_{k+q})\right] , \qquad (2.47)$$

which may be directly checked to be consistent with the canonical (anti)commutation relations and with

$$\mathscr{H}_0=\frac{1}{\pi}\int_0^\infty dk\, k(b_k^\dagger b_k+d_k^\dagger d_k)$$
$$=\frac{1}{\pi}\int_0^\infty dq\, q a_q^\dagger a_q . \qquad (2.48)$$

It is evident from (2.47) that the bosonic states form a subspace of the fermion Fock space. On the other hand, for a one-fermion state $|\,k\rangle$,

$$\left\langle k'\left|\frac{1}{\pi}\int_0^\infty dq\, a_q^\dagger a_q\right|k\right\rangle=\pi\delta(k-k')\int_0^k\frac{dq}{q} , \qquad (2.49)$$

so the total number of bosons is infinite.[63]

There is one further point worth mentioning when we identify ϕ as a Nambu-Goldstone mode. Instead of (2.38), we could also have bosonized as

$$\chi(r,t)=\left[\frac{c\mu}{2\pi}\right]^{1/2}\begin{bmatrix} i{:}\exp\left[i\sqrt{\pi}\left[\phi(r,t)-\int_0^r ds\,\dot\phi(s,t)+\frac{\theta}{2\sqrt{\pi}}\right]\right]{:} \\ {:}\exp\left[i\sqrt{\pi}\left[\phi(r,t)+\int_0^r ds\,\dot\phi(s,t)-\frac{\theta}{2\sqrt{\pi}}\right]\right]{:}\end{bmatrix} \qquad (2.50)$$

provided we choose a different BC for the scalar field

$$\phi'(0,t)=0 . \qquad (2.51)$$

For the currents, we would then find

$$\rho_V(r,t)=-\dot\phi(r,t)/\sqrt{\pi} ,$$
$$\rho_A(r,t)=\phi'(r,t)/\sqrt{\pi} , \qquad (2.52)$$

indicating that the vector current is associated with a Nambu-Goldstone mode. The situation, however, is not as bad as it seems to be. One reason is that with the BC (2.51), the propagator for the scalar field is now infrared divergent, and the field operator is ill defined as in the case of a full line.[61] If we use μ as an infrared regulator, $\langle\chi\chi\rangle$ is, in fact, nonvanishing, being of order μ. Another reason is that again because of the BC (2.51), $\phi(r,t)/r$ is no longer a massless field in the (3 + 1)-dimensional sense. In other words, (2.40) as well as (2.44) are necessary to establish the interpretation of ϕ as a Nambu-Goldstone mode.

Let us now proceed to the case where an electrostatic potential is present in addition to the background monopole field. If the potential is spherically symmetric, the equation for the lowest partial wave

$$\left[-i\gamma_5\frac{d}{dr}+eA_0(r)\right]\tilde u_k(r)=k\tilde u_k(r) \qquad (2.53)$$

is easily solved:

$$\tilde u_{k\theta}(r)=\exp\left[-ie\gamma_5\int_0^r ds\, A_0(s)\right]u_{k\theta}(r) \qquad (2.54)$$

(and similarly for $\tilde v$). From (2.26) and (2.27) it follows that $j_{\mu5}^{\rm inv}$ and $j_{\mu5}^{\rm sym}$ as defined by (2.13) and (2.14) now become unequal:

$$j_{05}^{\rm inv}=j_{05}^{\rm sym} , \quad j_{k5}^{\rm inv}=j_{k5}^{\rm sym}-\frac{e\hat{\mathbf{x}}_k}{4\pi^2 r^2}A_0 . \qquad (2.55)$$

On the other hand, the anomalous commutator (2.26) and the order parameter[13] (2.37) will remain unchanged.

We may regard (2.55) as the analog of the London equations.[64] According to (2.54), A_0 does not affect the energy levels, but will change the (chiral) phase of the wave functions; i.e., its effect is magnetic rather than electric. Further taking the divergence, we also find[55]

$$\partial_\mu(j_{\mu5}^{\rm inv}-j_{\mu5}^{\rm sym})=-\frac{e}{4\pi^2 r^2}A_0'(r)-\frac{e}{\pi}A_0(0)\delta(\mathbf{x}) , \qquad (2.56)$$

The first term is just the familiar $F_{\mu\nu}\tilde F_{\mu\nu}$, whereas the second term is the extra contribution found by Nair[36] (apart from a discrepancy in the coefficient). Again, the apparent noninvariance of (2.56) may be explained by the fact that the ground state is only covariant under gauge transformations of the background field. (See, however, Sec. IV.)

So far, our examples have been rather trivial dynamically, although (hopefully) instructive. In particular, the anomaly was at most a c number. Let us now consider a case where it becomes a q number. It is sufficient to take the Hamiltonian[23]

$$\mathscr{H}=\int d\mathbf{x}{:}\Psi^\dagger\gamma_5\boldsymbol{\sigma}\cdot(-i\nabla-e\mathbf{A})\Psi{:} + \tfrac{1}{2}\int d\mathbf{x}\int d\mathbf{y}\, j_0(\mathbf{x})\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}j_0(\mathbf{y}) , \tag{2.57}$$

where :: indicates normal ordering with respect to the decomposition (2.25) at some fixed time. (We have suppressed the θ dependence.) As before,

$$[j_0(\mathbf{x},t),\ {:}\Psi^\dagger(\mathbf{y},t)\gamma_5\Psi(\mathbf{y},t){:}] = \frac{-e}{4\pi^2 r^2 r'^2}(\Omega,\Omega')[\delta'(r-r')-\delta'(r+r')] \tag{2.58}$$

and

$$i[\mathscr{H}-\mathscr{H}_0,\ {:}\Psi^\dagger\gamma_5\Psi{:}] = -\frac{e}{4\pi^2 r^2}\frac{\partial A_0}{\partial r} ,$$
$$A_0(\mathbf{x})=\int\frac{d\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}j_0(\mathbf{y}) , \tag{2.59}$$

since A_0 vanishes at spatial infinity. As expected for a positive-definite Hilbert space, we find that $j_{\mu 5}^{\rm sym}$ becomes nonlocal

$$j_{05}^{\rm sym}={:}\Psi^\dagger\gamma_5\Psi{:} ,$$
$$j_{k5}^{\rm sym}={:}\Psi^\dagger\sigma_k\Psi{:}+\frac{e^2}{2\pi}\epsilon_{ijk}A_i\nabla_j A_0 . \tag{2.60}$$

The dynamics being nontrivial, it is no longer possible to solve the system exactly. However, the similarity of (2.57) with the electron plasma suggests the use of the random-phase approximation. We first expand the field operator as before

$$\Psi(\mathbf{x})=\frac{1}{\pi}\int_0^\infty dk\sum_{\alpha jm}[b_{kjm}^{(\alpha)}u_{kjm}^{(\alpha)}(\mathbf{x})+d_{kj-m}^{(\alpha)\dagger}v_{kjm}^{(\alpha)}(\mathbf{x})] . \tag{2.61}$$

(See Appendix A for details of the notation.) For fermion bilinears, the equations of motion read as

$$[\mathscr{H},\ b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\beta)}]=(k-k')b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\beta)}+\tfrac{1}{2}\int d\mathbf{x}\{eA_0(\mathbf{x}),\ \Psi^\dagger(\mathbf{x})u_{kjm}^{(\alpha)}(\mathbf{x})b_{k'jm}^{(\beta)}-b_{kjm}^{(\alpha)\dagger}u_{k'jm}^{(\beta)\dagger}(\mathbf{x})\Psi(\mathbf{x})\} , \tag{2.62}$$

$$[\mathscr{H},\ d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\beta)}]=(k-k')d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\beta)}+\tfrac{1}{2}\int d\mathbf{x}\{eA_0(\mathbf{x}),\ v_{kjm}^{(\alpha)\dagger}(\mathbf{x})\Psi(\mathbf{x})d_{k'jm}^{(\beta)}-d_{kjm}^{(\alpha)\dagger}\Psi^\dagger(\mathbf{x})v_{k'jm}^{(\beta)}(\mathbf{x})\} , \tag{2.63}$$

$$[\mathscr{H},\ b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\beta)\dagger}]=(k+k')b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\beta)\dagger}+\tfrac{1}{2}\int d\mathbf{x}\{eA_0(\mathbf{x}),\ \Psi^\dagger(\mathbf{x})u_{kjm}^{(\alpha)}(\mathbf{x})d_{k'jm}^{(\beta)\dagger}-b_{kjm}^{(\alpha)\dagger}v_{k'jm}^{(\beta)\dagger}(\mathbf{x})\Psi(\mathbf{x})\} . \tag{2.64}$$

Replacing $\Psi^\dagger(x)d^\dagger$, etc., by their expectation values, and summing over the indices m and α,

$$\left[\mathscr{H},\ \sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\alpha)}\right]=(k-k')\sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\alpha)} , \tag{2.65}$$

$$\left[\mathscr{H},\ \sum_{\alpha m}d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)}\right]=(k-k')\sum_{\alpha m}d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)} , \tag{2.66}$$

$$\left[\mathscr{H},\ \sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)\dagger}\right]=(k+k')\sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)\dagger}+e\int_0^\infty dr\,r^2\int_0^\infty dr'\int d\Omega\, j_0(\mathbf{x})\frac{2j+1}{4\pi r_>}\sum_\alpha v_{k'j}^{(\alpha)\dagger}(r')u_{kj}^{(\alpha)}(r') , \tag{2.67}$$

where $r_>=\max\{r,r'\}$ and

$$\sum_m v_{k'jm}^{(\beta)\dagger}(\mathbf{x})u_{kjm}^{(\alpha)}(\mathbf{x})=\frac{2j+1}{4\pi r^2}\delta^{\alpha\beta}v_{k'j}^{(\alpha)\dagger}(r)u_{kj}^{(\alpha)}(r) . \tag{2.68}$$

Equations (2.65)—(2.67) form a closed eigenvalue problem, since

$$r^2\int d\Omega\, j_0(\mathbf{x})=\frac{e}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\sum_{\alpha jm}\{b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\alpha)}u_{kj}^{(\alpha)\dagger}(r)u_{k'j}^{(\alpha)}(r)-d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)}v_{k'j}^{(\alpha)\dagger}(r)v_{kj}^{(\alpha)}(r)$$
$$+[b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)\dagger}u_{kj}^{(\alpha)\dagger}(r)v_{k'j}^{(\alpha)}(r)+\text{H.c.}]\} . \tag{2.69}$$

The kernel is non-Hermitian; however, since we do not expect spurious zero modes, there should still exist a basis $\{O_q\}$ such that[65]

$$[\mathscr{H},O_q]=-qO_q , \tag{2.70}$$

$$\sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)\dagger}=\sum_q c_j(q,k,k')O_q , \tag{2.71}$$

$$\sum_{\alpha m}b_{kjm}^{(\alpha)\dagger}b_{k'jm}^{(\alpha)} ,\ \sum_{\alpha m}d_{kjm}^{(\alpha)\dagger}d_{k'jm}^{(\alpha)}\propto O_{k'-k} . \tag{2.72}$$

If all the eigenvalues q are real, a modified ground state may be introduced by

$$O_q|\ \rangle = O^\dagger_{-q}|\ \rangle = 0 \quad (q>0) , \tag{2.73}$$

which is stable within the approximation; otherwise, the original ground state is unstable.

To obtain some insight into the equations, we further throw away all the terms from the higher partial waves, which leaves us with

$$[\mathscr{H}, b_k^\dagger d_{k'}^\dagger] = (k+k')b_k^\dagger d_{k'}^\dagger + \frac{e^2}{4\pi}\int_0^\infty dr \int_0^\infty dr' \rho_V(r)\frac{1}{r_>}\cos(k+k')r' , \tag{2.74}$$

$$\rho_V(r) = \frac{1}{\pi^2}\int_0^\infty dk \int_0^\infty dk' [(b_k^\dagger b_{k'} - d_k^\dagger d_{k'})\cos(k-k')r + (b_k^\dagger d_{k'}^\dagger + d_{k'} b_k)\cos(k+k')r] . \tag{2.75}$$

In particular, for a_q defined by (2.47)

$$[\mathscr{H}, a_q^\dagger] = q a_q^\dagger + \frac{e^2}{4\pi}\left[\frac{q}{\pi}\right]^{1/2}\int_0^\infty dr \int_0^\infty dr' \rho_V(r)\frac{1}{r_>}\cos qr' . \tag{2.76}$$

However, this is just what we would get from the replacement (2.21) and (2.40) and (2.44)

$$\mathscr{H} \to -i\int_0^\infty dr\, \chi^\dagger(r)\gamma_5 \frac{d}{dr}\chi(r) + \frac{e^2}{8\pi}\int_0^\infty dr \int_0^\infty dr' \rho_V(r)\frac{1}{r_>}\rho_V(r')$$

$$= \tfrac{1}{2}\int_0^\infty dr [(\dot\phi)^2 + (\phi')^2] + \frac{e^2}{8\pi^2}\int_0^\infty \frac{dr}{r^2}\phi^2 . \tag{2.77}$$

The reduced Hamiltonian (2.77) has already been discussed extensively in the literature beginning with Rubakov and Callan, and we may immediately write down the solution for the boson fields[66]

$$\phi(r,t) = \left[\frac{r}{2\pi}\right]^{1/2}\int_0^\infty dq\, J_\nu(qr)(O_q e^{-iqt} + O_{-q}e^{iqt}) , \quad \nu = \left[\frac{1}{4} + \frac{e^2}{8\pi^2}\right]^{1/2} , \tag{2.78}$$

$$[\mathscr{H}, O_q] = -qO_q , \quad [O_q, O_{q'}] = \pi\delta(q+q') . \tag{2.79}$$

Evidently, the Nambu-Goldstone modes persist in the presence of a dynamical (q-number) anomaly, at least in the s-wave approximation.

It is trivial to extend the analysis to two or more species, provided the BC at the origin conserves each species separately. The result is, however, of importance, since it indicates that there is a Nambu-Goldstone mode associated with the nonanomalous axial current $j_{\mu 5}^{(-)} = j_{\mu 5}^{(1)} - j_{\mu 5}^{(2)}$, as well as another one associated with the anomalous current $j_{\mu 5}^{(+)} = j_{\mu 5}^{(1)} + j_{\mu 5}^{(2)}$. In fact, in Sec. III, we suggest that this is also what happens to two doublets of fermions interacting with a non-Abelian monopole.

III. THE NON-ABELIAN CASE

As for the chiral anomaly in the absence of a monopole, there is not much difference with the Abelian case for the first few equations. If we take the case of SU(2) with one massless (Dirac) isodoublet, Eqs. (2.1)–(2.3) are replaced by

$$\delta_h A_\mu = \partial_\mu h - ie[A_\mu, h] , \quad \delta_h \Psi = ieh\Psi , \tag{3.1}$$

$$\delta_h j_{\mu 5}^{\rm inv} = 0 , \quad \partial_\mu j_{\mu 5}^{\rm inv} = \frac{e^2}{16\pi^2}[F_{\mu\nu}^a \widetilde{F}^a_{\mu\nu}] , \tag{3.2}$$

$$\delta_h j_{\mu 5}^{\rm sym} = -\frac{e^2}{8\pi^2}\epsilon_{\mu\nu\alpha\beta}\partial_\nu(h^a \partial_\alpha A_\beta^a) , \quad \partial_\mu j_{\mu 5}^{\rm sym} = 0 , \tag{3.3}$$

where

$$h = h^a \frac{\tau^a}{2} , \quad A_\mu = A_\mu^a \frac{\tau^a}{2} \tag{3.4}$$

are SU(2) matrices.

In terms of the charges

$$Q_5^{\rm sym} = Q_5^{\rm inv} + 2W[A] , \tag{3.5}$$

where $W[A]$ is the Chern-Simons secondary invariant[10,67]

$$-\frac{e^2}{32\pi^2}\int d\mathbf{x}\, \epsilon_{ijk}\left[F_{ij}^a A_k^a - \frac{e}{3}\epsilon_{abc}A_i^a A_j^h A_k^c\right] . \tag{3.6}$$

Again, it is $Q_5^{\rm sym}$ which defines the operator chiralities as in (2.4).

The difference between the Abelian and non-Abelian case, however, shows up in the fact that there are proper gauge transformations

$$e^{ieh(\mathbf{x})} \to 1 , \quad |\mathbf{x}| \to \infty , \tag{3.7}$$

which are topologically nontrivial[68] ("large") as well as those which are topologically trivial ("small"). An example of the former is given by

$$G = e^{2\pi i \hat{\mathbf{x}}\cdot\boldsymbol{\tau}\mathscr{F}(r)} \tag{3.8}$$

with

$$\mathscr{F}(0) = 0 , \quad \mathscr{F}(\infty) = 1 . \tag{3.9}$$

Under a gauge transformation, the Chern-Simons term transforms as

$$W[A^G]=W[A]+w[G] - \frac{ie}{8\pi^2}\int d\mathbf{x}\,\epsilon_{ijk}\mathrm{tr}\nabla_i(G^{-1}A_j\nabla_k G)\ , \tag{3.10}$$

where the second term is the winding number of the transformation

$$w[G]=\frac{1}{24\pi^2}\int d\mathbf{x}\,\epsilon_{ijk}\mathrm{tr}(G^{-1}\nabla_i G)(G^{-1}\nabla_j G)(G^{-1}\nabla_k G)\ . \tag{3.11}$$

For (3.8) and (3.9), $w[G]=2$. On the other hand, the third term will vanish under mild regularity conditions such as

$$A_k(\mathbf{x})=O(r^{-1})\ ,\quad G=1+O(r^{-\eta})\ ,\quad \nabla_k G=O(r^{-1-\eta}) \quad (\eta>0)\ . \tag{3.12}$$

We find that $Q_5^{\rm sym}$ is no longer invariant under proper gauge transformations if they are "large":

$$Q_5^{\rm sym}\rightarrow Q_5^{\rm sym}+2w[G]\ . \tag{3.13}$$

Nevertheless, since $w[G]$ is again only a c number, the chirality of a gauge-invariant operator remains well defined, and we may still meaningfully talk of a spontaneous breaking of the anomalous chiral symmetry if

$$\langle O^{\rm inv}\rangle\neq 0\ ,\quad n_O\neq 0\ . \tag{3.14}$$

Furthermore, if we demand that the physical vacuum $|\theta\rangle$ is invariant under "large" transformations up to a phase[69]

$$|\theta\rangle\rightarrow e^{-iw[G]\theta}|\theta\rangle\ , \tag{3.15}$$

it follows from (3.13) that $Q_5^{\rm sym}$ cannot annihilate the vacuum. Therefore, one expects that chiral symmetry is in fact spontaneously broken in the sense of (3.14), a conclusion confirmed by an instanton calculation.[17,68] (See, however, Sec. IV and Appendix B.)

In the presence of a monopole, we find that the same arguments go through without any essential changes, unlike the Abelian case. At the classical level, the monopole (dyon) solution has the asymptotic form[70–72]

$$eA_0(\mathbf{x})=(\hat{\mathbf{x}}\cdot\tau)\frac{J(r)}{r}\sim\tfrac{1}{2}(\hat{\mathbf{x}}\cdot\tau)\times\text{const}\ , \tag{3.16}$$

$$e\,\mathbf{A}(\mathbf{x})=(\hat{\mathbf{x}}\times\tau)\frac{K(r)-1}{2r}\sim-\frac{1}{2r}(\hat{\mathbf{x}}\times\tau)\ , \tag{3.17}$$

$$e\phi(\mathbf{x})=(\hat{\mathbf{x}}\cdot\tau)\frac{H(r)}{2r}\sim\tfrac{1}{2}(\hat{\mathbf{x}}\cdot\tau)\times\text{const}\ . \tag{3.18}$$

Explicitly, in the Prasad-Sommerfield limit,[72]

$$\begin{aligned} J(r)&=\sinh\gamma(m_W r\coth m_W r-1)\ ,\\ K(r)&=m_W r/\sinh m_W r\ ,\\ H(r)&=\cosh\gamma(m_W r\coth m_W r-1)\ . \end{aligned} \tag{3.19}$$

Evidently, (3.17) satisfies the regularity condition (3.12).

We may also consider the Dirac equation with (3.16)–(3.18) as a background field.[21–25,73,74] Again, we find a difference with the Abelian case: Chirality is unbroken for all partial waves, implying that the chiral charge $Q_5^{\rm sym}$ is well defined, conserved, and annihilates the ground state. [As before, the ground state is only covariant in the sense of Sec. II, so there is no inconsistency with (3.13).]

In particular, for the lowest partial wave, we find

$$i\frac{\partial}{\partial t}\chi(r,t)=\left[-i\gamma_5\tau_3\frac{\partial}{\partial r}-\gamma_5\tau_2\frac{K(r)}{r}+\frac{\tau_3}{2}\frac{J(r)}{r}\right]\chi(r,t)\ , \tag{3.20}$$

where

$$\chi=\begin{bmatrix}\chi_+\\ \chi_-\end{bmatrix},\quad \tau_3=\begin{bmatrix}1_2 & \\ & -1_2\end{bmatrix},\quad \chi_\pm=\begin{bmatrix}R_\pm(r)\\ L_\pm(r)\end{bmatrix},\quad \gamma_5=\begin{bmatrix}1 & \\ & -1\end{bmatrix}. \tag{3.21}$$

After eliminating the electrostatic potential

$$\chi(r)\rightarrow\exp\left[\frac{-i\gamma_5}{2}\int_0^r\frac{dr'}{r'}J(r')\right]\chi(r) \tag{3.22}$$

explicit solutions can be found[22,74] in the Prasad-Sommerfield limit (3.19)

$$R_\pm(r)=\frac{m_W}{(8k^2+2m_W{}^2)^{1/2}}\left[-\frac{e^{\mp ikr}}{\sinh m_W r}+\left[\mp\frac{2ik}{m_W}+\coth m_W r\right]e^{\pm ikr}\right]\ , \tag{3.23}$$

$$u_k^{(R)}(r)=\begin{bmatrix}R_+(r)\\0\\R_-(r)\\0\end{bmatrix},\quad u_k^{(L)}(r)=\begin{bmatrix}0\\R_+^*(r)\\0\\R_-^*(r)\end{bmatrix},\quad v_k^{(R)}(r)=\begin{bmatrix}R_+^*(r)\\0\\R_-^*(r)\\0\end{bmatrix},\quad v_k^{(L)}(r)=\begin{bmatrix}0\\R_+(r)\\0\\R_-(r)\end{bmatrix}. \tag{3.24}$$

Here we note one similarity with the Abelian case: Eq. (3.22) implies that

$$\langle j_{k5}^{\rm inv}\rangle\rightarrow-\frac{1}{4\pi^2r^2}\frac{d}{dr}\frac{J(r)}{r}\ . \tag{3.25}$$

Even then, however, comparison with

$$\sum_{i,a}E_i^aB_i^a=\frac{1}{e^2r^2}\frac{d}{dr}\frac{J(r)}{r}[K^2(r)-1] \tag{3.26}$$

shows that the anomalous divergence is no longer saturated by the lowest partial wave, so that the higher partial waves must contribute near the core. In other words, the conservation law (3.3) for $j_{\mu5}^{\rm sym}$ admits a consistent reduction to the lowest partial wave, but that for $j_{\mu 5}^{\rm inv}$ does not. (See, however, Appendix C.)

Equations (3.16)—(3.26) above essentially constitute the first two terms in the semiclassical expansion.[52] However, it is not satisfactory on various accounts. One is that it fails to incorporate charge conservation. In particular, in the point limit $r_0 = m_W{}^{-1} \to 0$, the solutions obey

$$i\frac{\partial}{\partial t}\chi(r,t) = \left[-i\gamma_5\tau_3\frac{\partial}{\partial r} + \frac{\tau_3}{2} \right]\chi(r,t) \tag{3.27}$$

with the charge-mixing BC's

$$R_+(0,t) = R_-(0,t)\ ,\quad L_+(0,t) = L_-(0,t)\ , \tag{3.28}$$

and there is a constant stream of charge flowing into the origin.[75] Another unsatisfactory aspect is that the anomaly features as a c number rather than a q number.

To remedy these shortcomings, it is further necessary to treat the electrostatic potential J as a dynamical variable. The proper treatment of the dyon degree of freedom φ in the s-wave approximation then leads to

$$\begin{aligned} \frac{J(r)}{r} &= -\mathcal{J}(r)\dot{\varphi} + \frac{e^2}{4\pi}\int_0^\infty dr'\,\mathcal{G}(r,r')\rho(r')\ , \\ \rho(r) &= \lim_{r'\to r}\tfrac{1}{2}\left[\chi^\dagger(r'), \frac{\tau_3}{2}\chi(r)\right] \end{aligned} \tag{3.29}$$

where $\mathcal{J}(r)$ is a homogeneous solution of Gauss's law

$$r\frac{d^2}{dr^2}r\mathcal{J}(r) - 2\mathcal{J}(r)K^2(r) = 0 \tag{3.30}$$

obeying the BC (3.9), and $\mathcal{G}(r,r')$ is the associated Green's function which vanishes as $r, r' \to \infty$. The corresponding Hamiltonian is given by[23,27]

$$\mathcal{H} = \mathcal{H}_F + \mathcal{H}_C + \mathcal{H}'_C\ , \tag{3.31}$$

$$\mathcal{H}_F = \int_0^\infty dr\,\chi^\dagger(r)\left[i\gamma_5\tau_3\frac{d}{dr} + \gamma_0 M - \gamma_5\tau_2\frac{K(r)}{r} \right]\chi(r)\ . \tag{3.32}$$

$$\mathcal{H}_C = \frac{I}{2}\dot{\varphi}^2 = \frac{1}{2I}\left[Q_{\rm tot} - \int_0^\infty dr\,\mathcal{J}(r)\rho(r) \right]^2\ , \tag{3.33}$$

$$\mathcal{H}'_C = \frac{e^2}{8\pi^2}\int_0^\infty dr\int_0^\infty dr'\rho(r)\mathcal{G}(r,r')\rho(r')\ , \tag{3.34}$$

where I is a positive constant of order r_0/e^2, and

$$Q_{\rm tot} = \frac{n}{2} - \frac{\bar{\theta}}{2\pi} = I\dot{\varphi} + \int_0^\infty dr\,\mathcal{J}(r)\rho(r) \tag{3.35}$$

is the total charge of the system for $M \neq 0$.[23,24,76]

For (3.31)—(3.34) to provide a sensible approximation, it is necessary that $e^2 \ll 1$ and that the relevant energies do not exceed m_W. In that case, we may as well neglect $\mathcal{H}'_C$, since it is of order e^2 and nonsingular as $r_0 \to 0$. This is also justified by the results of the previous section, since $\mathcal{H}'_C$ is just the self-energy of the fermions. On the other hand, unlike the Abelian case, the anomaly acquires a q-number character even without $\mathcal{H}'_C$, because of the dyon degree of freedom

$$i[\mathcal{H}_C, Q_5[h]] = \frac{\dot{\varphi}}{\pi}\int_0^\infty dr\,\mathcal{J}'(r)h(r)\ ,\quad h(\infty) = 0\ . \tag{3.36}$$

For $M = 0$, we also expect $Q_{\rm tot} = 0$, and hence the Hamiltonian will be

$$\int_0^\infty dr\,\chi^\dagger(r)\left[-i\gamma_5\tau_3\frac{d}{dr} - \gamma_5\tau_2\frac{K(r)}{r} \right]\chi(r) + \frac{1}{2I}\left[\int_0^\infty dr\,\mathcal{J}(r)\rho(r) \right]^2\ . \tag{3.37}$$

The quadratic term alone gives (3.20) as before, so we may take it to define the chirally symmetric ground state $|\,{\rm sym}\rangle$ by the usual Fock construction. The quartic term can be treated in the random-phase approximation to test for stability, but a simpler procedure is available in this case. We only need to compute

$$\langle {\rm sym}\,|\,\mathcal{H}_C\,|\,{\rm sym}\rangle = \frac{1}{8\pi^2 I}\int_0^\Lambda dk\int_0^\Lambda dk' \sum_{\alpha=R,L}\left| \int_0^\infty dr\,u_k^{(\alpha)\dagger}(r)\tau_3 v_{k'}^{(\alpha)}(r)\mathcal{J}(r) \right|^2\ . \tag{3.38}$$

The integral over r may be explicitly evaluated in the point limit (3.28) and (3.29) or the Prasad-Sommerfield limit (3.23) and (3.24), the latter which gives

$$\begin{aligned} \frac{i}{m_W}&\left[\frac{\pi^2}{4} + 2(\epsilon_+{}^2 + \epsilon_-{}^2) + (\epsilon_+{}^2 - \epsilon_-{}^2)^2 \right]^{-1/2} \left[\left[\frac{\pi^2}{4} + \epsilon_+{}^2 + \epsilon_-{}^2 \right]\coth\epsilon_+ + \left[\frac{\pi^2}{4} - \epsilon_+{}^2 + \epsilon_-{}^2 \right] \right. \\ &\left. + 2\epsilon_+ \ln\left[\frac{\sinh\epsilon_+}{\cosh\epsilon_-} \right] + 2\epsilon_+\epsilon_-\tanh\epsilon_- \right]\ ,\quad \epsilon_\pm = \frac{\pi(k \pm k')}{2m_W}\ . \end{aligned} \tag{3.39}$$

Therefore, as first noticed by Besson,[22] the integration over k is infrared divergent, and $|\,{\rm sym}\rangle$ is unstable for arbitrary weak coupling $e^2 > 0$.

As for the nature of the true ground state, there have been two proposals so far. One is that of our previous paper[23] (II), where fermionic BC's are effectively changed from (3.34), and the other is that of Callan[31] where bosonic BC's are effectively changed after the Hamiltonian is bosonized.

The reasoning behind our proposal is the following. If we split $\mathcal{H}_F$ as

$$\mathcal{H}_F^{\rm point} = \int_0^\infty dr\,\chi^\dagger(r)\left[-i\gamma_5\tau_3\frac{d}{dr} + \gamma_0 M \right]\chi(r)\ , \tag{3.40}$$

$$\mathscr{H}_F^{\text{mix}} = -\int_0^\infty dr\, \chi^\dagger(r)\gamma_5\tau_2\chi(r)\frac{K(r)}{r}\,, \tag{3.41}$$

we find that $\mathscr{H}_C$ is formally of order m_W, $\mathscr{H}_F^{\text{point}}$ of order 1, and $\mathscr{H}_F^{\text{mix}}$ of order $m_W{}^{-1}$ but singular. Therefore in the limit $m_W \to \infty$, we may expect

$$\left\langle \left| \left[Q_{\text{tot}} - \int_0^\infty dr\, \rho(r) \right]^2 \right| \right\rangle = 0\,, \tag{3.42}$$

or equivalently

$$\int_0^\infty dr\, \rho(r) \,|\ \rangle = Q_{\text{tot}}\,|\ \rangle\,, \tag{3.43}$$

which requires a charge-conserving BC for $\mathscr{H}_F^{\text{point}}$. Compatibility with the singularity of $K(r)/r$ further imposes

$$\chi^\dagger(0)\gamma_5\tau_2\chi(0) = 0\,, \tag{3.44}$$

which leads to either

$$R_\pm(0) = e^{i\bar{\theta}}L_\pm(0)\,, \tag{3.45}$$

or

$$R_\pm(0) = -e^{i\bar{\theta}}L_\pm(0)\,. \tag{3.46}$$

The existence of two solutions (3.45) and (3.46) implies the breakdown of fermion-number conjugation

$$\chi \to \rho_2\tau_1\chi^*\,,\quad F \to -F\,. \tag{3.47}$$

On the other hand, for a Higgs-boson mass

$$\mathscr{H}_F = \int_0^\infty dr\, \chi^\dagger(r) \left[-i\gamma_5\tau_3\frac{d}{dr} - \gamma_5\tau_2\frac{K(r)}{r} - \gamma_0\tau_3 M \right] \chi(r)\,, \tag{3.48}$$

we find only one stable solution

$$R_\pm(0) = -ie^{i\bar{\theta}}L_\pm(0)\,, \tag{3.49}$$

and fermion-number conjugation

$$\chi \to \gamma_0\tau_1\chi^*\,,\quad F \to -F \tag{3.50}$$

is unbroken. Also, because of the changed BC's we find that there is no zero mode[52,77] within the region of stability $|Q_{\text{tot}}| < \frac{1}{4}$.

Let us now turn to the other proposal. Callan observes that after bosonization, the currents are linear in the scalar fields (2.44), and therefore proposes that the BC's should be imposed in terms of them. Furthermore, on the basis of the Dirac equation (3.20), it is argued that the charge-mixing BC (3.28) will still hold for the fermion fields [in spite of the nonlinearity (3.29) and (3.33)], and the bosonization is taken to be

$$R_\pm(r,t) = \left[\frac{c\mu}{2\pi} \right]^{1/2} :\exp\left[i\sqrt{\pi}\left[\phi_R(r,t) \mp \int_0^r ds\, \dot{\phi}_R(s,t) \right] \right]:\,, \tag{3.51}$$

$$L_\pm(r,t) = \left[\frac{c\mu}{2\pi} \right]^{1/2} (-1)^F :\exp\left[i\sqrt{\pi}\left[\phi_L(r,t) \pm \int_0^r ds\, \dot{\phi}_L(s,t) \right] \right]:\,, \tag{3.52}$$

with

$$\phi_R'(0,t) = \phi_L'(0,t) = 0\,. \tag{3.53}$$

The bilinears are then given by

$$\chi^\dagger\chi = -\dot{A}(r,t)/\sqrt{2\pi}\,,\quad \chi^\dagger\gamma_5\tau_3\chi = A'(r,t)/\sqrt{2\pi}\,, \tag{3.54}$$

$$\chi^\dagger\gamma_5\chi = -\dot{B}(r,t)/\sqrt{2\pi}\,,\quad \chi^\dagger\tau_3\chi = B'(r,t)/\sqrt{2\pi}\,, \tag{3.55}$$

where

$$\begin{aligned} \sqrt{2}A(r,t) &= \phi_R(r,t) + \phi_L(r,t)\,, \\ \sqrt{2}B(r,t) &= \phi_R(r,t) - \phi_L(r,t)\,. \end{aligned} \tag{3.56}$$

Equations (3.53)—(3.55) imply that fermion number and chirality are conserved, since the associated fluxes vanish at the origin (see Table I). On the other hand, the fermion electric charge is not conserved by itself, in conflict with (3.43). The resolution offered at this point is that the nonlinearity $\mathscr{H}_C$ here comes into play, so that the original BC (3.53) is modified to a new effective BC,

$$A'(0,t) = \dot{B}(0,t) = 0\,, \tag{3.57}$$

which conserves fermion number and fermion electric charge, but no longer chirality.

Callan's proposal appears to be *ad hoc* in its handling of the nonlinear interaction of the fermions. This we believe is not because of a false light on our part, but rather the effect of a clear-cut treatment of the dyon degree of freedom. Let us, however, postpone further discussion of this point to Sec. IV, and turn to a problem which is common to both.

TABLE I. Fermion bilinears.

Fermion-number density	$\Psi^\dagger\Psi$	$\chi^\dagger\chi$
Radial fermion-number current	$\Psi^\dagger\boldsymbol{\alpha}\cdot\hat{\mathbf{x}}\Psi$	$\chi^\dagger\gamma_5\tau_3\chi$
Axial charge density	$\Psi^\dagger\gamma_5\Psi$	$\chi^\dagger\gamma_5\chi$
Radial axial-vector current	$\Psi^\dagger\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}\Psi$	$\chi^\dagger\tau_3\chi$
Fermion electric charge density	$-e\Psi^\dagger\frac{\tau^a}{2}\Psi$	$e\hat{\mathbf{x}}_a\chi^\dagger\frac{\tau_3}{2}\chi$
Radial fermion electric current	$-e\Psi^\dagger\boldsymbol{\alpha}\cdot\hat{\mathbf{x}}\frac{\tau^a}{2}\Psi$	$e\hat{\mathbf{x}}_a\chi^\dagger\frac{\gamma_5}{2}\chi$

The manner in which we have described both proposals leaves room for ambiguity, since we had invoked the notion of an effective BC for a field operator. To recast the proposals in a more precise manner, we have found it useful to adopt a variational framework.

For our proposal, we first take the c-number solutions of the point-monopole equation

$$\left[-i\gamma_5\tau_3\frac{d}{dr}+\gamma_0 M\right]u_{kU}(r)=(k^2+M^2)^{1/2}u_{kU}(r)\ , \tag{3.58}$$

$$\left[-i\gamma_5\tau_3\frac{d}{dr}+\gamma_0 M\right]v_{kU}(r)=-(k^2+M^2)^{1/2}v_{kU}(r) \tag{3.59}$$

with the BC parametrized by a 2×2 unitary matrix

$$\begin{bmatrix}R_+(0)\\L_-(0)\end{bmatrix}=iU\begin{bmatrix}L_+(0)\\R_-(0)\end{bmatrix}\ ,\quad U^\dagger U=1\ . \tag{3.60}$$

The wave functions then form a complete orthonormal set for given U, and we may expand the field operator as[78]

$$\chi(r)=\frac{1}{\pi}\int_0^\infty dk[b_{kU}u_{kU}(r)+d_{kU}^\dagger v_{kU}(r)]\ . \tag{3.61}$$

Defining the ground states $|U\rangle$ as before by

$$b_{kU}|U\rangle=d_{kU}|U\rangle=0\ , \tag{3.62}$$

we find that $\langle U|\mathscr{H}|U\rangle$ is minimized when $m_W/M\to\infty$ with

$$U=\pm i\begin{bmatrix}e^{i\bar\theta} & 0\\ 0 & e^{-i\bar\theta}\end{bmatrix} \tag{3.63}$$

corresponding to (3.45) and (3.46).

So far, we have implicitly assumed that $M\neq 0$. For $M=0$, we find that the discontinuity in $Q_{\rm tot}$ cancels against the discontinuity in[23,24]

$$\left\langle U\left|\int_0^\infty dr\,\rho(r)\right|U\right\rangle \tag{3.64}$$

so the ground states are continuously connected as $M\to 0$. However, $\bar\theta$ in the BC (3.63) becomes arbitrary for $M=0$, and the ground states are invariant under fermion-number conjugation, after a suitable redefinition.

Also, if we bosonize the theory in the interaction picture defined by $\mathscr{H}_F^{\rm point}$ and (3.63) for $M=0$, we find two massless fields $\phi_\pm$ corresponding to χ_+ and χ_-. One linear combination $\phi_++\phi_-$ corresponds to a genuine Nambu-Goldstone mode associated with the breaking of the anomalous chiral symmetry, and the other $\phi_+-\phi_-$ is a pseudo-Nambu-Goldstone mode in the sense of Seo,[32] corresponding to $\chi\to e^{i\alpha\gamma_5\tau_3}\chi$ outside the core region.

At this point, one may be worried about the fact that we have been approximating regular expressions by singular ones, which do not have well-defined gauge transformation properties at the origin. To examine this point, we write the three-dimensional wave function ψ (suppressing spin-isospin factors) as

$$\psi(\mathbf{x})=\frac{r}{r^2+a^2}u_{kU}(r)\ . \tag{3.65}$$

We then find that there are no singular contributions to $\langle\mathscr{H}\rangle$ as $a\to 0$, so long as (3.44) and (3.60) are obeyed.

We may now turn to Callan's proposal, which may be reformulated in a similar manner.[79] Instead of expanding the field operators as

$$A(r,t)=\frac{1}{\pi}\int_0^\infty\frac{dq}{\sqrt{q}}\cos qr(c_q^A e^{-iqt}+c_q^{A\dagger}e^{iqt})\ , \tag{3.66}$$

$$B(r,t)=\frac{1}{\pi}\int_0^\infty\frac{dq}{\sqrt{q}}\cos qr(c_q^B e^{-iqt}+c_q^{B\dagger}e^{iqt})\ , \tag{3.67}$$

and taking essentially the previous $|{\rm sym}\rangle$ as the approximate (trial) ground state[80]

$$c_q^A|{\rm sym}\rangle=c_q^B|{\rm sym}\rangle=0\ , \tag{3.68}$$

we expand B as

$$B(r,t)=\frac{1}{\pi}\int_0^\infty\frac{dq}{\sqrt{q}}\sin qr(a_q^B e^{-iqt}+a_q^{B\dagger}e^{iqt})\ , \tag{3.69}$$

$$\frac{1}{\sqrt{k}}a_k^B=\frac{1}{\pi}\int_0^\infty\frac{dq}{\sqrt{q}}\left[\frac{\mathscr{P}}{k-q}c_q^B+\frac{\mathscr{P}}{k+q}c_q^{B\dagger}\right] \tag{3.70}$$

and take the new ground state as

$$c_q^A|{\rm mod}\rangle=a_q^B|{\rm mod}\rangle=0\ . \tag{3.71}$$

To distinguish whether we normal order with respect to $|{\rm sym}\rangle$ or $|{\rm mod}\rangle$, we shall also introduce the notation $:\,:_{\rm sym}$ and $:\,:_{\rm mod}$, e.g., the normal ordering in (3.51) and (3.52) would now be $:\,:_{\rm sym}$.

Using (2.42) and (2.43) we find

$$R_+=e^{i\pi/4}e^{i\pi Q_5/2}\xi_+^A\xi_+^B\ , \tag{3.72}$$

$$L_+=e^{-i\pi/4}e^{i\pi Q_5/2}(-1)^F\xi_-^A\xi_-^B\ ,$$

$$R_-=e^{-i\pi/4}e^{-i\pi Q_5/2}\xi_-^A\xi_-^{B\dagger}\ , \tag{3.73}$$

$$L_-=e^{i\pi/4}e^{-i\pi Q_5/2}(-1)^F\xi_+^A\xi_+^{B\dagger}\ ,$$

where

$$\xi_\pm^A(r,t)=\left[\frac{c\mu}{2\pi}\right]^{1/4}:\exp\left[i\left[\frac{\pi}{2}\right]^{1/2}\left[A(r,t)\mp\int_0^r ds\,\dot A(s,t)\right]\right]:_{\rm mod}\ , \tag{3.74}$$

$$\xi_\pm^B(r,t)=\left[\frac{c\mu}{2\pi}\right]^{1/4}:\exp\left[i\left[\frac{\pi}{2}\right]^{1/2}\left[\int_r^\infty ds\,e^{-\mu s/2}\dot B(s,t)\pm B(r,t)\right]\right]:_{\rm mod}\ , \tag{3.75}$$

are the "half-soliton" operators[81] which obey

$$\xi_\pm^A(r,t)\xi_\pm^A(r',t)=\mp i\,\mathrm{sgn}(r-r')\xi_\pm^A(r',t)\xi_\pm^A(r,t)\ , \quad (3.76)$$

$$\xi_\pm^A(r,t)\xi_\mp^A(r',t)=\mp i\xi_\mp^A(r',t)\xi_\pm^A(r,t)\ , \quad (3.77)$$

$$\xi_\pm^A(r,t)\xi_\pm^{A\dagger}(r',t)=\pm i\,\mathrm{sgn}(r-r')\xi_\pm^{A\dagger}(r',t)\xi_\pm^A(r,t)\ , \quad (3.78)$$

$$\xi_\pm^A(r,t)\xi_\mp^{A\dagger}(r',t)=\pm i\xi_\mp^{A\dagger}(r',t)\xi_\pm^A(r,t) \quad (3.79)$$

(and similarly for ξ^B).

According to (3.72) and (3.73), a massless fermion consists of two half-solitons. However, the compositeness turns out to be unobservable in this case. If we take the combinations

$$\chi_+=e^{i\theta\gamma_5/2}\begin{bmatrix} i\xi_+^A\xi_+^B \\ \xi_-^A\xi_-^B \end{bmatrix}\ ,\qquad \chi_-=e^{i\theta\gamma_5/2}\begin{bmatrix} i\xi_-^A\xi_-^{B\dagger} \\ \xi_+^A\xi_+^{B\dagger} \end{bmatrix}\ , \quad (3.80)$$

we find that χ gives the same propagators as the point-monopole equation (3.58) and (3.59) with the Abelian BC

$$R_\pm(0)=ie^{i\theta}L_\pm(0)\ . \quad (3.81)$$

Therefore, the difference between our procedure and Callan's only amounts to a Klein transformation, and the two are essentially equivalent, as found by Polchinski.[43]

Actually, there is one possible fly in the ointment, since the chiral charge

$$Q_5=-\left[\frac{2}{\pi}\right]^{1/2}\int_0^\infty dr\, e^{-\mu r/2}\dot{B}(r,t)$$
$$=\tfrac{1}{2}\int_0^\infty dr\, e^{-\mu r/2}[\chi^\dagger(r,t),\gamma_5\chi(r,t)] \quad (3.82)$$

which appears in the Klein factor tends to become ill defined in the limit $\mu\to 0$. Fortunately, however, this does not cause much harm. For a product of an even number of χ's, the Klein factor becomes $e^{\pm i\pi Q_5}$, which changes both $R_\pm$ and $L_\pm$ by -1, and hence does not affect the BC's or the Hilbert space.

We may now also see the reason for the confusion concerning fermion-number conjugation and zero modes (see also Appendix C). Previously, it was assumed that the original BC (3.28) for the fermions continues to be obeyed, even after the boson BC's have been changed from (3.53) to (3.57). Therefore, Callan[31] and Harvey[33] had argued that the Jackiw-Rebbi zero mode[52] persists for a Higgs-boson mass, and we had concluded that the two procedures lead to different results. As we have seen however, the correct conclusion is that the fermion BC is also effectively changed, and hence both procedures lead to the same result as given[23] in (II), i.e., fermion-number conjugation is broken for a Dirac mass, and there is no zero-energy state for a Higgs-boson mass within the region of stability. In particular, for the latter CP is unbroken for $Q_{\rm tot}=0$, and we may set $\bar{\theta}=\theta$, in agreement with Niemi, Paranjape, and Semenoff.[11]

Let us now consider the case of two Dirac isodoublets $\chi^{(1)}$ and $\chi^{(2)}$. The generalization of our proposal to this case is straightforward. The BC's are now parametrized by a 4×4 unitary matrix

$$\begin{bmatrix} R_+^{(1)}(0)\\ R_+^{(2)}(0)\\ L_-^{(1)}(0)\\ L_-^{(2)}(0)\end{bmatrix}=iU\begin{bmatrix} L_+^{(1)}(0)\\ L_+^{(2)}(0)\\ R_-^{(1)}(0)\\ R_-^{(2)}(0)\end{bmatrix}\ ,\quad U^\dagger U=1\ . \quad (3.83)$$

For $M=0$, we find that $\langle U\,|\,\mathscr{H}_C\,|\,U\rangle$ is infrared finite if and only if U is charge conserving

$$U=\begin{bmatrix} U_2 & 0\\ 0 & V_2\end{bmatrix}\ . \quad (3.84)$$

Compatibility with the singularity of $\mathscr{H}_F^{\rm mix}$ further requires

$$V_2=-U_2^\dagger\ . \quad (3.85)$$

Therefore, by a suitable redefinition of the fields

$$U_2=\begin{bmatrix} ie^{i\theta_1} & 0\\ 0 & ie^{i\theta_2}\end{bmatrix}\ . \quad (3.86)$$

Bosonization now leads to four fields $\phi_\pm^{(1)},\phi_\pm^{(2)}$. Two linear combinations correspond to genuine Nambu-Goldstone modes associated with the breaking of both the anomalous and the nonanomalous chiral symmetries, whereas the remaining two are pseudo-Nambu-Goldstone modes in the sense of Seo.[32]

The reformulation of Callan's proposal also proceeds without essential change. One starts again from the bosonic representation

$$R_\pm^{(1)}=\left[\frac{c\mu}{2\pi}\right]^{1/2}:\left[\exp i\sqrt{\pi}\left[\phi_R^{(1)}(r,t)\mp\int_0^r ds\,\dot{\phi}_R^{(1)}(s,t)\right]\right]:_{\rm sym}\ , \quad (3.87)$$

$$L_\pm^{(1)}=\left[\frac{c\mu}{2\pi}\right]^{1/2}(-1)^{F_1}:\exp\left[i\sqrt{\pi}\left[\phi_L^{(1)}(r,t)\pm\int_0^r ds\,\dot{\phi}_L^{(1)}(s,t)\right]\right]:_{\rm sym}\ , \quad (3.88)$$

$$R_\pm^{(2)}=\left[\frac{c\mu}{2\pi}\right]^{1/2}(-1)^{F_1}:\exp\left[i\sqrt{\pi}\left[\phi_R^{(2)}(r,t)\mp\int_0^r ds\,\dot{\phi}_R^{(2)}(s,t)\right]\right]:_{\rm sym}\ , \quad (3.89)$$

$$L_\pm^{(2)}=\left[\frac{c\mu}{2\pi}\right]^{1/2}(-1)^{F_1+F_2}:\exp\left[i\sqrt{\pi}\left[\phi_L^{(2)}(r,t)\pm\int_0^r ds\,\dot{\phi}_L^{(2)}(s,t)\right]\right]:_{\rm sym}\ , \quad (3.90)$$

with the BC's

$$\phi_R^{(i)\prime}(0,t)=\phi_L^{(i)\prime}(0,t)=0 \quad (i=1,2) . \tag{3.91}$$

The currents are the same as (3.54) and (3.55) apart from the extra index i, and we find that fermion number and chirality are conserved for each doublet, but not fermion electric charge. The suggestion therefore is that the BC's are effectively modified to

$$\begin{aligned} &A^{(1)\prime}(0,t)=A^{(2)\prime}(0,t)=0 , \\ &B^{(1)\prime}(0,t)-B^{(2)\prime}(0,t)=\dot B^{(1)}(0,t)+\dot B^{(2)}(0,t)=0 \end{aligned} \tag{3.92}$$

so that what is conserved is the fermion number of each doublet, the sum of the fermion electric charge, and the nonanomalous chiral charge.

To analyze the implications of the modified BC's for the fermionic fields, it is convenient to introduce

$$2C^{(1)}(r,t)=B^{(1)}(r,t)+B^{(2)}(r,t)-\int_0^r ds\, \dot B^{(1)}(s,t) + \int_0^r ds\, \dot B^{(2)}(s,t) , \tag{3.93}$$

$$2C^{(2)}(r,t)=B^{(1)}(r,t)+B^{(2)}(r,t) + \int_0^r ds\, \dot B^{(1)}(s,t)-\int_0^r ds\, \dot B^{(2)}(s,t) , \tag{3.94}$$

The ground state corresponding to (3.92) is then given by

$$c_q^{A(i)}\,|\,\mathrm{mod}\rangle = a_q^{C(i)}\,|\,\mathrm{mod}\rangle=0 \quad (i=1,2) \tag{3.95}$$

and we find up to Klein factors

$$\begin{bmatrix} R_+^{(1)} \\ R_+^{(2)} \\ R_-^{(1)} \\ R_-^{(2)} \end{bmatrix} = \begin{bmatrix} \xi_+^{A(1)}\xi_+^{C(1)} \\ \xi_+^{A(2)}\xi_+^{C(2)} \\ \xi_-^{A(1)}\xi_-^{C(2)\dagger} \\ \xi_-^{A(2)}\xi_-^{C(1)\dagger} \end{bmatrix} , \quad \begin{bmatrix} L_+^{(1)} \\ L_+^{(2)} \\ L_-^{(1)} \\ L_-^{(2)} \end{bmatrix} = \begin{bmatrix} \xi_-^{A(1)}\xi_-^{C(2)} \\ \xi_-^{A(2)}\xi_-^{C(1)} \\ \xi_+^{A(1)}\xi_+^{C(1)\dagger} \\ \xi_+^{A(2)}\xi_+^{C(2)\dagger} \end{bmatrix} . \tag{3.96}$$

Unlike the previous case with one doublet, it is now impossible to group the various components into a Dirac field, in agreement with the analysis of Callan and Das.[32] Therefore, the two proposals lead to different physical results, and it is necessary to make a choice.

For $M=0$, however, it is easy to see that the two classes of ground states are degenerate. The conservation of fermion electric charge (3.43) implies that

$$\langle \mathscr{H}_F^{\mathrm{mix}} \rangle =0 , \tag{3.97}$$

whereas bosonization shows that $\langle \mathscr{H}_F^{\mathrm{point}}\rangle + \langle \mathscr{H}_C \rangle$ are equal, since the first term does not depend on the BC's, whereas the second term is evaluated with the same charge-conserving BC.

To resolve this degeneracy, the natural procedure would be to add a mass term. In that case, we are no longer able to construct the ground state according to Callan's proposal, since the mass term cannot be written as a bilinear in a Dirac field. Nevertheless, there are good reasons to believe that our construction $|\,U\rangle$ will give a lower value for $\langle \mathscr{H} \rangle$, and hence is a better candidate for the ground state.

Whatever $|\,\mathrm{mod}\rangle$ may be, we do not expect it to give a better value for $\langle \mathscr{H}_F^{\mathrm{point}}\rangle$, and we may still assume (3.97) if the dyon degree of freedom is not to be excited. Therefore, the only hope for $|\,\mathrm{mod}\rangle$ lies with $\langle \mathscr{H}_C\rangle$. However,

$$\left\langle U \left| \left[Q_{\mathrm{tot}} - \int_0^\infty dr\, \mathscr{f}(r)\rho(r) \right]^2 \right| U \right\rangle = O(1) , \tag{3.98}$$

so one must do better, clearly a difficult task in view of the fact that $\langle U\,|\,\mathscr{H}_C\,|\,U\rangle$ and $\langle \mathrm{mod}\,|\,\mathscr{H}_C\,|\,\mathrm{mod}\rangle$ are equal for $M=0$. The difficulty is compounded by the fact that

$$\langle U\,|\,\mathscr{H}_F^{\mathrm{point}}\,|\,U\rangle = -\frac{1}{\pi}\int_0^\infty dr \int_0^\Lambda dk\,(k^2+M^2)^{1/2} \tag{3.99}$$

is proportional to the volume, and the requirement[23,24] that the infinite-volume limit must be taken before the ultraviolet cutoff Λ is lifted. Therefore, barring the possibility that $|\,\mathrm{mod}\rangle$ gives a lower (or at least equal) energy density for the quadratic Hamiltonian $\mathscr{H}_F^{\mathrm{point}}$ than the Fock construction $|\,U\rangle$, we may conclude that $|\,\mathrm{mod}\rangle$ is not the correct ground state for $M\neq 0$, and the nonanomalous chiral symmetry remains broken as $M\to 0$.

As noted in the Introduction, the conclusion is consistent with the infrared divergence of the chiral perturbation series around the Rubakov-Callan ground state, as found by Polchinski.[43] According to our results, $|\,\mathrm{mod}\rangle$ should be unstable against emission of light particles in the presence of mass perturbations.

IV. DISCUSSIONS

The result of the previous section raises the serious possibility that grand-unified-theory (GUT) monopoles (if they exist) actually do not catalyze proton decay at strong-interaction rates, since the case of SU(5) is formally equivalent to that of SU(2) with the identification

$$\chi^{(1)} = \begin{bmatrix} u_1^c \\ u_2 \end{bmatrix} , \quad \chi^{(2)} = \begin{bmatrix} d_3 \\ e^+ \end{bmatrix} . \tag{4.1}$$

On the other hand, several arguments have been advanced that monopoles do catalyze proton decay at strong-interaction rates, so let us reexamine those arguments in light of our results.

The first argument is based on conservation laws[32] associated with

$$F^{(i)} = \int_0^\infty dr\, \chi^{(i)\dagger}(r)\chi^{(i)}(r) \quad (i=1,2) \tag{4.2}$$

$$Q_5^{(-)} = \int_0^\infty dr [\chi^{(1)\dagger}(r)\gamma_5\chi^{(1)}(r) - \chi^{(2)\dagger}(r)\gamma_5\chi^{(2)}(r)] , \tag{4.3}$$

$$Q_F^{(+)} = \int_0^\infty dr \left[\chi^{(1)\dagger}(r)\frac{\tau_3}{2}\chi^{(1)}(r) + \chi^{(2)\dagger}(r)\frac{\tau_3}{2}\chi^{(2)}(r) \right] . \tag{4.4}$$

The case for catalysis is based on the assumption that all four charges (4.2)—(4.4) are conserved in the $j=0$ sector, the justification being that the first three are associated with conserved currents in the full theory, and the last one is dynamically required as in (3.43), if the dyon degree of freedom is not to be excited in the limit $m_X \gg M$. Together with the kinematics of the $j=0$ partial wave, it then follows that processes such as

$$u_{1L} + \text{monopole} \to u_{1R} + \text{monopole} \quad (4.5)$$

are forbidden, and the simplest type of an allowed process is

$$u_{1L} + u_{2L} + \text{monopole} \to d^c_{3R} + e^+_R + \text{monopole} \quad (4.6)$$

which conserves fermion color hypercharge as well as (4.2)–(4.4), but not baryon number.

The weakness of the argument however lies in the well-known fact that the integrated charge $Q_5^{(-)}$ need not be conserved, even if the associated current is. In the monopole sector of the fermion—Yang-Mills—Higgs system, the dyon degree of freedom induces a four-Fermi interaction (3.33), which may lead to a spontaneous breaking of the nonanomalous as well as the anomalous chiral symmetry. Indeed, the variational framework we have adopted as well as the results we have obtained are quite standard for this class of problems since the classic works of Nambu and Jona-Lasinio[12] and Vaks and Larkin.[12]

Recent work on symmetry breaking in vectorlike gauge theories[82] also lends support to our (counter)argument. The s-wave approximation is equivalent to taking the functional integral for a SU(2) gauge theory with (massless) Dirac fermions

$$\int [dA][d\Psi][d\overline{\Psi}]e^{-S_E[A,\Psi,\overline{\Psi}]} \quad (4.7)$$

and restricting the measure to U(1) transforms of the 't Hooft—Polyakov solution[71] and the corresponding s-wave fermions. (Note that the Higgs field plays only a marginal role in this formulation.) The restriction, however, preserves the positivity of the Euclidean measure, which is the essential ingredient in the investigations of Ref. 82, so we would expect that the pattern of symmetry breaking with the s-wave Hamiltonian is the same as for QCD [with SU(2) color], i.e., unbroken vector symmetries but broken chiral symmetries.[83]

The second argument for catalysis is based on the modification of current algebra in the presence of a monopole.[36] However, the extra contribution (2.56) which formed the basis of the argument was seen to be present for the Dirac phase as well, so the argument can serve at most as a necessary condition for catalysis, not a sufficient one.

We are thus left only with recourse to actual dynamical calculations. However, the "orthodox" result was obtained only by replacing the charge-mixing term $\mathscr{H}_F^{\text{mix}}$ with the charge-mixing BC (3.28) for the fermions, a procedure which, despite appearances, can be far from innocuous. As we have seen, (1) results derived in the semiclassical expansion treating the dyon as a background field cannot be carried over into the regime $e^2 m_X \gg M$, especially when $M=0$, (2) if the dyon degree of freedom is properly quantized, the Dirac equation for the fermions will have a strong nonlinearity, which cannot be neglected with impunity, and (3) the charge-mixing BC is wiped out anyhow, even with the orthodox calculation.

Another aspect of the "orthodox" calculation also provides cause for reappraisal. If we calculate the functional integral (4.7) with two doublets, restricting the measure to a dilute gas of instantons,[17,78] we would find a four-body condensate which breaks the anomalous chiral symmetry, but no two-body condensates which break the nonanomalous chiral symmetry. If this result were taken at face value, the inference might be that the nonanomalous chiral symmetry is realized in the Wigner-Weyl mode for massless QCD [with SU(2) color], and that the nonanomalous chiral charge must be conserved for any scattering process. To us, the similarity between this line of reasoning and the orthodox position on catalysis is too close for comfort.

Nevertheless, one may question the applicability of our results to GUT monopoles, since some differences exist between SU(2) and SU(5) which may turn out to be significant, once we go beyond the s-wave approximation. One is that the analog of the fermion number no longer needs to be conserved, since in terms of Weyl fields, three come from the **10** representation and one comes from the **5***. Similarly, the origin of the fermion mass is different, coming from the Higgs **5**. Another difference is the existence of strong[29] and weak[30,64] interactions. Also, the relation between the vacuum sector and the monopole sector becomes different for SU(5), since global color ceases to be defined.[84]

Actually, however, the last circumstance is presumably a blessing as far as an extrapolation of our results is concerned. Grossman[24] has suggested that the fermion-monopole system is analogous to a Kondo-type system, with the dyon degree of freedom providing the analog of the impurity spin. A Hartree-Fock type of approach such as ours would then be ill suited for the problem.

The objection would indeed be quite serious if the dyon degree of freedom were non-Abelian, since the two essential ingredients of the Kondo effect are the noncommutativity of the impurity spin and the existence of a Fermi surface. However, in our case, the dyon degree of freedom is Abelian, and we have seen that it may be eliminated through the conservation law (3.35) in favor of a simple four-Fermi interaction. Therefore, it would seem that superconductivity would be a better analog than the Kondo effect, as suggested by Srivastava and Widom.[26]

In short, although there is nothing wrong with the basic idea that fermions in the s-wave can probe the monopole core, there is also nothing wrong either with GUT monopoles behaving like Dirac monopoles,[54] particularly since the Abelian BC mimics all the physics of the underlying non-Abelian theory: the Witten effect,[76] chiral condensates,[13] and Nambu-Goldstone modes.[12] To determine which possibility is actually realized, it is essential to study the effect of mass terms; the evidence so far points strongly to the Dirac scenario.

Even if the reader disagrees with this assessment, we may still call attention to the following issues which have a simple resolution within the Dirac scenario, but to our knowledge, has not been resolved for the orthodox scenario.

(i) How do the "half-solitons" combine to form a full fermion?

(ii) How do we recover the semiclassical results when $m_X \lesssim M$?

(iii) What are the total charge and the charge distribution of the stable dyons if there are no massless fermions?

This would seem to be a case where common sense as well as normal scientific practice would dictate that one should not accept the exotic without ruling out the mundane.

Aside from implications for proton-decay catalysis, we believe our analysis is also interesting from a general field-theoretical point of view. One feature of interest is that Nambu-Goldstone bosons can exist in $1+1$ dimensions, if space is restricted to a half-line. This could be of relevance to materials with a boundary, or theories such as the quantum Liouville model where spontaneous compactification may occur.[85]

The other feature of interest is that a physical Nambu-Goldstone mode can accompany the spontaneous breaking of an anomalous chiral symmetry. To our knowledge, this is the first nontrivial example of such a phenomenon. In particular, our result clearly shows that the Kogut-Susskind mechanism[16] depends on the specific dynamics of the theory, and not just the general fact that $j^{\text{inv}}_{\mu 5}$ is not conserved or Q_5^{sym} is not invariant under "large" gauge transformations.

Another area where our analysis is of interest is of quark models. It has already been recognized[31,86] that the BC (2.24) for the monopole is quite similar to that of the bag model. If we take a spherical bag with free quarks on the outside, we may bosonize each partial wave separately, which would give us a Nambu-Goldstone mode associated with chiral-symmetry breaking, i.e., pions [and a U(1) boson]. Equation (2.49) then suggests that one may now consistently delete the quark states from the spectrum.

However, to return to the fermion-monopole system, it must be mentioned that we have left unsolved many problems, some already noted[23] in (II). One is that the variational approach, although conceptually unambiguous and qualitatively satisfactory, is quite unsatisfactory in its quantitative aspects. Simple energetics suggests that the semiclassical approximations should break down when

$$Q_{\text{tot}}/I \gtrsim 2M \ . \tag{4.8}$$

However, our previous results were that the breakdown occurs when

$$\frac{e^2}{4\pi}\ln\frac{m_W}{M} \gtrsim 1 \ . \tag{4.9}$$

Another problem was that we have ignored the back reaction of the electric fields on the magnetic fields, an effect which exists even in a static situation, owing to the non-Abelian nature of the monopole. The back reaction is negligible in the semiclassical approximation; however, so is the back reaction of the fermions, which we have seen is actually quite important when the semiclassical expansion is no longer valid.[87]

In fact, if the back reaction on the magnetic field is significant, there is no reason for monopoles to be heavy; for all we know, they could even be as light as quarks and leptons. On the other hand, there are some reasons to believe that the back reaction may not be important. We have already noted[23] in (II) that the ground state carries charge

$$\langle [\overline{\Psi}, \gamma_0 \tau^a \Psi] \rangle_{s\ \text{wave}} \neq 0 \tag{4.10}$$

but not current

$$\langle [\overline{\Psi}, \gamma_k \tau^a \Psi] \rangle_{s\ \text{wave}} = 0 \ . \tag{4.11}$$

Also, the back reaction of the electrostatic potential J and the Higgs field H cancel in the Prasad-Sommerfield limit,[72] whereas the Montonen-Olive-Witten argument[88] suggests that the mass of the monopole is still governed by the Higgs field in supersymmetric theories.

The third problem is the validity of the s-wave approximation, particularly with respect to the emission of soft photons or gluons.[48] If we consider a classical charge-monopole system, we find that the charged particle experiences infinite acceleration in the minimum angular-momentum state. Similarly, for the quantum system, the expression for the phase shift[23,24]

$$\tan\delta(E) = \frac{k}{E+M}\tan\left[\frac{\theta}{2}+\frac{\pi}{4}\right] \tag{4.12}$$

implies that the collision time $d\delta(E)/dE$ tends to zero as $E/M \to 0$.

The fourth problem is the behavior of a multimonopole system interacting with fermions. Here the one-particle approach runs into conceptual as well as technical difficulties. The (extended) charge quantization condition[93] requires that θ must be common to all monopoles (dyons), and it is not at all clear how that may come about. A related problem is how the various symmetries are broken or restored in a monopole-antimonopole system (monium) as the antimonopole is moved off to infinity.[89] In particular, the behavior of the Nambu-Goldstone mode associated with the anomalous chiral symmetry is of great interest in connection with the U(1) problem.[13–19]

The last and also related problem is what happens to the chiral currents when a Dirac monopole is itself a dynamical object, since (2.55) suggests that $j^{\text{inv}}_{\mu 5}$ actually ceases to be invariant. The major difficulty here is that a satisfactory formalism for a dynamical point monopole is lacking so far. The fiber-bundle description[90,91] does not adapt well to creation and annihilation of monopoles, whereas the traditional "string" formulation[92,93] does not seem to accommodate θ properly. The most promising approach would seem to be a lattice formulation.[94] Even then, however, nontrivial obstacles are expected when adding fermions.[7]

Evidently, a satisfactory approach to these problems would be most welcome.

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APPENDIX A: PARTIAL-WAVE ANALYSIS OF THE ANOMALY

We wish to compute the commutator $[Q[f],\ Q_5[g]]$, where

$$Q[f]=\int d\mathbf{x}\, f(r):\Psi^\dagger(\mathbf{x})\Psi(\mathbf{x}): , \tag{A1}$$

$$Q_5[g]=\int d\mathbf{x}\, g(r):\Psi^\dagger(\mathbf{x})\gamma_5\Psi(\mathbf{x}): . \tag{A2}$$

Since we are interested in the short-distance singularities, we set the fermion mass to zero. As usual, Ψ and $\Psi^\dagger$ are expressed in terms of eigenfunctions of the Dirac Hamiltonian:

$$\Psi(\mathbf{x},t)=\frac{1}{\pi}\int_0^\infty dk \sum_{\alpha jm}[b_{kjm}^{(\alpha)}u_{kjm}^{(\alpha)}(\mathbf{x})e^{-ikt} + d_{kj,-m}^{(\alpha)\dagger}v_{kjm}^{(\alpha)}(\mathbf{x})e^{ikt}] , \tag{A3}$$

$$\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})u_{kjm}^{(\alpha)}(\mathbf{x})=ku_{kjm}^{(\alpha)}(\mathbf{x}) , \tag{A4}$$

$$\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})v_{kjm}^{(\alpha)}(\mathbf{x})=-kv_{kjm}^{(\alpha)}(\mathbf{x}) ,$$

$$\begin{aligned}&\mathbf{J}^2u_{kjm}^{(\alpha)}(\mathbf{x})=j(j+1)u_{kjm}^{(\alpha)}(\mathbf{x}) ,\quad J_zu_{kjm}^{(\alpha)}(\mathbf{x})=mu_{kjm}^{(\alpha)}(\mathbf{x}) ,\\ &\mathbf{J}^2v_{kjm}^{(\alpha)}(\mathbf{x})=j(j+1)v_{kjm}^{(\alpha)}(\mathbf{x}) ,\quad J_zv_{kjm}^{(\alpha)}(\mathbf{x})=mv_{kjm}^{(\alpha)}(\mathbf{x}) ,\end{aligned} \tag{A5}$$

$$\{b_{kjm}^{(\alpha)},b_{k'j'm'}^{(\beta)\dagger}\}=\{d_{kjm}^{(\alpha)},d_{k'j'm'}^{(\beta)\dagger}\} =\pi\delta_{jj'}\delta_{mm'}\delta_{\alpha\beta}\delta(k-k') . \tag{A6}$$

For higher partial waves corresponding to given (E,j,m), there exist two eigenfunctions:[95]

$$u_{kjm}^{(1)}(\mathbf{x})=\frac{1}{r}\begin{bmatrix}F_{kj}^{(1)}(r)\xi_{jm}^{(1)}(\Omega)\\ G_{kj}^{(1)}(r)\xi_{jm}^{(2)}(\Omega)\end{bmatrix} ,\quad u_{kj}^{(1)}(r)=\begin{bmatrix}F_{kj}^{(1)}(r)\\ G_{kj}^{(1)}(r)\end{bmatrix} , \tag{A7}$$

$$u_{kjm}^{(2)}(\mathbf{x})=\frac{1}{r}\begin{bmatrix}F_{kj}^{(2)}(r)\xi_{jm}^{(2)}(\Omega)\\ G_{kj}^{(2)}(r)\xi_{jm}^{(1)}(\Omega)\end{bmatrix} ,\quad u_{kj}^{(2)}(r)=\begin{bmatrix}F_{kj}^{(2)}(r)\\ G_{kj}^{(2)}(r)\end{bmatrix} \tag{A8}$$

(and similarly for v). We then have

$$\langle[Q[f],\ Q_5[g]]\rangle_{j\neq0}=\frac{e}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\sum_{\alpha jm\beta j'm'}\sum[(v_{kjm}^{(\alpha)},fu_{k'j'm'}^{(\beta)})(u_{k'j'm'}^{(\beta)},g\gamma_5v_{kjm}^{(\alpha)})-(u\leftrightarrow v)] , \tag{A9}$$

where

$$(v,fu)=\int d\mathbf{x}\, v^\dagger(\mathbf{x})f(r)u(\mathbf{x}) . \tag{A10}$$

Higher partial waves do not survive the angular integrations. By orthogonality of $\xi_{jm}^{(\alpha)}$

$$(v_{kjm}^{(\alpha)},fu_{k'j'm'}^{(\beta)})\propto\delta_{jj'}\delta_{mm'}\delta_{\alpha\beta} \tag{A11}$$

while

$$(u_{k'j'm'}^{(\alpha)},g\gamma_5 vv_{kjm}^{(\beta)})\propto\delta_{jj'}\delta_{mm'}\tau_{\alpha\beta} ,\quad \tau_{\alpha\beta}=\begin{bmatrix}0&1\\1&0\end{bmatrix} . \tag{A12}$$

The contribution for $j\geq1$ is then proportional to $\tau_{\alpha\beta}\delta_{\alpha\beta}=0$ and thus vanishes.[96] On the other hand, the lowest partial-wave contributes

$$\langle[Q[f],\ Q_5[g]]\rangle=\frac{ie}{\pi}\int_0^\infty dr\, f'(r)g(r) \tag{A13}$$

as shown in the text.

For the anomalous divergence, we wish to consider a dyon with an electrostatic potential $A_0(r)$, which is regular at the origin $A_0(0)<\infty$. Then the wave functions satisfy

$$\begin{aligned}&[\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})+eA_0(r)]u_{kjm}^{(\alpha)}(\mathbf{x})=ku_{kjm}^{(\alpha)}(\mathbf{x}) ,\\ &[\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})+eA_0(r)]v_{kjm}^{(\alpha)}(\mathbf{x})=-kv_{kjm}^{(\alpha)}(\mathbf{x}) .\end{aligned} \tag{A14}$$

We calculate

$$\partial_\mu\langle j_{\mu5}^{\rm inv}\rangle=\nabla\cdot\int_0^\infty\frac{dk}{2\pi}\sum_{\alpha jm}[v_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}v_{kjm}^{(\alpha)}(\mathbf{x}) -u_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}u_{kjm}^{(\alpha)}(\mathbf{x})] . \tag{A15}$$

Point splitting in the time direction with the standard phase factor

$$\nabla\cdot\int_0^\infty\frac{dk}{2\pi}\sum_{\alpha jm}[v_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}v_{kjm}^{(\alpha)}(\mathbf{x})e^{ikt} -u_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}u_{kjm}^{(\alpha)}(\mathbf{x})e^{-ikt}]e^{ieA_0t} . \tag{A16}$$

Using properties of monopole harmonics, this reduces to[96]

$$\frac{1}{2\pi r}\frac{d}{dr}r^2\int_0^\infty dk\sum_{\alpha jm}[v_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}v_{kjm}^{(\alpha)}(\mathbf{x})e^{ikt} -u_{kjm}^{(\alpha)\dagger}(\mathbf{x})\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}u_{kjm}^{(\alpha)}(\mathbf{x})e^{-ikt}]e^{ieA_0t} . \tag{A17}$$

We can now see that higher partial waves will not contribute to this expression, since they involve the vanishing quantity

$$\sum_m\xi_{jm}^{(\alpha)\dagger}\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}\xi_{jm}^{(\alpha)} . \tag{A18}$$

Only the lowest partial wave remains, giving

$$\partial_\mu\langle j_{\mu5}^{\rm inv}\rangle=-\frac{e}{4\pi^2r^2}A_0'(r)-\frac{e}{\pi}A_0(0)\delta(\mathbf{x}) . \tag{A19}$$

APPENDIX B: RENORMALIZATION OF THE CHIRAL ANOMALY

In this section, we shall mainly follow the paper of Bardeen.[3] Bare quantities and renormalized quantities will be distinguished by the superscripts B and R.

Let us consider the bare chiral current

$$j^B_{\mu 5}=\bar{\Psi}^B\gamma_\mu\gamma_5\Psi^B=Z_2\bar{\Psi}^R\gamma_\mu\gamma_5\Psi^R \tag{B1}$$

in a SU(2) theory with a massless Dirac isodoublet. If we use a gauge-invariant regularization, such as the Fujikawa-Vergeles prescription,[4] we arrive at the unrenormalized form of the anomalous divergence equation

$$\partial_\mu j^{\text{inv}B}_{\mu 5}=\frac{(e^B)^2}{16\pi^2}F^{aB}_{\mu\nu}\tilde{F}^{aB}_{\mu\nu} =\frac{(e^R)^2}{16\pi^2}F^{aR}_{\mu\nu}\tilde{F}^{aR}_{\mu\nu}\ . \tag{B2}$$

Introducing the (bare) Chern-Simons current[10,67]

$$C^B_\mu=-\frac{(e^B)^2}{32\pi^2}\epsilon_{\mu\alpha\beta\gamma}\left[F^{aB}_{\alpha\beta}A^{aB}_\gamma-\frac{e^B}{3}\epsilon_{abc}A^{aB}_\alpha A^{bB}_\beta A^{cB}_\gamma\right] \tag{B3}$$

we may rewrite (B2) as

$$\partial_\mu j^{\text{sym}B}_{\mu 5}=0\ ,\quad j^{\text{sym}B}_{\mu 5}=j^{\text{inv}B}_{\mu 5}+2C^B_\mu\ . \tag{B4}$$

Since $j^{\text{sym}B}_{\mu 5}$ is an external conserved current, it follows that its matrix elements are actually finite,[97] i.e.,

$$j^{\text{sym}B}_{\mu 5}=j^{\text{inv}B}_{\mu 5}\ . \tag{B5}$$

On the other hand, $j^{\text{inv}B}_{\mu 5}$ is not finite, since $F^{aR}_{\mu\nu}\tilde{F}^{aR}_{\mu\nu}$ is not. However, since it is the only gauge-invariant pseudovector with dimension three, it is still multiplicatively renormalizable

$$j^{\text{inv}R}_{\mu 5}=Z_A j^{\text{inv}B}_{\mu 5}\ . \tag{B6}$$

Therefore, we may introduce the renormalized quantities

$$C^R_\mu=C^B_\mu-\tfrac{1}{2}(Z_A-1)j^{\text{inv}B}_{\mu 5}\ , \tag{B7}$$

$$-\frac{(e^R)^2}{32\pi^2}[F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}]^R=\partial_\mu C^R_\mu\ . \tag{B8}$$

We find that renormalization does not affect the properties under gauge transformations

$$\delta_h C^R_\mu=\delta_h C^B_\mu=-\frac{(e^R)^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\partial_\nu(h^a\partial_\alpha A^{aR}_\beta)\ , \tag{B9}$$

$$\delta_h[F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}]^R=0\ . \tag{B10}$$

Also, $[F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}]^R$ may still be regarded as a topological charge density, since it is a total derivative. However, as emphasized by Crewther[17] and Witten,[18,51] its spectrum will depend on the fermions, as expected from (B7) and (B8).

The addition of a Dirac mass does not significantly affect the situation, since it will only add a "soft" divergence

$$2iM^B\bar{\Psi}^B\gamma_5\Psi^B \tag{B11}$$

to (B2). In particular,

$$Q^{\text{sym}}_5=\int d\mathbf{x}\, j^{\text{sym}}_{05} \tag{B12}$$

will still generate chiral transformations at equal times.

On the other hand, a Higgs-boson mass leads to significant changes, since it will add the "hard" divergence

$$2iG^B\bar{\Psi}^B\gamma_5\frac{\tau^a}{2}\Psi^B\phi^{aB}\ . \tag{B13}$$

In particular, one no longer has a meaningful chiral symmetry,[98] and it is possible to generate a Dirac mass starting from $M^B=0$ but $G^B\neq 0$.[99]

APPENDIX C: TECHNICAL ISSUES

In this section, we compare our results with the existing literature on some technical issues.

(i) Anomalous commutators

For an Abelian monopole, we have seen that there exists an ambiguity (2.8) and (2.11) associated with the extrapolation of the anomalous commutator from the vacuum sector. Similarly, the commutator between the electric charge and the axial charge was found to be ill defined, a point which seems to be frequently overlooked.

On the other hand, a careful treatment showed that

$$[Q,\, Q_5[g]]=0\ ,\quad g(\infty)=0\ , \tag{C1}$$

indicating that the electric charge is a chiral singlet, contrary to Wilczek[30] and Callan [second citation in Ref. 22, Eq. (4.7)].

The source of the discrepancy with the latter is clear: Callan has not retained the $\delta'(r+r')$ term in (2.29). As for the former, we believe Wilczek has not correctly taken into account the discontinuity associated with the limit $M\to 0$. If there fermions are strictly massless, the electric charge of the ground state is zero independently of the chiral angle. On the other hand, if the fermions are massive, the electric charge is proportional to θ; however, the different ground states are no longer related by (active) chiral transformations.

The argument is also consistent with our previous results for the non-Abelian case:[23] In the gauge (3.16)—(3.18), the total charge is given by the momentum p_φ conjugate to the dyon degree of freedom, which commutes with the fermion field at equal times.

(ii) Two forms of bosonization

For the Abelian case, we had noted the existence of two dual forms of bosonization (2.38) and (2.50), both which, however, gave the same results as $\mu\to 0$. In particular, no question was involved of the dyon degree of freedom, in contrast with Kazàma and Sen.[38]

The source of the discrepancy seems to lie in their expression for the electric field (3.16). As we have noted[23] in (III), the correct expression should contain the surface term $\phi_L(\infty)-\phi_R(\infty)$, which in turn leads to a change in their commutator (3.20).[100]

The authors have also attempted to justify the charge-mixing BC for the fermions on the basis of a decoupling argument (p. 194). It is easy to see, however, that their argument is not valid precisely when our (3.44) is satisfied.

(iii) Reduction to 1 + 1 dimensions

As we have noted, the consistency of the reduction to 1 + 1 dimensions requires that

$$\chi^\dagger(0)\gamma_5\tau_3\chi(0)=0 \tag{C2}$$

as well as (3.44) must be satisfied in the point limit. The first condition (C2) is incompatible with transformations of the type

$$\begin{bmatrix} \tau_1 R^\dagger \\ L \end{bmatrix} \to U \begin{bmatrix} \tau_1 R^\dagger \\ L \end{bmatrix}, \quad U^\dagger U = 1 , \tag{C3}$$

a point which seems to be overlooked by Craigie and Nahm.[29]

Also, for a non-Abelian monopole treated as a background field, we have found that the higher partial waves contribute to the anomalous divergence. Strictly speaking therefore, the equation

$$\partial_\mu j^{\text{inv}}_{\mu 5} = \frac{e^2}{8\pi^2}[F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu}] \tag{C4}$$

is inconsistent with the s-wave approximation, a point which is sometimes ignored in the literature.

On the other hand, it follows from (3.25) and (3.26) that the integral version of (C4) does not suffer from such inconsistencies, so this is a relatively minor problem.

(iv) The Witten charge for a Higgs-boson mass

On this issue, we shall merely add the following observations.

(1) As noted in previous papers,[23] the proper quantization of the dyon degree of freedom shows that the bosonized Hamiltonian used by Callan[31] and Harvey[33] lacks certain surface terms which are crucial for the problem at hand.

(2) If $m_W \lesssim M$, we expect that the semiclassical approximation should give the correct results, treating the monopole as a background field. The Jackiw-Rebbi zero mode[52] then satisfies the charge-mixing BC, and carries fermion number $F=\pm\frac{1}{2}$, but no net electric charge. Our procedure recovers this result, whereas Callan and Harvey do not.

(3) Harvey has also argued that a chiral rotation in the lowest partial wave will transform a Higgs-boson mass into a Dirac mass, and hence $\bar{\theta}$ should be shifted away from θ. This argument we believe is doubly incorrect. As we have seen, higher partial waves contribute to the anomalous divergence, so it is necessary to include these contributions, as done by Niemi *et al.*[11] Furthermore, even in the lowest partial wave, a chiral transformation will change the charge mixing term $\gamma_5\tau_2 K(r)/r$ to $\tau_2 K(r)/r$ as well, so the case of a Higgs-boson mass cannot be reduced to that of a Dirac mass.

Finally, we wish to record our opinion that among the various expositions of the orthodox calculation, the one by Polchinski[43] is technically the most reliable.

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